CO-ORDINATED CONVEX FUNCTIONS OF THREE VARIABLES AND SOME ANALOGOUS INEQUALITIES WITH APPLICATIONS

Rashida Hussain[a,*], Asghar Ali[b], Asia Latif[c] and Ghazala Gulshan[d]

Department of Mathematics, Mirpur University of Science and Technology (MUST), Mirpur-10250 (AJK), Pakistan.

[a]drrashida@must.edu.pk [b]drali.maths@must.edu.pk
[c]asialatif.maths@must.edu.pk [d]ghazala.maths@must.edu.pk

Abstract: Co-ordinated convex function of two variables and corresponding inequalities have been studied before by many researchers. This paper deals with Co-ordinated convex function of three variables. In the present paper the idea of co-ordinated convex function of two variables in a rectangle from the plane R^2 is extended to that of three variables in a rectangle from space R^3. Moreover corresponding extended right handed Hermite-Hadamard type inequalities are also incorporated. At the end some applications of resulting inequalities to special means are also given.

Key Words: Hermite-Hadamard's inequality, co-ordinated convex function, Hölder's integral inequality, power mean inequality, arithmetic mean, logarithmic mean.

1. Introduction

It is said that the notion *convex* was pioneered by Archimedes. While estimating value of π, he noticed that the perimeter of a convex figure is smaller than that of any other convex figure, encompassing it. As per J. L. Jensen, idea of convex function is as primeval as an increasing function or a positive function. A documented result spontaneously identified with convex function is Hermite-Hadamard (HH) inequality

$$g\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du \leq \frac{g(\alpha)+g(\beta)}{2}, \tag{1.1}$$

where g is a real valued convex function on the real interval I and $\alpha, \beta \in I$ with $\alpha < \beta$. The idea of co-ordinated convex (CC) function on a rectangle from the plane was coined by Dragomir and Pearce in 2000, see [6]. Co-ordinated convexity is more general than convexity. It only requires a function to be convex on each

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*Corresponding Author.

coordinate individually and independently, not simultaneously as in case of convexity. Thereafter Dragomir and many other mathematicians worked in this field, see [1, 5, 8, 9, 11, 13, 14, 16–18, 20–22]. The idea of co-ordinated convexity was joined with other classes of convex functions like s-convex, m-convex, (α, m)-convex, h-convex, log-convex, $((s, m), QC)$-convex functions and many interesting results were obtained, see [2–4, 10, 12, 15, 19] and references there in.

Dragomir and Pearce [6] defined CC function of two variables as follows. The principal outcomes of the present article are actuated by this definition.

Definition 1.1. *A function $g : \triangle \longrightarrow R$ is called CC on $\triangle$ if the partial functions $g_u : [\gamma, \delta] \longrightarrow R$ and $g_v : [\alpha, \beta] \longrightarrow R$, defined as $g_u(y) = g(u, y)$ and $g_v(x) = g(x, v)$ respectively, are convex. Here $\triangle$ is bi-dimensional interval given by $\triangle = [\alpha, \beta] \times [\gamma, \delta]$.*

2. Main Results

In this section we mainly establish results on the basis of definition of CC function of three variables on a rectangle from the space R^3. We derive HH type inequality for CC functions of three variables and then establish inequalities related to the algebraic combination of middle term and right side terms of this inequality, to be called right HH type inequalities. Let us first define CC function of three variables. Motivated by Definition 1.1, CC function of three variables is defined as:

Definition 2.1. *A function $g : \Delta \longrightarrow R$ is called CC on Δ if the partial functions $g_{u,v} : [\zeta, \eta] \longrightarrow R$, $g_{u,w} : [\gamma, \delta] \longrightarrow R$ and $g_{v,w} : [\alpha, \beta] \longrightarrow R$, defined as $g_{u,v}(z) = g(u, v, z), g_{u,w}(y) = g(u, y, w)$ and $g_{v,w}(x) = g(x, v, w)$ respectively, are convex. Here and on wards Δ is tri-dimensional interval given by $\Delta = [\alpha, \beta] \times [\gamma, \delta] \times [\zeta, \eta]$.*

Lemma 2.2. *If a function $g : \Delta \longrightarrow R$ is CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
& g\left(lc + (1-l)d, me + (1-m)h, nr + (1-n)s\right) \\
& \leq lmng(c, e, r) + lm(1-n)g(c, e, s) + l(1-m)ng(c, h, r) \\
& \quad + l(1-m)(1-n)g(c, h, s) + (1-l)mng(d, e, r) + (1-l)m(1-n)g(d, e, s) \\
& \quad + (1-l)(1-m)ng(d, h, r) + (1-l)(1-m)(1-n)g(d, h, s),
\end{aligned}
$$

for all $l, m, n \in [0, 1]$ and $(c, e, r), (c, e, s), (c, h, r)$ e.t.c. are in Δ.

Proof. The Ineq. (**??**) can be proved simply by applying Definition 2.1, to the given function g. □

Lemma 2.3. *Every convex function is CC. The converse in general is not true.*

Proof. It can be proved by using Definition 2.1 of CC function. Moreover the function $g : [0, 1]^3 \longrightarrow [0, \infty)$ given by $g(u, v, w) = uvw$, for all $(u, v, w) \in [0, 1]^3$ is CC function but is not convex on $[0, 1]^3$. □

Motivated by Dragomir [7], we extend HH Ineq. (1.1) for CC functions of three variables as follows:

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Theorem 2.4. *Suppose that $g : \Delta \subset R^3 \longrightarrow R$ is CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
& g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right) \\
& \leq \frac{1}{3}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta g\left(u, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right)du + \frac{1}{\delta-\gamma}\int_\gamma^\delta g\left(\frac{\alpha+\beta}{2}, v, \frac{\zeta+\eta}{2}\right)dv\right. \\
& \quad \left. + \frac{1}{\eta-\zeta}\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, w\right)dw\right] \qquad (2.1) \\
& \leq \frac{1}{3}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\gamma^\delta\int_\alpha^\beta g\left(u, v, \frac{\zeta+\eta}{2}\right)dudv + \frac{1}{(\delta-\gamma)(\eta-\zeta)}\right. \\
& \quad \left. \int_\gamma^\delta\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, v, w\right)dvdw + \frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\alpha^\beta\int_\zeta^\eta g\left(u, \frac{\gamma+\delta}{2}, w\right)dudw\right] \qquad (2.2) \\
& \leq \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw \qquad (2.3) \\
& \leq \frac{1}{6}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta \{g(u,v,\zeta)+g(u,v,\eta)\}dudv\right. \\
& \quad + \frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta \{g(\alpha,v,w)+g(\beta,v,w)\}dvdw \\
& \quad \left. + \frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\alpha^\beta\int_\zeta^\eta \{g(u,\gamma,w)+g(u,\delta,w)\}dudw\right] \qquad (2.4) \\
& \leq \frac{1}{12}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}du\right. \\
& \quad + \frac{1}{\delta-\gamma}\int_\gamma^\delta \{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}dv \\
& \quad \left. + \frac{1}{\eta-\zeta}\int_\zeta^\eta \{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}dw\right] \qquad (2.5) \\
& \leq \frac{1}{8}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)\right. \\
& \quad \left. +g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)\right]. \qquad (2.6)
\end{aligned}
$$

Proof. Since $g : \Delta \longrightarrow R$ is CC function, it follows that the function $g_{u,v} : [\zeta, \eta] \longrightarrow R$, defined by $g_{u,v}(z) = g(u, v, z)$ is convex on $[\zeta, \eta]$. Then from Inequality (1.1), we have

$$
\begin{aligned}
& g_{\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}}\left(\frac{\zeta+\eta}{2}\right) \leq \frac{1}{\eta-\zeta}\int_\zeta^\eta g_{\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}}(w)dw, \quad \text{where } w \in [\zeta,\eta] \\
& g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right) \leq \frac{1}{\eta-\zeta}\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, w\right)dw.
\end{aligned}
$$

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Similar arguments applied to the functions $g_{u,w}$ and $g_{v,w}$ respectively, we get

$$g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\leq\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv$$

and

$$g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\leq\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g\left(u,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du,$$

adding above three inequalities, we get Ineq. (2.1) as follows

$$\begin{aligned}&g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\\&\leq\frac{1}{3}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g\left(u,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\right.\\&\quad\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},w\right)dw\right].\end{aligned}$$

Now consider the following inequality

$$\begin{aligned}&g_{\frac{\alpha+\beta}{2},v}\left(\frac{\zeta+\eta}{2}\right)\leq\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{\frac{\alpha+\beta}{2},v}(w)dw,\qquad\text{where } w\in[\zeta,\eta],\\&g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)\leq\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},v,z\right)dw,\\&\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\leq\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},v,w\right)dvdw,\end{aligned}$$

adding all such inequalities, we get Ineq. (2.2) as follows

$$\begin{aligned}&\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f\left(x,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}f\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\\&+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}f\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},w\right)dw\leq\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}f\left(u,v,\frac{\zeta+\eta}{2}\right)dudv\\&+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}f\left(\frac{\alpha+\beta}{2},v,w\right)dvdw\\&+\frac{1}{(\eta-\zeta)(\beta-\alpha)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}f\left(u,\frac{\gamma+\delta}{2},w\right)dwdu.\end{aligned}$$

Considering

$$g_{u,v}\left(\frac{\zeta+\eta}{2}\right)\leq\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{u,v}(w)dw\leq\frac{g_{u,v}(m)+g_{u,v}(n)}{2},\ \text{ where } w\in[\zeta,\eta],$$

gives

$$g\left(u,v,\frac{\zeta+\eta}{2}\right)\leq\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g(u,v,w)dw\leq\frac{g(u,v,\zeta)+g(u,v,\eta)}{2},\tag{2.7}$$

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integrating Ineq. (2.7) w.r.t u and v over the intervals $[\alpha,\beta]$ and $[\gamma,\delta]$ respectively, then multiplying by $\frac{1}{(\beta-\alpha)(\delta-\gamma)}$, we have

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g\left(u,v,\frac{\zeta+\eta}{2}\right)dudv\\
&\le \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw\\
&\le \frac{1}{2(\beta-\alpha)(\delta-\gamma)}\left[\int_\alpha^\beta\int_\gamma^\delta g(u,v,\zeta)dudv+\int_\alpha^\beta\int_\gamma^\delta g(u,v,\eta)dudv\right],
\end{aligned}$$

by the same argument applied on the functions $g_{v,w}$ and $g_{u,w}$, we tend to get two additional such inequalities. Then by adding these inequalities, we have Ineqs. (2.3) and (2.4) as follows

$$\begin{aligned}
&\frac{1}{3}\Bigg[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g\left(u,v,\frac{\zeta+\eta}{2}\right)dudv+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\\
&\int_\gamma^\delta\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2},v,w\right)dvdw+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\zeta^\eta\int_\alpha^\beta g\left(u,\frac{\gamma+\delta}{2},w\right)dudw\Bigg]\\
&\le \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw\\
&\le \frac{1}{6}\Bigg[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g(u,v,\zeta)dudv+\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g(u,v,\eta)dudv\\
&\quad+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\alpha,v,w)dvdw+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\beta,v,w)dvdw\\
&\quad+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\eta^\zeta\int_\alpha^\beta g(u,v,\zeta)dudw+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\eta^\zeta\int_\alpha^\beta g(u,v,\eta)dudw\Bigg].
\end{aligned}$$

By convexity of $g_{\alpha,v}$, we have

$$\begin{aligned}
\frac{1}{\eta-\zeta}\int_\zeta^\eta g_{\alpha,v}(w)dw &\le \frac{g_{\alpha,v}(\zeta)+g_{\alpha,v}(\eta)}{2}, \quad \text{where } w\in[\zeta,\eta]\\
\frac{1}{\eta-\zeta}\int_\zeta^\eta g(\alpha,v,w)dw &\le \frac{g(\alpha,v,\zeta)+g(\alpha,v,\eta)}{2},
\end{aligned}$$

integrating this inequality w.r.t. v over the interval $[\gamma,\delta]$, we have

$$\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\alpha,v,w)dvdw\le\frac{1}{2(\delta-\gamma)}\int_\gamma^\delta\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)\}dv,$$

similarly by convexity of $g_{\beta,v}$ we have

$$\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\beta,v,w)dvdw\le\frac{1}{2(\delta-\gamma)}\int_\gamma^\delta\{g(\beta,v,\zeta)+g(\beta,v,\eta)\}dv,$$

after further calculation we have six such inequalities, adding these we get Ineq. (2.5) as follows

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\{g(u,v,\zeta)+g(u,v,\eta)\}\,dudv\\
&+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}\{g(\alpha,v,w)+g(\beta,v,w)\}\,dvdw\\
&+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}\{g(u,v,\zeta)+g(u,v,\eta)\}\,dudw\\
&\le\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du\right.\\
&+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv\\
&\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\right].
\end{aligned}$$

Now to get last inequality we use the convexity of $g_{\alpha,\gamma}$, which implies

$$\begin{aligned}
&\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{\alpha,\gamma}(w)dw\le\frac{g_{\alpha,\gamma}(\zeta)+g_{\alpha,\gamma}(\eta)}{2},\ \text{where } w\in[\zeta,\eta],\\
&\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g(\alpha,\gamma,w)dw\le\frac{g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)}{2},
\end{aligned}$$

adding such inequalities, we get Ineq. (2.6) as follows

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\left[\int_{\alpha}^{\beta}g(u,\gamma,\zeta)du+\int_{\alpha}^{\beta}g(u,\gamma,\eta)du+\int_{\alpha}^{\beta}g(u,\delta,\zeta)du+\int_{\alpha}^{\beta}g(u,\delta,\eta)du\right]\\
&+\frac{1}{\delta-\gamma}\left[\int_{\gamma}^{\delta}g(\alpha,v,\zeta)dv+\int_{\gamma}^{\delta}g(\alpha,v,\eta)dv+\int_{\gamma}^{\delta}g(\beta,v,\zeta)dv+\int_{\gamma}^{\delta}g(\beta,v,\eta)dv\right]\\
&+\frac{1}{\eta-\zeta}\left[\int_{\zeta}^{\eta}g(\alpha,\gamma,w)dw+\int_{\zeta}^{\eta}g(\alpha,\delta,w)dw+\int_{\zeta}^{\eta}g(\beta,\gamma,w)dw+\int_{\zeta}^{\eta}g(\beta,\delta,w)dw\right]\\
&\le\frac{3}{2}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)\right.\\
&\left.+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)\right].
\end{aligned}$$

Hence proved. □

Motivated by notion given in [22], we now present right handed HH type inequalities related to inequality given in Theorem 2.4, for differentiable CC functions on rectangle from the space R^3. In order to establish further results we need the following lemma.

Note: From here onwards we use '$\mathcal{A}$' to represent the following algebraic combination of middle and the right sided terms of HH type inequality given in Theorem

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2.4.

$$\frac{1}{8}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)+g(\beta,\delta,\zeta)\right.$$
$$\left.+g(\beta,\delta,\eta)\right]-\frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du\right.$$
$$+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv$$
$$\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\right]+\frac{1}{2}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\right.$$
$$\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\{g(u,v,\zeta)+g(u,v,\eta)\}dudv+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}\{g(\alpha,v,w)+g(\beta,v,w)\}dvdw$$
$$\left.+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}\{g(u,\gamma,w)+g(u,\delta,w)\}\,dudw\right]$$
$$-\frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}g(u,v,w)dudvdw.$$

Lemma 2.5. *Let $g:\Delta\subset R^3\longrightarrow R$ be partial differentiable function on Δ. If $g_{lmn}\in L(\Delta)$, then the subsequent identity is induced*

$$\mathcal{A}=\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1(1-2l)(1-2m)(1-2n)$$
$$g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn.$$

Proof. Considering the following triple integral and integrating it by parts w.r.t. l,m and n respectively, we get

$$\int_0^1\int_0^1\int_0^1(1-2l)(1-2m)(1-2n)$$
$$g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\,dldmdn$$
$$=\int_0^1\int_0^1(1-2m)(1-2n)\left[\frac{(1-2l)}{\alpha-\beta}g_{mn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\Big|_0^1\right.$$
$$\left.-\int_0^1\frac{(-2)}{\alpha-\beta}g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dl\right]dmdn$$
$$=\int_0^1\int_0^1\frac{(1-2m)(1-2n)}{\beta-\alpha}[g_{mn}(\alpha,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)+g_{mn}(\beta,m\gamma+(1-m)\delta,$$
$$n\zeta+(1-n)\eta)-2\int_0^1 g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dl]dmdn$$
$$=\frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\{[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)$$
$$+g(\beta,\gamma,\eta)+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)]-2\left[\int_0^1\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)\right.$$

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$$+g(u,\delta,\eta)\}\,du+\int_0^1\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv$$
$$+\int_0^1\Big\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\Big\}dw\Big]$$
$$+4\Big[\int_0^1\!\!\int_0^1\{g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\zeta)+g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\eta)\}dldm$$
$$+\int_0^1\!\!\int_0^1\{g(\alpha,m\gamma+(1-m)\delta,\zeta n+(1-n)\eta)+g(\beta,m\gamma+(1-m)\delta,\zeta n+(1-n)\eta)\}\,dmdn$$
$$+\int_0^1\!\!\int_0^1\{g(l\alpha+(1-l)\beta,\gamma,\zeta n+(1-n)\eta)+g(l\alpha+(1-l)\beta,\delta,\zeta n+(1-n)\eta)\}\,dldn\Big]$$
$$-8\int_0^1\int_0^1\int_0^1 g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn\Big\}, \qquad (2.8)$$

on multiplying both sides Eq. (2.8) by $\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}$, we get

$$\frac{1}{8}\,[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)$$
$$+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)]-\frac{1}{4}\Big[\int_0^1\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du$$
$$+\int_0^1\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv$$
$$+\int_0^1\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\Big]$$
$$+\frac{1}{2}\Big[\int_0^1\!\!\int_0^1\{g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\zeta)+g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\eta)\}dldm$$
$$+\int_0^1\!\!\int_0^1\{g(\alpha,m\gamma+(1-m)\delta,n\zeta+(1-\zeta n)\eta)+g(\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\}dmdn$$
$$+\int_0^1\!\!\int_0^1\{g(l\alpha+(1-l)\beta,\gamma,n\zeta+(1-n)\eta)+g(l\alpha+(1-l)\beta,\delta,n\zeta+(1-n)\eta)\}\,dldn\Big]$$
$$-\int_0^1\int_0^1\int_0^1 g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn$$
$$=\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1[(1-2l)(1-2m)(1-2n)$$
$$g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)]dldmdn. \qquad (2.9)$$

By changing variable on left hand side of Eq. (2.9), we get the required result. □

Theorem 2.6. *Let $g:\Delta\subset R^3\longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|$ is a CC function on Δ, then the subsequent inequality is induced*

$$|\mathcal{A}|\le\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{512}\,[|g_{lmn}(\alpha,\gamma,\zeta)|+|g_{lmn}(\alpha,\gamma,\eta)+|g_{lmn}(\alpha,\delta,\zeta)$$
$$+|g_{lmn}(\alpha,\delta,\eta)|+|g_{lmn}(\beta,\gamma,\zeta)|+|g_{lmn}(\beta,\gamma,\eta)|+|g_{lmn}(\beta,\delta,\zeta)|+|g_{lmn}(\beta,\delta,\eta)|].$$

Proof. From Lemma 2.5, properties of modulus, co-ordinated convexity of $|g_{lmn}|$ and integrating by parts w.r.t. l, m and n respectively, we have

$$\begin{aligned}
&|\mathcal{A}|\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |1-2l||1-2m||1-2n|\,[l|g_{lmn}(\alpha, m\gamma+(1-m)\delta,\\
&\quad \zeta n+(1-n)\eta)| + (1-l)|g_{lmn}(\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|]\,dldmdn\\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8\times 4}\int_0^1\int_0^1 |1-2m||1-2n|\,[|g_{mn}(\alpha, m\gamma+(1-m)\delta,\\
&\quad n\zeta+(1-n)\eta)| + |g_{mn}(\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|]\,dmdn\\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8\times 4\times 4}\int_0^1 |1-2n|\,[|g_n(\alpha,\gamma,\zeta n+(1-n)\eta)|\\
&\quad +|g_n(\alpha,\delta,n\zeta+(1-n)\eta)|+|g_n(\beta,\gamma,n\zeta+(1-n)\eta)||g_n(\beta,\delta,n\zeta+(1-n)\eta)|]dn\\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{512}\,[|g_{lmn}(\alpha,\gamma,\zeta)| + |g_{lmn}(\alpha,\gamma,\eta) + |g_{lmn}(\alpha,\delta,\zeta)\\
&\quad +|g_{lmn}(\alpha,\delta,\eta)| + |g_{lmn}(\beta,\gamma,\zeta)| + |g_{lmn}(\beta,\gamma,\eta)| + |g_{lmn}(\beta,\delta,\zeta)| + |g_{lmn}(\beta,\delta,\eta)|].
\end{aligned}$$

Hence proved. □

Theorem 2.7. *Let $g : \Delta \subset R^3 \longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|^q$ with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, is a CC function on Δ, then the subsequent inequality is induced*

$$\begin{aligned}
|\mathcal{A}| \leq\ & \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8(p+1)^{\frac{3}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}}\\
& [|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q\\
& + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2.5, properties of modulus, Hölder's integral inequality and Lemma 2.2, we have

$$\begin{aligned}
&|\mathcal{A}|\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|\\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|dldmdn\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|^p dldmdn\right)^{\frac{1}{p}}\\
&\quad \left(\int_0^1\int_0^1\int_0^1 |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|^q dldmdn\right)^{\frac{1}{q}}\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |1-2l|^p\,|1-2m|^p\,|1-2n|^p dldmdn\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&(\int_0^1\int_0^1\int_0^1 (lmn|g_{lmn}(\alpha,\gamma,\zeta)|^q + lm(1-n)|g_{lmn}(\alpha,\gamma,\eta)|^q \\
&+ l(1-m)n|g_{lmn}(\alpha,\delta,\zeta)|^q + l(1-m)(1-n)|g_{lmn}(\alpha,\delta,\eta)|^q \\
&+ (1-l)mn|g_{lmn}(\beta,\gamma,\zeta)|^q + (1-l)m(1-n)|g_{lmn}(\beta,\gamma,\eta)|^q \\
&+(1-l)(1-m)n|g_{lmn}(\beta,\delta,\zeta)|^q + (1-l)(1-m)(1-n)|g_{lmn}(\beta,\delta,\eta)|^q)\, dldmdn)^{\frac{1}{q}} \\
=& \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{(p+1)^3}\right)^{\frac{1}{p}}\left(\frac{1}{8}\right)^{\frac{1}{q}}(|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q \\
&+ |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q \\
&+|g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q)^{\frac{1}{q}}\,.
\end{aligned}
$$

Hence proved. □

Theorem 2.8. *Let $g : \Delta \subset R^3 \longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|^q$, $q \geq 1$ is a CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
|\mathcal{A}| \leq& \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{64}\left(\frac{1}{8}\right)^{\frac{1}{q}} \\
&[|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q \\
&+ |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2.5, properties of modulus, power mean inequality and Lemma 2.2, we have

$$
\begin{aligned}
&|\mathcal{A}| \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)| \\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|dldmdn \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|dldmdn\right)^{1-\frac{1}{q}} \\
&\quad (\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)| \\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|^q dldmdn)^{\frac{1}{q}} \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}(\int_0^1\int_0^1\int_0^1 |1-2l||1-2m||1-2n| \\
&\quad (lmn|g_{lmn}(\alpha,\gamma,\zeta)|^q + lm(1-n)|g_{lmn}(\alpha,\gamma,\eta)|^q + l(1-m)n|g_{lmn}(\alpha,\delta,\zeta)|^q \\
&\quad + l(1-m)(1-n)|g_{lmn}(\alpha,\delta,\eta)|^q + (1-l)mn|g_{lmn}(\beta,\gamma,\zeta)|^q + \\
&\quad (1-l)m(1-n)|g_{lmn}(\beta,\gamma,\eta)|^q + (1-l)(1-m)n|g_{lmn}(\beta,\delta,\zeta)|^q \\
&\quad + (1-l)(1-m)(1-n)|g_{lmn}(\beta,\delta,\eta)|^q)dldmdn)^{\frac{1}{q}}
\end{aligned}
$$

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$$= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{1}{64}\right)^{\frac{1}{q}}$$
$$(|g_{lmn}(\alpha,\gamma,\zeta)|^q+|g_{lmn}(\alpha,\gamma,\eta)|^q+|g_{lmn}(\alpha,\delta,\zeta)|^q+|g_{lmn}(\alpha,\delta,\eta)|^q$$
$$+|g_{lmn}(\beta,\gamma,\zeta)|^q+|g_{lmn}(\beta,\gamma,\eta)|^q+|g_{lmn}(\beta,\delta,\zeta)|^q+|g_{lmn}(\beta,\delta,\eta)|^q)^{\frac{1}{q}},$$

after simplification we get the required result. □

Remark. *Since $\frac{1}{8} < \frac{1}{(p+1)^{\frac{3}{p}}}$ for $p > 1$, therefore estimation given in Theorem 2.8 with an improved and simplified constant is even better than that of Theorem 2.7.*

3. Some Applications to special Means

In Section 2, we established some inequalities based on CC functions of three variables. Now we apply these inequalities to get estimates for absolute values of different patterns of some special means. For this let us first have a look at following special means of positive real numbers ϵ, κ ($\epsilon \neq \kappa$).

Arithmetic mean

$$A(\epsilon,\kappa) = \frac{\epsilon+\kappa}{2}.$$

Harmonic Mean

$$H(\epsilon,\kappa) = \frac{2\epsilon\kappa}{\epsilon+\kappa}.$$

Logarithmic mean

$$L(\epsilon,\kappa) = \frac{\kappa-\epsilon}{ln(\kappa)-ln(\epsilon)}.$$

Genralised Log-mean

$$L_i(\epsilon,\kappa) = \left[\frac{\kappa^{i+1}-\epsilon^{i+1}}{(i+1)(\kappa-\epsilon)}\right]^{\frac{1}{i}},\ i \in Z \setminus \{-1,0\}.$$

Proposition 3.1. *Let $\alpha,\beta,\gamma,\delta,\zeta,\eta \in R^+$ such that $\alpha<\beta, \gamma<\delta, \zeta<\eta$, we have*

$$\left|A_iA_jA_k-\left(L_i^iA_jA_k+A_iL_j^jA_k+A_iA_jL_k^k\right)+\left(L_i^iL_j^jA_k+L_i^iA_jL_k^k+A_iL_j^jL_k^k\right)-L_i^iL_j^jL_k^k\right|$$
$$\le (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}A_{i-1}A_{j-1}A_{k-1}. \tag{3.1}$$

$$\left|A_iA_jA_k-\left(L_i^iA_jA_k+A_iL_j^jA_k+A_iA_jL_k^k\right)+\left(L_i^iL_j^jA_k+L_i^iA_jL_k^k+A_iL_j^jL_k^k\right)-L_i^iL_j^jL_k^k\right|$$
$$\le (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{8(p+1)^{\frac{3}{p}}}\left(A_{q(i-1)}\,A_{q(j-1)}\,A_{q(k-1)}\right)^{\frac{1}{q}}. \tag{3.2}$$

$$\left|A_iA_jA_k-\left(L_i^iA_jA_k+A_iL_j^jA_k+A_iA_jL_k^k\right)+\left(L_i^iL_j^jA_k+L_i^iA_jL_k^k+A_iL_j^jL_k^k\right)-L_i^iL_j^jL_k^k\right|$$
$$\le (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}\left(A_{q(i-1)}\,A_{q(j-1)}\,A_{q(k-1)}\right)^{\frac{1}{q}}. \tag{3.3}$$

Where $A_i = A\left(\alpha^i,\beta^i\right)$, $A_j = A\left(\gamma^j,\delta^j\right)$, $A_k = A\left(\zeta^k,\eta^k\right)$, $L_i^i = L_i^i(\alpha,\beta)$, $L_j^j = L_j^j(\gamma,\delta)$, $L_k^k = L_k^k(\zeta,\eta)$, $A_{i-1} = A\left(\alpha^{i-1},\beta^{i-1}\right)$, $A_{q(i-1)} = A\left(\alpha^{q(i-1)},\beta^{q(i-1)}\right)$ etc.

Proof. The assertions in Proposition 3.1 follow by taking $g(u, v, w) = u^i v^j w^k$ as the CC function in Theorem 2.6, Theorem 2.7 and Theorem 2.8 respectively. Where $u, v, w \in R^+$ and $i, j, k \in Z^+$. □

Proposition 3.2. *Let $\alpha, \beta, \gamma, \delta, \zeta, \eta \in R^+$ such that $\alpha < \beta, \gamma < \delta, \zeta < \eta$, we have*

$$\Big|H_i H_j H_k - \left(L_{-i}^i H_j H_k + H_i L_{-j}^j H_k + H_i H_j L_{-k}^k\right) + \left(L_{-i}^i L_{-j}^j H_k + L_{-i}^i H_j L_{-k}^k + H_i L_{-j}^j L_{-k}^k\right) - L_{-i}^i L_{-j}^j L_{-k}^k\Big| \leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}(H_i H_j H_k)\left(H_{i+1}^{-1} H_{j+1}^{-1} H_{k+1}^{-1}\right)\left(L_{-i}^i L_{-j}^j L_{-k}^k\right).$$

$$\Big|H_i H_j H_k - \left(L_{-i}^i H_j H_k + H_i L_{-j}^j H_k + H_i H_j L_{-k}^k\right) + \left(L_{-i}^i L_{-j}^j H_k + L_{-i}^i H_j L_{-k}^k + H_i L_{-j}^j L_{-k}^k\right) - L_{-i}^i L_{-j}^j L_{-k}^k\Big| \leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{8(p+1)^{\frac{3}{p}}}(H_i H_j H_k)\left(H_{q(i+1)}^{-1} H_{q(j+1)}^{-1} H_{q(k+1)}^{-1}\right)^{\frac{1}{q}}\left(L_{-i}^i L_{-j}^j L_{-k}^k\right).$$

$$\Big|H_i H_j H_k - \left(L_{-i}^i H_j H_k + H_i L_{-j}^j H_k + H_i H_j L_{-k}^k\right) + \left(L_{-i}^i L_{-j}^j H_k + L_{-i}^i H_j L_{-k}^k + H_i L_{-j}^j L_{-k}^k\right) - L_{-i}^i L_{-j}^j L_{-k}^k\Big| \leq (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}(H_i H_j H_k)\left(H_{q(i+1)}^{-1} H_{q(j+1)}^{-1} H_{q(k+1)}^{-1}\right)^{\frac{1}{q}}\left(L_{-i}^i L_{-j}^j L_{-k}^k\right).$$

Where $H_i = H\left(\alpha^i, \beta^i\right), H_j = H\left(\gamma^j, \delta^j\right), H_k = H\left(\zeta^k, \eta^k\right), L_i^i = L_{-i}^i(\alpha, \beta)$, $L_{-j}^j = L_j^j(\gamma, \delta), L_{-k}^k = L_k^k(\zeta, \eta), H_{i+1}^{-1} = H^{-1}\left(\alpha^{i+1}, \beta^{i+1}\right)$, $H_{q(i+1)}^{-1} = H^{-1}\left(\alpha^{q(i+1)}, \beta^{q(i+1)}\right)$ *etc.*

Proof. The assertions in Proposition 3.2 follow by taking $g(u, v, w) = \frac{1}{u^i v^j w^k}$ as CC function in Theorem 2.6, Theorem 2.7 and Theorem 2.8 respectively. Where $u, v, w \in R^+$ and $i, j, k \in Z^+$. □

4. Conclusion

In this paper the idea of CC function of two variables in a rectangle from the plane R^2 is extended to that of three variables in a rectangle from the space R^3. Then HH type inequality for CC function of three variables is established. Consequently by using this inequality the analogous extended right handed HH type inequalities for CC functions of three variables are obtained. Thus obtained right handed inequalities are utilized to give bounds for algebraic combinations of some special means.

Motivated by these results one can also find extensions of existing left handed HH type inequalities. Furthermore it is asserted that in the same way idea of CC function of three variables in a rectangle from the space R^3 can be further extended to CC functions of n variables in a rectangle from n-dimensional Euclidean space R^n.

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