On the Order, Type and Zeros of Meromorphic Functions and Analytic Functions of $[p, q]$-Order in the Unit Disc

Jin Tu[1*], Ke Qi Hu[1], Hong Zhang[2]

1. College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, 330022, China
2. School of the Tourism and Urban Management, Jiangxi University of Finance and Economics, Nanchang 330032, China

Corresponding author: tujin2008@sina.com

Abstract

In this paper, the authors investigate the $[p, q]$-order and $[p, q]$-type of $f_1 + f_2$, $f_1 f_2$, f_1/f_2, where f_1, f_2 are meromorphic functions or analytic functions with the same $[p, q]$-order and different $[p, q]$-type in the unit disc, and the authors also study the $[p, q]$-order and $[p, q]$-type of f and its derivative. At the end, the authors investigate the relationship between two different $[p, q]$-convergence exponents of f. The obtained results are the improvements and supplements to many previous results.

Key words:meromorphic function; analytic function; unit disc; $[p, q]$-order; $[p, q]$-type
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1. Notations and Results

We use $\mathbb{C}$ to denote the complex plane and $\Delta = \{z : |z| < 1\}$ to denote the unit disc. By a meromorphic function f, we mean a meromorphic function in the complex plane or a meromorphic function in the unit disc. We shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory in the complex plane or in the unit disc (*see* $[4, 10, 14-17, 19, 20]$). Firstly for $r \in (0, +\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r), i \in \mathbb{N}$, for all r sufficiently large in $(0, +\infty)$, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r), i \in \mathbb{N}$, we also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the logarithmic measure of a set $E \subset [0, 1)$ by $m_l E = \int_E \frac{dt}{1-t}$. Throughout this paper, we use p, q to denote positive integers satisfying $1 \le q \le p$. Secondly, we recall some notations about meromorphic functions and analytic functions.

Definition 1.1 (*see* $[4, 17, 19, 20]$). The order $\sigma(f)$ and lower order $\mu(f)$ of a meromorphic function f in the complex plane are respectively defined by

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}, \qquad \mu(f) = \underline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r},$$

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where $T(r,f)$ is the characteristic function of a meromorphic function f in the complex plane or in the unit disc.

Definition 1.2 (*see* [4, 19, 20]). Let f be a meromorphic function in the complex plane or an entire function satisfying $0<\sigma(f)<\infty$, then the type of f is respectively defined by

$$\tau(f)=\overline{\lim_{r\to\infty}}\frac{T(r,f)}{r^{\sigma(f)}},\qquad \tau_M(f)=\overline{\lim_{r\to\infty}}\frac{\log M(r,f)}{r^{\sigma(f)}}.$$

Definition 1.3 (*see* [8, 9, 11, 13]). The $[p,q]$-order of a meromorphic function f in the complex plane is defined by

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q r}.$$

If f is a transcendental entire function, the $[p,q]$-order of f is defined by (*see* [11, 13])

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q r}=\overline{\lim_{r\to\infty}}\frac{\log_{p+1} M(r,f)}{\log_q r}.$$

If f is a polynomial, then $\sigma_{[p,q]}(f)=0$ for any $p\geq q\geq 1$. From Definition 1.3, if $q=1$, we denote $\sigma_{[1,1]}=\sigma_1(f)=\sigma(f)$, and $\sigma_{[p,1]}=\sigma_p(f)$. Similar with Definition 1.2, we can also give the definitions of $\tau_p(f)$ and $\tau_{M,p}(f)$ when $p>1$. In order to keep accordance with Definition 1.1, we give Definition 1.3 by making a small change to the original definition of entire functions of $[p,q]$-order (*see* [8, 9]).

Definition 1.4 (*see* [3, 7]). The iterated p-order of a meromorphic function f in Δ is defined by

$$\sigma_p(f)=\overline{\lim_{r\to 1^-}}\frac{\log_p T(r,f)}{-\log(1-r)}\qquad (p\in\mathbb{N}).$$

For an analytic function f in Δ, we also define

$$\sigma_{M,p}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_{p+1} M(r,f)}{-\log(1-r)}.$$

Remark 1.1. If $p=1$, then we denote $\sigma_1(f)=\sigma(f)$ and $\sigma_{M,1}(f)=\sigma_M(f)$, and we have $\sigma(f)\leq\sigma_M(f)\leq\sigma(f)+1$ (*see* [6, 12, 16, 17]) and $\sigma_{M,p}(f)=\sigma_p(f)$ $(p\geq 2)$ (*see* [3, 7]).

Definition 1.5 (*see* [2]). Let f be a meromorphic function in Δ, then the $[p,q]$-order and lower $[p,q]$-order of f are respectively defined by

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to 1^-}}\frac{\log_p T(r,f)}{\log_q\left(\frac{1}{1-r}\right)},\qquad \mu_{[p,q]}(f)=\underline{\lim_{r\to 1^-}}\frac{\log_p T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Definition 1.6 (*see* [2]). Let f be an analytic function in Δ, then the $[p, q]$-order and lower $[p, q]$-order about maximum modulus of f are respectively defined by

$$\sigma_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r,f)}{\log_q \left(\frac{1}{1-r}\right)}, \qquad \mu_{M,[p,q]}(f) = \underline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r,f)}{\log_q \left(\frac{1}{1-r}\right)}.$$

Definition 1.7 (*see* [2]). The $[p, q]$-type of a meromorphic function f of $[p, q]$-order in Δ with $0 < \sigma_{[p,q]}(f) = \sigma_1 < \infty$ is defined by

$$\tau_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p-1} T(r,f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\sigma_1}}.$$

For an analytic function f in Δ, and the $[p, q]$-type about maximum modulus of f of $[p, q]$-order with $0 < \sigma_{M,[p,q]}(f) = \sigma_2 < \infty$ is defined by

$$\tau_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p M(r,f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\sigma_2}}.$$

Definition 1.8 The lower $[p, q]$-type of a meromorphic function f of lower $[p, q]$-order in Δ with $0 < \mu_{[p,q]}(f) = \mu_1 < \infty$ is defined by

$$\underline{\tau}_{[p,q]}(f) = \underline{\lim_{r\to \infty}} \frac{\log_{p-1} T(r,f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\mu_1}}.$$

Similarly for an analytic function f in Δ, and the lower $[p, q]$-type about maximum modulus of f of lower $[p, q]$-order with $0 < \mu_{M,[p,q]}(f) = \mu_2 < \infty$ is defined by

$$\underline{\tau}_{M,[p,q]}(f) = \underline{\lim_{r\to \infty}} \frac{\log_p M(r,f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\mu_2}}.$$

Remark 1.2. From Definitions 1.7 and 1.8, it is easy to see that $\tau_{[p,q]}(f) \le \tau_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f) \le \underline{\tau}_{M,[p,q]}(f)$.

Definition 1.9. For any $a \in \mathbb{C} \cup \{\infty\}$, we use $n\left(r, \frac{1}{f-a}\right)$ to denote the unintegrated counting function for the sequence of a-point of a meromorphic function f in Δ. Then the $[p, q]$-exponents of convergence of a-point of f about $n\left(r, \frac{1}{f-a}\right)$ is defined by

$$\lambda^n_{[p,q]}(f,a) = \overline{\lim_{r\to 1^-}} \frac{\log_p n\left(r, \frac{1}{f-a}\right)}{\log_q \left(\frac{1}{1-r}\right)}.$$

Definition 1.10. Let $N\left(r, \frac{1}{f-a}\right)$ be the integrated counting function for the sequence of a-point of a meromorphic function f in Δ. Then the $[p,q]$-exponents of convergence of a-point of f about $N\left(r, \frac{1}{f-a}\right)$ is defined by

$$\lambda^{N}_{[p,q]}(f,a) = \overline{\lim_{r\to 1^-}} \frac{\log_p N\left(r, \frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Remark 1.3. Similar with Definitions 1.9 and 1.10, we can also give the definitions of the $[p,q]$-exponents of convergence of distinct a-point of f about $n\left(r, \frac{1}{f-a}\right)$ and $N\left(r, \frac{1}{f-a}\right)$, i.e., $\overline{\lambda}^{n}_{[p,q]}(f,a)$ and $\overline{\lambda}^{N}_{[p,q]}(f,a)$.

The order and type are two important indicators in revealing the growth of the entire functions or meromorphic functions, many authors have investigated the growth of entire functions or meromorphic functions in the complex plane or in the unit disc $(e.g., see[4, 8-10, 14-20])$ since the first half of the twentieth century. In the following, we list some classic results in the complex plane.

Theorem A(*see* [4, 10, 19, 20]). If f_1 and f_2 are meromorphic functions of finite order with $\sigma(f_1) = \sigma_3$ and $\sigma(f_2) = \sigma_4$, then $\sigma(f_1 + f_2) \le \max\{\sigma_3, \sigma_4\}, \sigma(f_1 f_2) \le \max\{\sigma_3, \sigma_4\}, \sigma(f_1/f_2)$ $\le \max\{\sigma_3, \sigma_4\}$; if $\sigma_3 < \sigma_4$, then $\sigma(f_1 + f_2) = \sigma(f_1 f_2) = \sigma(f_1/f_2) = \sigma_4$.

Theorem B (*see* [20]). If f_1 and f_2 are meromorphic functions of finite order, then $\mu(f_1 + f_2) \le \min\{\max\{\sigma(f_1), \mu(f_2)\}, \max\{\mu(f_1), \sigma(f_2)\}\}$, $\mu(f_1 f_2) \le \min\{\max\{\sigma(f_1), \mu(f_2)\}, \max\{\mu(f_1), \sigma(f_2)\}\}$. Furthermore, if $\sigma(f_1) < \mu(f_2)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_2)$; or if $\sigma(f_2) < \mu(f_1)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_1)$.

Theorem C (*see* [10]). If f_1 and f_2 are entire functions of finite order satisfying $\sigma(f_1) = \sigma(f_2) = \sigma_5$, then the following two statements hold:
(i) If $\tau_M(f_1) = 0$ and $0 < \tau_M(f_2) < \infty$, then $\sigma(f_1 f_2) = \sigma_5, \tau_M(f_1 f_2) = \tau_M(f_2)$.
(ii) If $\tau_M(f_1) < \infty$ and $\tau_M(f_2) = \infty$, then $\sigma(f_1 f_2) = \sigma_5, \tau_M(f_1 f_2) = \infty$.

Theorem D (*see* [18]). Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma_6 < \infty,\ 0 \le \tau_{M,p}(f_1) < \tau_{M,p}(f_2) \le \infty$. Then the following statements hold:
(i) If $p \ge 1$, then $\sigma_p(f_1 + f_2) = \sigma_6,\ \tau_{M,p}(f_1 + f_2) = \tau_{M,p}(f_2)$;
(ii) If $p > 1$, then $\sigma_p(f_1 f_2) = \sigma_6,\ \tau_{M,p}(f_1 f_2) = \tau_{M,p}(f_2)$.

Theorem E (*see* [18]). Let $p \ge 1,\ f(z)$ be an entire function or a meromorphic function in the complex plane satisfying $0 < \sigma_p(f) < \infty$. If $p \ge 1$, then $\sigma_p(f) = \sigma_p(f'), \tau_{M,p}(f') = \tau_{M,p}(f)$; if $p > 1$, then $\sigma_p(f) = \sigma_p(f'), \tau_p(f') = \tau_p(f)$.

From Theorems A-E, we can easily obtain the following similar propositions in the unit disc.

Propositions (i) If f_1 and f_2 are meromorphic functions satisfying $\sigma_{[p,q]}(f_1)=\sigma_6$ and $\sigma_{[p,q]}(f_2)=\sigma_7$ in Δ, then $\sigma_{[p,q]}(f_1\pm f_2)\le\max\{\sigma_6,\ \sigma_7\},\sigma_{[p,q]}(f_1f_2)\le\max\{\sigma_6,\sigma_7\}$ and $\sigma_{[p,q]}(f_1/f_2)\le\max\{\sigma_6,\sigma_7\}$.

(ii) If $\sigma_6\ne\sigma_7$ in Proposition (i), then $\sigma_{[p,q]}(f_1\pm f_2)=\sigma_{[p,q]}(f_1f_2)=\sigma_{[p,q]}(f_1/f_2)=\max\{\sigma_6,\sigma_7\}$.

(iii) If f_1 and f_2 are meromorphic functions in Δ, then $\mu_{[p,q]}(f_1+f_2)\le\max\{\sigma_{[p,q]}(f_1),\mu_{[p,q]}(f_2)\}$ or $\mu_{[p,q]}(f_1+f_2)\le\max\{\mu_{[p,q]}(f_1),\sigma_{[p,q]}(f_2)\}$ and $\mu_{[p,q]}(f_1f_2)\le\max\{\sigma_{[p,q]}(f_1),\mu_{[p,q]}(f_2)\}$ or $\mu_{[p,q]}(f_1f_2)\le\max\{\mu_{[p,q]}(f_1),\sigma_{[p,q]}(f_2)\}$.

(iv) If f_1 and f_2 are meromorphic functions in Δ satisfying $\sigma_{[p,q]}(f_1)<\mu_{[p,q]}(f_2)\le\infty$, then $\mu_{[p,q]}(f_1+f_2)=\mu_{[p,q]}(f_1f_2)=\mu_{[p,q]}(f_1/f_2)=\mu_{[p,q]}(f_2)$.

(v) If f_1 and f_2 are analytic functions in Δ satisfying $\sigma_{M,[p,q]}(f_1)=\sigma_8$ and $\sigma_{M,[p,q]}(f_2)=\sigma_9$, then $\sigma_{M,[p,q]}(f_1\pm f_2)\le\max\{\sigma_8,\sigma_9\}$ and $\sigma_{M,[p,q]}(f_1f_2)\le\max\{\sigma_8,\sigma_9\}$. If $\sigma_8\ne\sigma_9$, then $\sigma_{M,[p,q]}(f_1\pm f_2)=\max\{\sigma_8,\sigma_9\}$.

(vi) If f_1 and f_2 are analytic functions in Δ, then $\max\{\mu_{M,[p,q]}(f_1\pm f_2),\mu_{M,[p,q]}(f_1f_2)\}\le\max\{\sigma_{M,[p,q]}(f_1),\mu_{M,[p,q]}(f_2)\}$ or $\max\{\mu_{M,[p,q]}(f_1\pm f_2),\mu_{M,[p,q]}(f_1f_2)\}\le\max\{\mu_{M,[p,q]}(f_1),\sigma_{M,[p,q]}(f_2)\}$.

(vii) If f_1 and f_2 are analytic functions of $[p,q]$-order in Δ, for any $r\in[0,1)$, by the inequality $T(r,f)\le\log^+M(r,f)\le\frac{4}{1-r}T\left(\frac{1+r}{2},f\right)$ $(see\ [4,17])$, we easily obtain that if $p=q\ge2$ and $\sigma_{[p,q]}(f)>1$, or $p>q\ge1$, then $\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f)$ and $\tau_{[p,q]}(f)=\tau_{M,[p,q]}(f)$. Similarly, we have $\mu_{[p,q]}(f)=\mu_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f)=\underline{\tau}_{M,[p,q]}(f)$ if $p=q\ge2$ and $\mu_{[p,q]}(f)>1$, or $p>q\ge1$.

Combining Theorems D and E, a natural question is: Can we get the similar results with Theorems D, E for meromorphic functions or analytic functions of $[p,q]$ order in Δ? In fact, we obtain the following results:

Theorem 1.1. Let f_1 and f_2 be meromorphic functions in Δ satisfying $0<\sigma_{[p,q]}(f_1)=\sigma_{[p,q]}(f_2)=\sigma_{10}<\infty$ and $0\le\tau_1=\tau_{[p,q]}(f_1)<\tau_{[p,q]}(f_2)=\tau_2\le\infty$. Then $\sigma_{[p,q]}(f_1+f_2)=\sigma_{[p,q]}(f_1f_2)=\sigma_{[p,q]}(f_1/f_2)=\sigma_{10}$, and the following two statements hold:

(i) If $p>1$ and $p\ge q\ge1$, then $\tau_{[p,q]}(f_1+f_2)=\tau_{[p,q]}(f_1f_2)=\tau_{[p,q]}(f_1/f_2)=\tau_{[p,q]}(f_2)$.
(ii) If $p=q=1$, then $\tau_2-\tau_1\le\max\{\tau(f_1+f_2),\tau(f_1f_2),\tau(f_1/f_2)\}\le\tau_2+\tau_1$.

Theorem 1.2. Let f_1 and f_2 be meromorphic functions in Δ satisfying $0<\sigma_{[p,q]}(f_1)=\mu_{[p,q]}(f_2)<\infty$ and $0\le\tau_{[p,q]}(f_1)<\underline{\tau}_{[p,q]}(f_2)\le\infty$, then $\mu_{[p,q]}(f_1+f_2)=\mu_{[p,q]}(f_1f_2)=\mu_{[p,q]}(f_1/f_2)=\mu_{[p,q]}(f_2)$. And if $p>1$ and $p\ge q\ge1$, we have $\underline{\tau}_{[p,q]}(f_1+f_2)=\underline{\tau}_{[p,q]}(f_1f_2)=\underline{\tau}_{[p,q]}(f_1/f_2)=\underline{\tau}_{[p,q]}(f_2)$.

In the following, when f_1 and f_2 are analytic functions of $[p,q]$-order in the unit disc we have the similar results.

Theorem 1.3. Let f_1 and f_2 be analytic functions in Δ satisfying $0 < \sigma_{M,[p,q]}(f_1) = \sigma_{M,[p,q]}(f_2) = \sigma_{11} < \infty$ and $0 \leq \tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) \leq \infty$, then $\sigma_{M,[p,q]}(f_1+f_2) = \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) = \tau_{M,[p,q]}(f_2)$.

Remark 1.4. By Proposition (vii), we know that Theorem 1.3 is of the same with Theorem 1.1 for $p > q \geq 1$ and $p = q \geq 2, \sigma_{[p,q]}(f) > 1$. For the case $p = q = 1$, the result of Theorem 1.3 is better than that of Theorem 1.1.

Corollary 1.1. Let f_1 and f_2 be analytic functions in Δ satisfying $0 < \sigma_{M,[p,q]}(f_1) = \mu_{M,[p,q]}(f_2) < \infty$ and $0 \leq \tau_{M,[p,q]}(f_1) < \underline{\tau}_{M,[p,q]}(f_2) \leq \infty$, then $\mu_{M,[p,q]}(f_1 + f_2) = \mu_{M,[p,q]}(f_2)$ and $\underline{\tau}_{M,[p,q]}(f_1 + f_2) = \underline{\tau}_{M,[p,q]}(f_2)$.

Theorem 1.4. Let f be an analytic function of $[p, q]$-order in Δ, then $\sigma_{M,[p,q]}(f) = \sigma_{M,[p,q]}(f')$, $\mu_{M,[p,q]}(f) = \mu_{M,[p,q]}(f')$. If $0 < \sigma_{M,[p,q]}(f) < \infty$ or $0 < \mu_{M,[p,q]}(f) < \infty$, then $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$, $\underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f')$.

Theorem 1.5. Let f be a meromorphic function of $[p, q]$-order in Δ, then
(i) If $p \geq q \geq 2$ and $p > q = 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f'), \mu_{[p,q]}(f) = \mu_{[p,q]}(f')$ and $\tau_{[p,q]}(f) = \tau_{[p,q]}(f')$, $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q]}(f')$ for $0 < \sigma_{[p,q]}(f) < \infty$ or $0 < \mu_{[p,q]}(f) < \infty$.
(ii) If $p = q = 1$, then $\sigma(f) = \sigma(f'), \mu(f) = \mu(f')$ and $\tau_{[1,1]}(f') \leq 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \leq 2\underline{\tau}_{[1,1]}(f)$.

Theorem 1.6. Let f be a meromorphic function of $[p, q]$-order in Δ, $a \in \mathbb{C} \cup \{\infty\}$. Then the following statements hold:

(i) If $p > q \geq 1$, then $\lambda^N_{[p,q]}(f, a) = \lambda^n_{[p,q]}(f, a)$.
(ii) If $p = q = 1$, then $\lambda^N(f, a) \leq \lambda^n(f, a) \leq \lambda^N(f, a) + 1$ (*see* [12]).
(iii) If $p = q \geq 2$, then $\lambda^N_{[p,p]}(f, a) \leq \lambda^n_{[p,p]}(f, a) \leq \max\left\{\lambda^N_{[p,p]}(f, a), 1\right\}$. Furthermore, we have $\lambda^N_{[p,p]}(f, a) = \lambda^n_{[p,p]}(f, a)$ if $\lambda^N_{[p,p]}(f, a) \geq 1$, and if $\lambda^N_{[p,p]}(f, a) < 1$ then $\lambda^N_{[p,p]}(f, a) \leq \lambda^n_{[p,p]}(f, a) \leq 1$.

Remark 1.4. The conclusions of Theorem 1.6 also hold between $\overline{\lambda}^n_{[p,q]}(f, a)$ and $\overline{\lambda}^N_{[p,q]}(f, a)$.

2. Preliminary Lemmas

Lemma 2.1 (*see* [4, 19, 20]). Let f_1, f_2, $\cdots$, $f_m(z)$ be meromorphic functions in Δ, where $m \geq 2$ is a positive integer. Then

(i) $T(r, f_1 f_2 \cdots f_m) \leq \sum\limits_{i=1}^{m} T(r, f_i)$,

(ii) $T(r, f_1 + f_2 + \cdots + f_m) \leq \sum\limits_{i=1}^{m} T(r, f_i) + \log m$.

Lemma 2.2 (*see* [6, 17]). Let f be a meromorphic function in Δ, and let $k \geq 1$ be an integer.

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Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\left\{\log^{+} T(r,f) + \log\left(\frac{1}{1-r}\right)\right\}$, possibly outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dt}{1-t} < \infty$.

Lemma 2.3 (*see* [1]). Let $g : (0,1) \to R$ and $h : (0,1) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_2 \subset [0,1)$ for which $\int_{E_2} \frac{dt}{1-t} < \infty$. Then there exisits a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.

Lemma 2.4 (*see* [5,15]) Suppose that f is meromorphic in Δ with $f(0) = 0$. Then

$$m(r,f) \leq \left[1 + \varphi(\frac{r}{R})\right] T(R,f') + N(R,f'), \tag{3.13}$$

where $0 < r < R < 1, \varphi(t) = \frac{1}{\pi} \log \frac{1+t}{1-t}$.

3. Proofs of Theorems 1.1 -1.6

Proof of Theorem 1.1. Assume that $0 \leq \tau_1 = \tau_{[p,q]}(f_1) < \tau_{[p,q]}(f_2) = \tau_2 < \infty$, by Definition 1.7, it is easy to see that for any given $\varepsilon > 0$ and $r \to 1^-$, we have

$$T(r,f_1) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}, \tag{3.1}$$

$$T(r,f_2) \leq \exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}. \tag{3.2}$$

By using (3.1)-(3.2) and Lemma 2.1, we have

$$T(r, f_1 + f_2) \leq T(r,f_1) + T(r,f_2) + \log 2$$

$$\leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\} + \exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\} + \log 2$$

$$\leq 2\exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}.$$

Hence $\sigma_{[p,q]}(f_1 + f_2) \leq \sigma_{10}$. In addition, if $p = q = 1$, we can get $\tau_{[p,q]}(f_1 + f_2) \leq \tau_1 + \tau_2$, if $p > 1$, then $\tau_{[p,q]}(f_1 + f_2) \leq \tau_2$ for any $p \geq q \geq 1$. On the other hand, for any given $\varepsilon > 0$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$T(r_n,f_1) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}, \tag{3.3}$$

$$T(r_n,f_2)\geq \exp_{p-1}\left\{(\tau_2-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}. \tag{3.4}$$

By (3.3)-(3.4) and Lemma 2.1, we obtain

$$T(r_n,f_1+f_2)\geq T(r_n,f_2)-T(r_n,f_1)-\log 2$$

$$\geq \exp_{p-1}\left\{(\tau_2-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}-\exp_{p-1}\left\{(\tau_1+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}-\log 2. \tag{3.5}$$

By (3.5) we have $\sigma_{[p,q]}(f_1+f_2)\geq\sigma_{10}$. Furthermore, if $p=q=1$, then $\tau_{[p,q]}(f_1+f_2)\geq\tau_2-\tau_1$ and $\tau_{[p,q]}(f_1+f_2)\geq\tau_2$ for $p>1$ and $p\geq q\geq 1$.

Therefore, we have $\sigma_{[p,q]}(f_1+f_2)=\sigma_{10}$, and if $p>1$, then $\tau_{[p,q]}(f_1+f_2)=\tau_{[p,q]}(f_2)$, if $p=q=1$, then $\tau_2-\tau_1\leq\tau_{[p,q]}(f_1+f_2)\leq\tau_2+\tau_1$. Since $T(r,f_1f_2)\leq T(r,f_1)+T(r,f_2)$, $T(r,f_1f_2)\geq T(r,f_2)-T(r,f_1)-O(1)$ and $T\left(r,\frac{1}{f_2}\right)=T(r,f_2)+O(1)$, by the above proof, $\sigma_{[p,q]}(f_1f_2)=\sigma_{[p,q]}(f_1/f_2)=\sigma_{10}$, $\tau_{[p,q]}(f_1f_2)=\tau_{[p,q]}(f_1/f_2)=\tau_{[p,q]}(f_2)$ for $p>1$ and $\tau_2-\tau_1\leq\max\{\tau(f_1f_2),\tau(f_1/f_2)\}\leq\tau_2+\tau_1$ if $p=q=1$ also can hold. Moreover, Theorem 1.1 also holds for $\tau_{[p,q]}(f_2)=\tau_2=\infty$.

Proof of Theorem 1.2. Without loss of generality, we suppose that $0\leq\tau_3=\tau_{[p,q]}(f_1)<\underline{\tau}_{[p,q]}(f_2)=\tau_4<\infty$. Assume that $\sigma_{[p,q]}(f_1)=\mu_{[p,q]}(f_2)=\mu_3$, and by Definition 1.8, it is easy to see that for any given $\varepsilon>0$, there exists a sequence $\{r_n\}_{n=1}^{\infty}\to 1^-$ satisfying

$$T(r_n,f_1)<\exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}, \tag{3.6}$$

$$T(r_n,f_2)<\exp_{p-1}\left\{(\tau_4+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}. \tag{3.7}$$

By (3.6)-(3.7) and Lemma 2.1, we have

$$T(r_n,f_1+f_2)\leq T(r_n,f_1)+T(r_n,f_2)+\log 2$$

$$\leq \exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}+\exp_{p-1}\left\{(\tau_4+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}+\log 2$$

$$\leq 2\exp_{p-1}\left\{(\tau_4+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}.$$

Hence $\mu_{[p,q]}(f_1+f_2)\leq\mu_3$. In addition, if $p>1$, then $\underline{\tau}_{[p,q]}(f_1+f_2)\leq\tau_4$ for $p\geq q\geq 1$. On the other hand, for any given $\varepsilon>0$ and $r\to 1^-$, we have

$$T(r,f_1)\leq\exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\}, \tag{3.8}$$

$$T(r,f_2)\geq\exp_{p-1}\left\{(\tau_4-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\}. \tag{3.9}$$

By (3.8)-(3.9) and Lemma 2.1, we obtain

$$T(r, f_1+f_2) \geq T(r,f_2) - T(r,f_1) - \log 2$$

$$\geq \exp_{p-1}\left\{(\tau_4-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\} - \exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\} - \log 2. \quad (3.10)$$

By (3.10) we have $\mu_{[p,q]}(f_1+f_2) \geq \mu_3$ and $\underline{\tau}_{[p,q]}(f_1+f_2) \geq \tau_4$ for $p>1$ and $p \geq q \geq 1$. Thus we have $\mu_{[p,q]}(f_1+f_2) = \mu(f_2)$ and if $p>1$ and $p \geq q \geq 1$, then $\underline{\tau}_{[p,q]}(f_1+f_2) = \underline{\tau}_{[p,q]}(f_2)$. Since $T(r, f_1 f_2) \leq T(r,f_1) + T(r,f_2)$, $T(r,f_1f_2) \geq T(r,f_2) - T(r,f_1) - O(1)$ and $T\left(r, \frac{1}{f_2}\right) = T(r,f_2) + O(1)$, by the above proof, $\mu_{[p,q]}(f_1f_2) = \mu_{[p,q]}(f_1/f_2) = \mu_{[p,q]}(f_2)$ and $\underline{\tau}_{[p,q]}(f_1f_2) = \underline{\tau}_{[p,q]}(f_1/f_2) = \underline{\tau}_{[p,q]}(f_2)$ also hold if $p>1$ and $p \geq q \geq 1$.

The conclusions of Theorem 1.2 also hold for $\tau_3 = \tau_{[p,q]}(f_1) < \underline{\tau}_{[p,q]}(f_2) = \tau_4 = \infty$.

Proof of Theorem 1.3. Set $0 \leq \tau_5 = \tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) = \tau_6 < \infty$, by Definition 1.7, for any given ε $(0 < 2\varepsilon < \tau_6 - \tau_5)$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$M(r_n, f_1) \leq \exp_p\left\{(\tau_5+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}, \quad (3.11)$$

$$M(r_n, f_2) > \exp_p\left\{(\tau_6-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}. \quad (3.12)$$

We can choose a sequence $\{z_n\}_{n=1}^{\infty}$ satisfying $|z_n| = r_n (n = 1, 2, \cdots)$ and $|f_2(z_n)| = M(r_n, f_2)$, by (3.11)-(3.12) we have

$$M(r_n, f_1+f_2) \geq |f_1(z_n) + f_2(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \geq M(r_n, f_2) - M(r_n, f_1)$$

$$\geq \exp_p\left\{(\tau_6-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\} - \exp_p\left\{(\tau_5+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}$$

$$\geq \frac{1}{2}\exp_p\left\{(\tau_6-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\} \quad (r_n \to 1^-).$$

Hence $\sigma_{M,[p,q]}(f_1+f_2) \geq \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) \geq \tau_6$. On the other hand, we have

$$M(r, f_1+f_2) \leq M(r,f_1) + M(r,f_2)$$

$$\leq \exp_p\left\{(\tau_5+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\} + \exp_p\left\{(\tau_6+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\}$$

$$\leq 2\exp_p\left\{(\tau_6+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\},$$

therefore $\sigma_{M,[p,q]}(f_1+f_2) \leq \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) \leq \tau_6$. Thus we can get $\sigma_{M,[p,q]}(f_1+f_2) = \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) = \tau_{M,[p,q]}(f_2)$. Moreover, Theorem 1.3 also holds for $\tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) = \tau_6 = \infty$.

Proof of Theorem 1.4. Since f is an analytic function in the unit disc, from the formula

$$f(z) = f(0) + \int_0^z f'(\zeta)d\zeta \qquad (|z| = r < 1)$$

where the integral route is a line from 0 to z in the unit disc. We obtain that

$$M(r, f) \leq |f(0)| + |\int_0^z f'(\zeta)d\zeta| \leq |f(0)| + rM(r, f') \leq |f(0)| + M(r, f'),$$

i.e.

$$M(r, f') \geq M(r, f) - |f(0)|. \tag{3.13}$$

By (3.13), we have

$$\sigma_{M,[p,q]}(f') \geq \sigma_{M,[p,q]}(f), \quad \mu_{M,[p,q]}(f') \geq \mu_{M,[p,q]}(f).$$

On the other hand, in the circle $|z| = r \in (0, 1)$, we take a point z_0 satisfying $|f'(z_0)| = M(r, f')$. By the Cauchy inequality

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta,$$

where $C = \{\zeta : |\zeta - z_0| = s(r) - r\}$ and $s(r) = 1 - d(1 - r)$, $d \in (0, 1)$. We deduce that

$$M(r, f') = |f'(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f(\zeta)}{(\zeta - z_0)^2}(s(r) - r)\right| d\theta \leq \frac{M(s(r), f)}{s(r) - r},$$

i.e.

$$M(r, f') \leq \frac{M(s(r), f)}{(1 - d)(1 - r)}. \tag{3.14}$$

By (3.14), then $\sigma_{M,[p,q]}(f') \leq \sigma_{M,[p,q]}(f)$, $\mu_{M,[p,q]}(f') \leq \mu_{M,[p,q]}(f)$. Hence

$$\sigma_{M,[p,q]}(f) = \sigma_{M,[p,q]}(f'), \quad \mu_{M,[p,q]}(f') = \mu_{M,[p,q]}(f). \tag{3.15}$$

If $0 < \sigma_{M,[p,q]}(f) < \infty$ and by (3.13), (3.15), we can get $\tau_{M,[p,q]}(f') \geq \tau_{M,[p,q]}(f)$. Then by (3.14)-(3.15), if $p \geq q = 1$ we can obtain

$$\frac{\log_p M(r, f')}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f')}} \leq \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}, \frac{\log_p M(s(r), f)}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}\right\}$$

$$\leq \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}, \frac{\log_p M(s(r), f)}{\left[\frac{1}{1-s(r)}\right]^{\sigma_{M,[p,1]}(f)}} \cdot \left(\frac{1}{d}\right)^{\sigma_{M,[p,1]}(f)}\right\},$$

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let $d \to 1$, therefore $\tau_{M,[p,q]}(f') \le \tau_{M,[p,q]}(f)$, then $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$. If $p \ge q \ge 2$, then

$$\frac{\log_p M(r,f')}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f')}} \le \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f)}}, \frac{\log_p M(s(r),f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f)}}\right\},$$

thus we have $\tau_{M,[p,q]}(f') \le \tau_{M,[p,q]}(f)$ and $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$. If $0 < \mu_{M,[p,q]}(f) < \infty$, we can similarly obtain $\underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f')$.

Proof of Theorem 1.5. By Lemma 2.2, we have

$$T(r,f') = m(r,f') + N(r,f') \le m(r,f) + m\left(r,\frac{f'}{f}\right) + 2N(r,f)$$

$$\le 2T(r,f) + m\left(r,\frac{f'}{f}\right) \le (2+\varepsilon)T(r,f) + O\left\{\log\frac{1}{1-r}\right\} \quad (r \notin E_1). \tag{3.16}$$

From (3.16) and Lemma 2.3, we have $\sigma_{[p,q]}(f') \le \sigma_{[p,q]}(f), \mu_{[p,q]}(f') \le \mu_{[p,q]}(f)$ for $p \ge q \ge 1$, $\tau_{[p,q]}(f') \le \tau_{[p,q]}(f)$, $\underline{\tau}_{[p,q]}(f') \le \underline{\tau}_{[p,q]}(f)$, for $p > 1$ and $\tau_{[1,1]}(f') \le 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \le 2\underline{\tau}_{[1,1]}(f)$ for $p = q = 1$. On the other hand, set $R = s(r) = 1 - d(1-r), d \in (0,1)$ in Lemma 2.4, we have

$$T(r,f) < \left(2 + \frac{1}{\pi}\log\frac{3}{(1-d)(1-r)}\right)T(s(r),f'). \tag{3.17}$$

By (3.17) and by the similar proof in Theorem 1.4, we have $\sigma_{[p,q]}(f) \le \sigma_{[p,q]}(f')$, $\mu_{[p,q]}(f) \le \mu_{[p,q]}(f')$ for $p \ge q \ge 1$, $\tau_{[p,q]}(f) \le \tau_{[p,q]}(f')$, $\underline{\tau}_{[p,q]}(f) \le \underline{\tau}_{[p,q]}(f')$ for $p \ge q \ge 2$ and $\tau_{[p,q]}(f) \le (\frac{1}{d})^{\sigma_{[p,q]}(f)}\tau_{[p,q]}(f')$ for $p > q = 1$, letting $d \to 1$, therefore the following statements hold:

If $p \ge q \ge 2$ and $p > q = 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f'), \mu_{[p,q]}(f) = \mu_{[p,q]}(f')$ and $\tau_{[p,q]}(f) = \tau_{[p,q]}(f')$ for $0 < \sigma_{[p,q]}(f) < \infty$, $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q]}(f')$ for $0 < \mu_{[p,q]}(f) < \infty$.

If $p = q = 1$, then $\sigma(f) = \sigma(f'), \mu(f) = \mu(f')$ and $\tau_{[1,1]}(f') \le 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \le 2\underline{\tau}_{[1,1]}(f)$.

Proof of Theorem 1.6. Without loss of generality, assume that $f(a) \ne 0$, by

$$N\left(r,\frac{1}{f-a}\right) = \int_0^r \frac{n\left(t,\frac{1}{f-a}\right) - n\left(0,\frac{1}{f-a}\right)}{t}dt \quad (0 < r < 1),$$

we have

$$n\left(r,\frac{1}{f-a}\right) \le \frac{1}{\log\left(1+\frac{1-r}{2r}\right)}\int_r^{r+\frac{1-r}{2}} \frac{n\left(t,\frac{1}{f-a}\right)}{t}dt \le \frac{1}{\log\left(1+\frac{1-r}{2r}\right)}N\left(\frac{1+r}{2},\frac{1}{f-a}\right), \tag{3.18}$$

where $0 < r < 1, \log(1+\frac{1-r}{2r}) \sim \frac{1-r}{2r}, r \to 1^-$. By (3.18), we have

$$\overline{\lim_{r\to 1^-}}\frac{\log_p n\left(r,\frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)} \le \max\left\{\overline{\lim_{r\to 1^-}}\frac{\log_p N\left(\frac{1+r}{2},\frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}, \overline{\lim_{r\to 1^-}}\frac{\log_p\left(\frac{2r}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}\right\}. \tag{3.19}$$

By (3.19), we can obtain

(i) if $p > q \geq 1$, then $\lambda^n_{[p,q]}(f, a) \leq \lambda^N_{[p,q]}(f, a)$;

(ii) if $p = q = 1$, then $\lambda^n(f, a) \leq \lambda^N(f, a) + 1$;

(iii) if $p = q \geq 2$, then $\lambda^n_{[p,p]}(f, a) \leq \max\left\{\lambda^N_{[p,p]}(f, a), 1\right\}$.

On the other hand, by

$$N\left(r, \frac{1}{f-a}\right) = \int_{r_0}^{r} \frac{n\left(t, \frac{1}{f-a}\right)}{t} dt + N\left(r_0, \frac{1}{f-a}\right) \leq n\left(r, \frac{1}{f-a}\right) \log\left(\frac{r}{r_0}\right) + O(1), \quad (3.20)$$

where $0 < r_0 < r < 1$. By (3.20), we can get

(i) if $p > q \geq 1$, then $\lambda^N_{[p,q]}(f, a) \leq \lambda^n_{[p,q]}(f, a)$;

(ii) if $p = q = 1$, then $\lambda^N(f, a) \leq \lambda^n(f, a)$;

(iii) if $p = q \geq 2$, then $\lambda^N_{[p,p]}(f, a) \leq \lambda^n_{[p,p]}(f, a)$.

Therefore, the conclusions of Theorem 1.6 hold.

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