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On the Carlitz's type twisted (p,q)-Euler polynomials and twisted (p,q)-Euler zeta function

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Abstract : In this paper we construct Carlitz's type twisted (p,q)-Euler zeta function. In order to define Carlitz's type twisted (p,q)-Euler zeta function, we introduce the Carlitz's type twisted (p,q)-Euler numbers and polynomials by generalizing the Euler numbers and polynomials, Carlitz's type q-Euler numbers and polynomials. We also give some interesting properties, explicit formulas, a connection with Carlitz's type twisted (p,q)-Euler numbers and polynomials. Finally, we investigate the zeros of the Carlitz's type twisted (p,q)-Euler polynomials by using computer.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, (h,q)-Euler numbers and polynomials, Carlitz's type twisted (p,q)-Euler numbers and polynomials, (p,q)-Euler zeta function, twisted (p,q)-Euler zeta function.

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1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-10]). In this paper, we define Carlitz's type twisted (p,q)-Euler numbers and polynomials and study some properties of the Carlitz's type twisted (p,q)-Euler numbers and polynomials.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0,-1,-2,-3,\ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the following generating functions(see [1, 2, 3, 4, 5])

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

and

$$\left(\frac{2}{e^t+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.2}$$

respectively.

The (p,q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1}.$$

It is clear that (p,q)-number contains symmetric property, and this number is q-number when $p = 1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q} = n$ with $p = 1$.

By using (p,q)-number, we define the (p,q)-analogue of Euler polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Euler numbers and polynomials. We begin by recalling here the Carlitz's type q-Euler numbers and polynomials(see 1, 2, 3, 4, 5]).

Definition 1. The Carlitz's type q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function

$$F_q(t,x)=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_q t}. \tag{1.3}$$

and their values at $x=0$ are called the Carlitz's type q-Euler numbers and denoted $E_{n,q}$.

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-10]). Based on this idea, we generalize the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$. It follows that we define the following (p,q)-analogues of the the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$ (see [6, 7, 9, 10]).

Definition 2. For $0<q<p\le 1$, the Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_{p,q}t}. \tag{1.4}$$

and

$$F_{p,q}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}, \tag{1.5}$$

respectively.

In the following section, we define Carlitz's type twisted (p,q)-Euler zeta function. We introduce the Carlitz's type twisted (p,q)-Euler polynomials and numbers. After that we will investigate some their properties. Finally, we investigate the zeros of the Carlitz's type twisted (p,q)-Euler polynomials by using computer.

2. Twisted (p,q)-Euler numbers and polynomials

In this section, we define twisted (p,q)-Euler numbers and polynomials and provide some of their relevant properties. Let r be a positive integer, and let ω be rth root of 1.

Definition 2. For $0<q<p\le 1$, the Carlitz's type twisted (p,q)-Euler numbers $E_{n,p,q,\omega}$ and polynomials $E_{n,p,q,\omega}(x)$ are defined by means of the generating functions

$$F_{p,q,\omega}(t)=\sum_{n=0}^{\infty}E_{n,p,q,\omega}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m]_{p,q}t}. \tag{2.1}$$

and

$$F_{p,q,\omega}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q,\omega}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p=1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type twisted q-Euler number $E_{n,q,\omega}$ and q-Euler polynomials $E_{n,q,\omega}(x)$ respectively. Obviously, if we put $\omega=1$, then we have

$$E_{n,p,q,\omega}(x)=E_{n,p,q}(x),\quad E_{n,p,q,\omega}=E_{n,p,q}.$$

Putting $p=1$, we have

$$\lim_{q\to 1}E_{n,p,q,\omega}(x)=E_{n,\omega}(x),\quad \lim_{q\to 1}E_{n,p,q,\omega}=E_{n,\omega}.$$

By using above equation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,p,q,\omega}\frac{t^n}{n!} &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m]_{p,q}t}\\ &= \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{1}{1+\omega q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q,\omega} = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{1}{1+\omega q^{l+1}p^{n-l}}.$$

If we put $p = 1$ in the above theorem we obtain

$$E_{n,p,q,\omega} = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l \frac{1}{1+\omega q^{l+1}}.$$

By (2.2), we obtain

$$E_{n,p,q,\omega}(x) = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl} p^{(n-l)x}\frac{1}{1+\omega q^{l+1}p^{n-l}}. \tag{2.4}$$

By using (2.2) and (2.4), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,p,q,\omega}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\omega q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}\\ &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m+x]_{p,q}t}.\end{aligned} \tag{2.5}$$

Since $[x+y]_{p,q} = p^y[x]_{p,q} + q^x[y]_{p,q}$, we see that

$$E_{n,p,q,\omega}(x) = [2]_q \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left(\frac{1}{p-q}\right)^l \frac{1}{1+\omega q^{k+1}p^{n-k}}. \tag{2.6}$$

Next, we introduce Carlitz's type twisted (h,p,q)-Euler polynomials $E_{n,p,q,\omega}^{(h)}(x)$.

Definition 4. The Carlitz's type twisted (h,p,q)-Euler polynomials $E_{n,p,q,\omega}^{(h)}(x)$ are defined by

$$E_{n,p,q}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m p^{hm}\omega^m [m+x]_{p,q}^n. \tag{2.7}$$

By using (2.7) and (p,q)-number, we have the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$E_{n,p,q,\omega}^{(h)}(x) = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\omega q^{l+1}p^{n-l+h}}.$$

By (2.6) and Theorem 2.4, we have

$$E_{n,p,q,\omega}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}E_{l,p,q,\omega}^{(n-l)}$$

The following elementary properties of the (p,q)-analogue of Euler numbers $E_{n,p,q,\omega}$ and polynomials $E_{n,p,q,\omega}(x)$ are readily derived form (2.1) and (2.2). We, therefore, choose to omit details involved.

Theorem 6. (Distribution relation) For any positive integer m(=odd), we have

$$E_{n,p,q,\omega}(x) = \frac{[2]_q}{[2]_{q^m}}[m]_{p,q}^n \sum_{a=0}^{m-1}(-1)^a q^a \omega^a E_{n,p^m,q^m,\omega^m}\left(\frac{a+x}{m}\right), n \in \mathbb{Z}_+.$$

Theorem 7. (Property of complement) For $n \in \mathbb{Z}_+$, we have

$$E_{n,p^{-1},q^{-1},\omega^{-1}}(1-x) = (-1)^n \omega p^n q^n E_{n,p,q,\zeta}(x).$$

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$\omega q E_{n,p,q,\omega}(1) + E_{n,p,q,\omega} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By (2.1) and (2.2), we get

$$-[2]_q \sum_{l=0}^{\infty}(-1)^{l+n} q^{l+n}\omega^{l+n} e^{[l+n]_{p,q}t} + [2]_q \sum_{l=0}^{\infty}(-1)^l q^l \omega^l e^{[l]_{p,q}t} = [2]_q \sum_{l=0}^{n-1}(-1)^l q^l \omega^l e^{[l]_{p,q}t}. \tag{2.8}$$

Hence we have

$$(-1)^{n+1}q^n\omega^n \sum_{m=0}^{\infty} E_{m,p,q,\omega}(n)\frac{t^m}{m!} + \sum_{m=0}^{\infty} E_{m,p,q,\omega}\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left([2]_q \sum_{l=0}^{n-1}(-1)^l q^l \omega^l [l]_{p,q}^m\right)\frac{t^m}{m!}. \tag{2.9}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.9), we have the following theorem.

Theorem 9. For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-1}(-1)^l q^l \omega^l [l]_{p,q}^m = \frac{(-1)^{n+1}q^n\omega^n E_{m,p,q,\omega}(n) + E_{m,p,q,\omega}}{[2]_q}.$$

We investigate the zeros of the twisted (p,q)-Euler polynomials $E_{n,p,q,\omega}(x)$ by using a computer. We plot the zeros of the twisted (p,q)-Euler polynomials $E_{n,p,q,\omega}(x)$ for $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 20, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{2}}$. In Figure 1(top-right), we choose $n = 40, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{2}}$. In Figure 1(bottom-left), we choose $n = 20, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{4}}$. In Figure 1(bottom-right), we choose $n = 40, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{4}}$.

3. Twisted (p,q)-Euler zeta function

By using twisted (p,q)-Euler numbers and polynomials, (p,q)-Euler zeta function and Hurwitz (p,q)-Euler zeta function is defined. These functions interpolate the twisted (p,q)-Euler numbers $E_{n,p,q,\omega}$, and polynomials $E_{n,p,q,\omega}(x)$, respectively. From (2.1), we note that

$$\begin{aligned} \frac{d^k}{dt^k}F_{p,q,\omega}(t)\bigg|_{t=0} &= [2]_q \sum_{m=0}^{\infty}(-1)^n q^m \omega^m [m]_{p,q}^k \\ &= E_{k,p,q,\omega}, (k \in \mathbb{N}). \end{aligned}$$

By using the above equation, we are now ready to define twisted (p,q)-Euler zeta function.

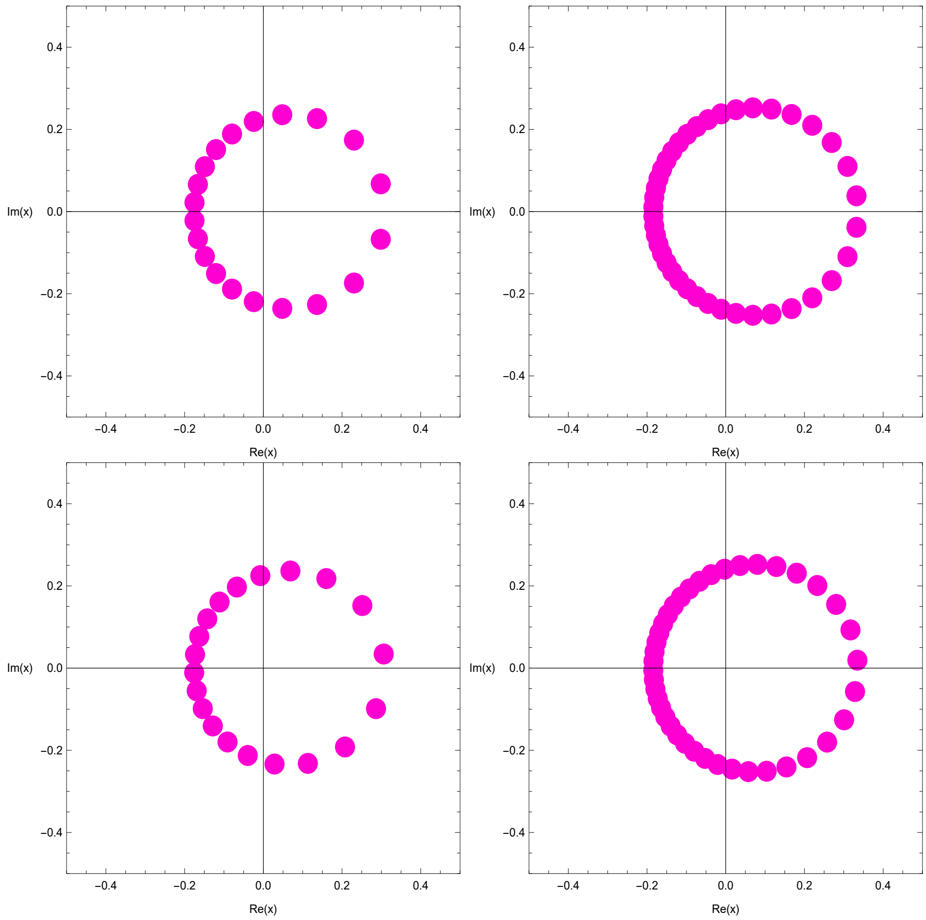

Figure 1: Zeros of $E_{n,p,q,\omega}(x)$

Definition 10. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta_{p,q,\omega}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n \omega^n}{[n]_{p,q}^s}. \tag{3.1}$$

Note that $\zeta_{p,q,\omega}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q,\omega}(s) = \zeta_E(s)$ which is the Euler zeta functions(see [4]). Relation between $\zeta_{p,q,\omega}(s)$ and $E_{k,p,q,\omega}$ is given by the following theorem.

Theorem 11. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q,\omega}(-k) = E_{k,p,q,\omega}.$$

Observe that $\zeta_{p,q,\omega}(s)$ function interpolates $E_{k,p,q,\omega}$ numbers at non-negative integers. By using (2.2), we note that

$$\left.\frac{d^k}{dt^k} F_{p,q,\omega}(t,x)\right|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \omega^m [m+x]_{p,q}^k \tag{3.2}$$

and

$$\left.\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!}\right)\right|_{t=0} = E_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz (p,q)-Euler zeta function.

Definition 12. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q,\omega}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n \omega^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $\zeta_{p,q,\omega}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q,\omega}(s,x) = \zeta_E(s,x)$ which is the Hurwitz Euler zeta functions(see [1, 3, 6]). Relation between $\zeta_{p,q,\omega}(s,x)$ and $E_{k,p,q,\omega}(x)$ is given by the following theorem.

Theorem 13. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q,\omega}(-k,x) = E_{k,p,q,\omega}(x).$$

Observe that $\zeta_{p,q,\omega}(-k,x)$ function interpolates $E_{k,p,q,\omega}(x)$ numbers at non-negative integers.

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