Some Generalized k-Fractional Integral Inequalities for Quasi-Convex Functions

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Abstract

Fractional integral operators generalize the concept of definite integration. Therefore these operators play a vital role in the advancement of subjects of sciences and engineering. The aim of this study is to establish the bounds of a generalized fractional integral operator via quasi-convex functions. These bounds behave as a formula in unified form, and estimations of almost all fractional integrals defined in last two decades can be obtained at once by choosing convenient parameters. Moreover, several related fractional integral inequalities are identified.

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1 Introduction

A function $f : I \to \mathbb{R}$ is said to be *convex* if the following inequality holds:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1.1}$$

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for all $a, b \in I$ and $t \in [0, 1]$.

If inequality (1.1) is reversed, then the function f will be the concave on $[a, b]$. Convex functions are very useful in mathematical analysis. A lot of integral inequalities have been established due to convex functions in literature (for details see, [2–6, 10, 11, 18–20]. Quasi-convexity is also class of convex functions which is defined as follows:

Definition 1.1. ([10]) A function $f : I \to \mathbb{R}$ is said to be *quasi-convex* if the following inequality holds:

$$f(ta + (1 - t)b) \leq \max\{f(a), f(b)\} \tag{1.2}$$

for all $a, b \in I$ and $t \in [0, 1]$.

Example 1.2. ([11, p. 83]) The function $f : [-2, 2] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & x \in [-2, -1] \\ x^2 & x \in (-1, 2] \end{cases}$$

is not a convex function on $[-2, 2]$ but it is quasi-convex function on $[-2, 2]$.

It is noted that class of quasi-convex functions contain the class of finite convex functions defined on finite closed intervals. For some recent citations and utilizations of quasi-convex functions one can see [2, 10, 11, 20] and references therein.

Fractional integral operators play an important role in generalizing the mathematical inequalities. In recent years, authors have proved various interesting mathematical inequalities due to different fractional integral operators, for example see [3–8 11, 15, 20]. The upcoming definitions and remark provide a detailed information of recent and classical fractional integral operators.

Definition 1.3. Let $f \in L_1[a, b]$ with $0 \leq a < b$. Then *Riemann-Liouville fractional integral operators* of order $\mu > 0$ are defined by

$${}^{\mu}I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x - t)^{\mu - 1} f(t) dt, \quad x > a \tag{1.3}$$

and

$${}^{\mu}I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t - x)^{\mu - 1} f(t) dt, \quad x < b, \tag{1.4}$$

where $\Gamma(\mu)$ is the Gamma function defined by $\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt$.

Definition 1.4. ([16]) Let $f \in L_1[a, b]$ with $0 \leq a < b$. Then *Riemann-Liouville k-fractional integral operators* of order $\mu, k > 0$ are defined by

$${}^{\mu}I^k_{a^+} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x - t)^{\frac{\mu}{k} - 1} f(t) dt, \quad x > a \tag{1.5}$$

and

$${}^{\mu}I^k_{b^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t - x)^{\frac{\mu}{k} - 1} f(t) dt, \quad x < b, \tag{1.6}$$

where $\Gamma_k(\mu)$ is the k-Gamma function defined as $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$.

Definition 1.5. ([14]) Let $f \in L_1[a,b]$ with $0 \leq a < b$. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). *The left-sided and right-sided fractional integrals of a function* f with respect to another function g on $[a,b]$ of order $\mu > 0$, are defined by

$$ {}_{g}^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x)-g(t))^{\mu-1} g'(t) f(t)dt, \quad x > a \tag{1.7}$$

and

$$ {}_{g}^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t)-g(x))^{\mu-1} g'(t) f(t)dt, \quad x < b. \tag{1.8}$$

Definition 1.6. ([15]) Let $f \in L_1[a,b]$ with $0 \leq a < b$. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). *The left-sided and right-sided fractional integrals of a function* f with respect to another function g on $[a,b]$ of order $\mu, k > 0$ are defined by

$$ {}_{g}^{\mu}I_{a^+}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} g'(t) f(t)dt, \quad x > a \tag{1.9}$$

and

$$ {}_{g}^{\mu}I_{b^-}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1} g'(t) f(t)dt, \quad x < b. \tag{1.10}$$

These are compact formulas which give almost all fractional integrals by choosing suitable formations of function g. In this context the following remark is important:

Remark 1.7. Fractional integrals elaborated in (1.9) and (1.10) particularly produce several known fractional integrals corresponding to different settings of k and g.

(i) For $k = 1$ (1.9) and (1.10) fractional integrals coincide with (1.7) and (1.8).

(ii) For taking g as identity function (1.9) and (1.10) fractional integrals coincide with (1.5) and (1.6).

(iii) For $k = 1$, along with g as identity function (1.9) and (1.10) fractional integrals coincide with (1.3) and (1.4).

(iv) For $k = 1$ and $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$, (1.9) and (1.10) produce Katugampola fractional integrals defined by Chen et al. in [1].

(v) For $k = 1$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, (1.9) and (1.10) produce generalized conformable fractional integrals defined by Khan et al. in [13].

(vi) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10), then conformable (k,s)-fractional integrals are achieved as defined by Sidra et al. in [9].

(vii) If we take $g(x) = \frac{x^{1+s}}{1+s}$, then conformable fractional integrals are achieved as defined by Sarikaya et al. in [17].

(viii) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10) with $k = 1$, then conformable fractional integrals are achieved as defined by Jarad et al. in [12].

The rest of paper is organized as follows:

In Section 2, the bounds of sum of left-sided and right-sided generalized fractional integrals via quasi-convex function are established. First result provides an upper bound for generalized fractional integrals, and some particular cases are elaborated. Then bounds along with particular cases, in modulus form have been presented. Furthermore, Hadamarad type bounds are formulated. In Section 3, applications of results of Section 2 are given. Moreover concluding remarks are included at the end.

In the next sections the notation $M_a^b(f) = \max\{f(a), f(b)\}$ has been used frequently.

2 Main Results

Firstly, the following theorem set the formula for upper bounds of fractional integrals via quasi-convex functions in a unified form.

Theorem 2.1. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be two functions such that g be differentiable and $f \in L[a,b]$ with $a < b$. Also let f be positive, quasi-convex and g be strictly increasing function with $g' \in L[a,b]$. Then for $x \in [a,b]$ and $\mu, \nu \geq k$, the following inequality holds:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) + {}_g^{\nu} I_{b^-}^{k} f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \tag{2.1}$$

Proof. As f is quasi-convex, therefore for $t \in [a,x]$, $f(t) \leq M_a^x(f)$. Under assumptions on function g, for all $x \in [a,b]$, $t \in [a,x]$ and $\mu \geq k$, the following inequality holds:

$$ g'(t)(g(x)-g(t))^{\frac{\mu}{k}-1} \leq g'(t)(g(x)-g(a))^{\frac{\mu}{k}-1}. \tag{2.2}$$

From aforementioned two inequalities, the following integral inequality is yielded:

$$ \int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} f(t) g'(t)dt \leq (g(x)-g(a))^{\frac{\mu}{k}-1} M_a^x(f) \int_a^x g'(t)dt. \tag{2.3}$$

By using (1.9) of Definition 1.6, the following bound of fractional integral defined in (1.9) is obtained:

$$ {}_g^{\mu}I_{a^+}^{k} f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f). \tag{2.4}$$

Again from quasi-convexity of f, for $t \in [x,b]$, $f(t) \leq M_x^b(f)$. Also for $x \in [a,b]$, $t \in [x,b]$ and $\nu \geq k$, the following inequality holds:

$$ g'(t)(g(t)-g(x))^{\frac{\nu}{k}-1} \leq g'(t)(g(b)-g(x))^{\frac{\nu}{k}-1}. \tag{2.5}$$

From aforementioned two inequalities, the following integral inequality is yielded:

$$ \int_x^b (g(t)-g(x))^{\frac{\nu}{k}-1} f(t) g'(t)dt \leq (g(b)-g(x))^{\frac{\nu}{k}-1} M_x^b(f) \int_x^b g'(t)dt. \tag{2.6}$$

By using (1.10) of Definition 1.6, the following bound of fractional integral defined in (1.10) is obtained:

$$ {}_g^{\nu}I_{b^-}^{k} f(x) \leq \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \tag{2.7}$$

From (2.4) and (2.7), the bound of sum of left-sided and right-sided fractional integrals is achieved. □

Special cases of Theorem 2.1, are discussed in the following corollaries.

Corollary 2.2. *If we take $\mu = \nu$ in (2.1), then we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) + {}_g^{\mu} I_{b^-}^{k} f(x) \leq \frac{1}{k\Gamma_k(\mu)} \left((g(x) - g(a))^{\frac{\mu}{k}} M_a^x(f) + (g(b) - g(x))^{\frac{\mu}{k}} M_x^b(f) \right). \quad (2.8) $$

Corollary 2.3. *If we take $k = 1$ in (2.1), then we get the following generalized (RL) fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+} f(x) + {}_g^{\nu} I_{b^-} f(x) \leq \frac{(g(x) - g(a))^{\mu}}{\Gamma(\mu)} M_a^x(f) + \frac{(g(b) - g(x))^{\nu}}{\Gamma(\nu)} M_x^b(f). \quad (2.9) $$

Corollary 2.4. *If we take $g(x) = x$ in (2.1), then we get the following (RL) k-fractional integral inequality:*

$$ {}^{\mu}I_{a^+}^{k} f(x) + {}^{\nu} I_{b^-}^{k} f(x) \leq \frac{(x - a)^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f) + \frac{(b - x)^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \quad (2.10) $$

Corollary 2.5. *If we take $g(x) = x$ and $k = 1$ in (2.1), then we get the following (RL) fractional integral inequality:*

$$ {}^{\mu}I_{a^+} f(x) + {}^{\nu} I_{b^-} f(x) \leq \frac{(x - a)^{\mu}}{\Gamma(\mu)} M_a^x(f) + \frac{(b - x)^{\nu}}{\Gamma(\nu)} M_x^b(f). \quad (2.11) $$

Corollary 2.6. *Under the assumptions of above theorem if f is increasing on $[a, b]$, then from (2.1), we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) + {}_g^{\nu} I_{b^-}^{k} f(x) \leq \frac{(g(x) - g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} f(x) + \frac{(g(b) - g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} f(b). \quad (2.12) $$

Corollary 2.7. *Under the assumptions of above theorem if f is decreasing on $[a, b]$, then from (2.1), we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) + {}_g^{\nu} I_{b^-}^{k} f(x) \leq \frac{(g(x) - g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} f(a) + \frac{(g(b) - g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} f(x). \quad (2.13) $$

Next theorem provides the bound of generalized fractional integrals in modulus form.

Theorem 2.8. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two differentiable functions with $a < b$. Also let $|f'|$ be quasi-convex and g be strictly increasing with $g' \in L[a, b]$. Then for $x \in [a, b]$ and $\mu, \nu, k > 0$, the following inequality holds:*

$$ \left| {}_g^{\mu}I_{a^+}^{k} f(x) + {}_g^{\nu} I_{b^-}^{k} f(x) - \left(\frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a) + \frac{(g(b) - g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu + k)} f(b) \right) \right| \quad (2.14) $$
$$ \leq \frac{(g(x) - g(a))^{\frac{\mu}{k}} (x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|) + \frac{(g(b) - g(x))^{\frac{\nu}{k}} (b - x)}{\Gamma_k(\nu + k)} M_x^b(|f'|). $$

Proof. As $|f'|$ is quasi-convex, therefore for $t \in [a, x]$, we have

$$|f'(t)| \le M_a^x(|f'|). \tag{2.15}$$

From (2.15), we have

$$f'(t) \le M_a^x(|f'|). \tag{2.16}$$

Under assumptions of the function g, the following inequality holds:

$$(g(x) - g(t))^{\frac{\mu}{k}} \le (g(x) - g(a))^{\frac{\mu}{k}} \tag{2.17}$$

for all $x \in [a, b]$, $t \in [a, x]$ and $\mu, k > 0$.

From (2.16) and (2.17), we have

$$\int_a^x (g(x) - g(t))^{\frac{\mu}{k}} f'(t)dt \le (g(x) - g(a))^{\frac{\mu}{k}} M_a^x(|f'|) \int_a^x dt, \tag{2.18}$$

the left hand side calculate as follows:

$$\begin{aligned}&\int_a^x (g(x) - g(t))^{\frac{\mu}{k}} f'(t)dt \\ &= f(t)(g(x) - g(t))^{\frac{\mu}{k}}|_a^x + \frac{\mu}{k}\int_a^x (g(x) - g(t))^{\frac{\mu}{k}-1} f(t)g'(t)dt \\ &= -f(a)(g(x) - g(a))^{\frac{\mu}{k}} + \Gamma_k(\mu + k){}_g^{\mu}I_{a^+}^k f(x).\end{aligned}$$

Using above calculation in (2.18), we get the following inequality:

$${}_g^{\mu}I_{a^+}^k f(x) - \frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a) \le \frac{(g(x) - g(a))^{\frac{\mu}{k}}(x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|). \tag{2.19}$$

Also from (2.15), we can write

$$f'(t) \ge -M_a^x(|f'|). \tag{2.20}$$

Following the same procedure as we did for (2.16), we also have

$$\frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a) -{}_g^{\mu} I_{a^+}^k f(x) \le \frac{(g(x) - g(a))^{\frac{\mu}{k}}(x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|). \tag{2.21}$$

From (2.19) and (2.21), we get the following modulus inequality:

$$\left|{}_g^{\mu}I_{a^+}^k f(x) - \frac{(g(x) - g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu + k)} f(a)\right| \le \frac{(g(x) - g(a))^{\frac{\mu}{k}}(x - a)}{\Gamma_k(\mu + k)} M_a^x(|f'|). \tag{2.22}$$

Again by using quasi-convexity of $|f'|$, for $t \in [x, b]$, we have

$$|f'(t)| \le M_x^b(|f'|). \tag{2.23}$$

Now for $x \in [a, b]$, $t \in [x, b]$ and $\nu, k > 0$, the following inequality holds:

$$(g(t) - g(x))^{\frac{\nu}{k}} \le (g(b) - g(x))^{\frac{\nu}{k}}. \tag{2.24}$$

By adopting the same way as we have done for (2.16), (2.17) and (2.20) one can get from (2.23) and (2.24) the following modulus inequality:

$$\left|{}_g^{\nu}I_{b-}^{k}f(x)-\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right|\leq\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}M_x^b(|f'|). \tag{2.25}$$

From (2.22) and (2.25) via triangular inequality, we get the modulus inequality in (2.14), which is required. □

Special cases of Theorem 2.8, are discussed in the following corollaries.

Corollary 2.9. *If we take $\mu=\nu$ in* (2.14), *then we get the following fractional integral inequality:*

$$\left|{}_g^{\mu}I_{a+}^{k}f(x)+{}_g^{\mu}I_{b-}^{k}f(x)-\frac{1}{\Gamma_k(\mu+k)}\left((g(x)-g(a))^{\frac{\mu}{k}}f(a)+(g(b)-g(x))^{\frac{\mu}{k}}f(b)\right)\right| \tag{2.26}$$
$$\leq\frac{1}{\Gamma_k(\mu+k)}\left((g(x)-g(a))^{\frac{\mu}{k}}(x-a)M_a^x(|f'|)+(g(b)-g(x))^{\frac{\mu}{k}}(b-x)M_x^b(|f'|)\right).$$

Corollary 2.10. *If we take $k=1$ in (2.14), then we get the following generalized (RL) fractional integral inequality:*

$$\left|{}_g^{\mu}I_{a+}f(x)+{}_g^{\nu}I_{b-}f(x)-\left(\frac{(g(x)-g(a))^{\mu}}{\Gamma(\mu+1)}f(a)+\frac{(g(b)-g(x))^{\nu}}{\Gamma(\nu+1)}f(b)\right)\right| \tag{2.27}$$
$$\leq\frac{(g(x)-g(a))^{\mu}(x-a)}{\Gamma(\mu+1)}M_a^x(|f'|)+\frac{(g(b)-g(x))^{\nu}(b-x)}{\Gamma(\nu+1)}M_x^b(|f'|).$$

Corollary 2.11. *If we take $g(x)=x$ in* (2.14), *then we get the following (RL) k-fractional integral inequality:*

$$\left|{}^{\mu}I_{a+}^{k}f(x)+{}^{\nu}I_{b-}^{k}f(x)-\left(\frac{(x-a)^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(b-x)^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \tag{2.28}$$
$$\leq\frac{(x-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}M_a^x(|f'|)+\frac{(b-x)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}M_x^b(|f'|).$$

Corollary 2.12. *If we take $g(x)=x$ and $k=1$ in* (2.14), *then we get the following (RL) fractional integral inequality:*

$$\left|{}^{\mu}I_{a+}f(x)+{}^{\nu}I_{b-}f(x)-\left(\frac{(x-a)^{\mu}}{\Gamma(\mu+1)}f(a)+\frac{(b-x)^{\nu}}{\Gamma(\nu+1)}f(b)\right)\right| \tag{2.29}$$
$$\leq\frac{(x-a)^{\mu+1}}{\Gamma(\mu+1)}M_a^x(|f'|)+\frac{(b-x)^{\nu+1}}{\Gamma(\nu+1)}M_x^b(|f'|).$$

Corollary 2.13. *Under the assumptions of above theorem if $|f'|$ is increasing on $[a,b]$, then from* (2.14), *we get the following fractional integral inequality:*

$$\left|{}_g^{\mu}I_{a+}^{k}f(x)+{}_g^{\nu}I_{b-}^{k}f(x)-\left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \tag{2.30}$$
$$\leq\frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)}|f'(x)|+\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}|f'(b)|.$$

Corollary 2.14. *Under the assumptions of above theorem if $|f'|$ is decreasing on $[a,b]$, then from* (2.14), *we get the following fractional integral inequality:*

$$\left|{}_{g}^{\mu}I_{a^+}^{k}f(x)+{}_{g}^{\nu}I_{b^-}^{k}f(x)-\left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \tag{2.31}$$
$$\leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)}|f'(a)|+\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}|f'(x)|.$$

We need the following lemma in the proof of next result.

Lemma 2.15. *Let $f:[0,\infty)\to R$ be a quasi-convex function. If $f(x)=f(a+b-x)$, then for $x\in[a,b]$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq f(x). \tag{2.32}$$

Proof. We have

$$\frac{a+b}{2}=\frac{1}{2}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)+\frac{1}{2}\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right). \tag{2.33}$$

As f is quasi-convex, therefore for $x\in[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\leq \max\{f(x),f(a+b-x)\}. \tag{2.34}$$

Using given condition $f(x)=f(a+b-x)$ in (2.34), then inequality in (2.32) is established. □

Theorem 2.16. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be two functions such that g be differentiable and $f\in L[a,b]$ with $a<b$. Also let f be positive, quasi-convex, $f(x)=f(a+b-x)$ and g be strictly increasing with $g'\in L[a,b]$. Then for $x\in[a,b]$ and $\mu,\nu,k>0$, the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \tag{2.35}$$
$$\leq {}_{g}^{\nu+k}I_{b^-}^{k}f(a)+{}_{g}^{\mu+k}I_{a^+}^{k}f(b)$$
$$\leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]M_a^b(f).$$

Proof. As f is quasi-convex, therefore for $x\in[a,b]$, we have

$$f(x)\leq M_a^b(f). \tag{2.36}$$

Under assumptions of the function g, the following inequality holds:

$$g'(x)(g(x)-g(a))^{\frac{\nu}{k}}\leq g'(x)(g(b)-g(a))^{\frac{\nu}{k}} \tag{2.37}$$

for all $x\in[a,b]$ and $\nu,k>0$.

From (2.36) and (2.37), we have

$$\int_a^b (g(x)-g(a))^{\frac{\nu}{k}} f(x)g'(x)dx \leq (g(b)-g(a))^{\frac{\nu}{k}} M_a^b(f) \int_a^b g'(x)dx.$$

By using (1.10) of Definition 1.6, we get

$$ {}_g^{\nu+k}I_{b^-}^k f(a) \leq \frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{k\Gamma_k(\nu+k)} M_a^b(f). \tag{2.38}$$

Now for $x \in [a,b]$ and $\mu, k > 0$, the following inequality inequality holds:

$$g'(x)(g(b)-g(x))^{\frac{\mu}{k}} \leq g'(x)(g(b)-g(a))^{\frac{\mu}{k}}. \tag{2.39}$$

From (2.36) and (2.39), we have

$$\int_a^b (g(b)-g(x))^{\frac{\mu}{k}} f(x)g'(x)dx \leq (g(b)-g(a))^{\frac{\mu}{k}} M_a^b(f) \int_a^b g'(x)dx.$$

By using (1.9) of Definition 1.6, we get

$$ {}_g^{\mu+k}I_{a^+}^k f(b) \leq \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{k\Gamma_k(\mu+k)} M_a^b(f). \tag{2.40}$$

Adding (2.38) and (2.40), we get the following inequality

$$ {}_g^{\nu+k}I_{b^-}^k f(a) + {}_g^{\mu+k} I_{a^+}^k f(b) \leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] M_a^b(f). \tag{2.41}$$

Now on the other hand multiplying (2.32) with $(g(x)-g(a))^{\frac{\nu}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b (g(x)-g(a))^{\frac{\nu}{k}} g'(x)dx \leq \int_a^b (g(x)-g(a))^{\frac{\nu}{k}} g'(x)f(x)dx. \tag{2.42}$$

By using (1.10) of Definition 1.6, we get

$$\frac{k(g(b)-g(a))^{\frac{\nu}{k}+1}}{\nu+k} f\left(\frac{a+b}{2}\right) \leq k\Gamma_k(\nu+k) {}_g^{\nu+k}I_{b^-}^k f(a). \tag{2.43}$$

Similarly, multiplying (2.32) with $(g(b)-g(x))^{\frac{\mu}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$\frac{k(g(b)-g(a))^{\frac{\mu}{k}+1}}{\mu+k} f\left(\frac{a+b}{2}\right) \leq k\Gamma_k(\mu+k) {}_g^{\mu+k}I_{a^+}^k f(b). \tag{2.44}$$

Adding (2.43) and (2.44), we get the following inequality

$$f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \leq {}_g^{\nu+k} I_{b^-}^k f(a) + {}_g^{\mu+k} I_{a^+}^k f(b). \tag{2.45}$$

From (2.41) and (2.45), we get the inequalities in (2.35), which is required. □

Special cases of Theorem 2.16, are discussed in the following corollaries.

Corollary 2.17. *If we take $\mu=\nu$ in (2.35), then we get the following fractional integral inequality:*

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] &\leq {}_g^{\mu+k}I_{b-}^k f(a)+{}_g^{\mu+k} I_{a+}^k f(b)\\ &\leq \frac{2}{k}\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]M_a^b(f).\end{aligned}$$

Corollary 2.18. *If we take $k=1$ in (2.35), then we get the following generalized (RL) fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\nu+1}}{\Gamma(\nu+2)}+\frac{(g(b)-g(a))^{\mu+1}}{\Gamma(\mu+2)}\right]\\ &\leq {}_g^{\nu+1}I_{b-}f(a)+{}_g^{\mu+1} I_{a+}f(b)\\ &\leq \left[\frac{(g(b)-g(a))^{\nu+1}}{\Gamma(\nu+1)}+\frac{(g(b)-g(a))^{\mu+1}}{\Gamma(\mu+1)}\right]M_a^b(f).\end{aligned}\tag{2.46}$$

Corollary 2.19. *If we take $g(x)=x$ in (2.35), then we get the following (RL) k-fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(b-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right]\\ &\leq {}^{\nu+k}I_{b-}^k f(a)+{}^{\mu+k} I_{a+}^k f(b)\\ &\leq \frac{1}{k}\left[\frac{(b-a)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(b-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]M_a^b(f).\end{aligned}\tag{2.47}$$

Corollary 2.20. *If we take $g(x)=x$ and $k=1$ in (2.35), then we get the following (RL) fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)}+\frac{(b-a)^{\mu+1}}{\Gamma(\mu+2)}\right]\\ &\leq {}^{\nu+1}I_{b-}f(a)+{}^{\mu+1} I_{a+}f(b)\\ &\leq \left[\frac{(b-a)^{\nu+1}}{\Gamma(\nu+1)}+\frac{(b-a)^{\mu+1}}{\Gamma(\mu+1)}\right]M_a^b(f).\end{aligned}\tag{2.48}$$

Corollary 2.21. *Under the assumptions of above theorem if f is increasing on $[a,b]$, then from (2.35), we get the following fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right]\\ &\leq {}_g^{\nu+k}I_{b-}^k f(a)+{}_g^{\mu+k} I_{a+}^k f(b)\\ &\leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]f(b).\end{aligned}\tag{2.49}$$

Corollary 2.22. *Under the assumptions of above theorem if f is decreasing on $[a,b]$, then from* (2.35), *we get the following fractional integral inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \\ \leq & \ {}_g^{\nu+k}I_{b^-}^k f(a) +_g^{\mu+k} I_{a^+}^k f(b) \\ \leq & \ \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] f(a). \end{aligned} \tag{2.50}$$

3 Applications

In this section we give applications of the results proved in the previous section. First we apply Theorem 2.1 and get the following result.

Theorem 3.1. *Under the assumptions of Theorem 2.1, we have the following fractional integral inequality:*

$${}_g^{\mu}I_{a^+}^k f(b) +_g^{\nu} I_{b^-}^k f(a) \leq \frac{1}{k}\left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu)}+\frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu)}\right) M_a^b(f). \tag{3.1}$$

Proof. If we put $x=a$ in (2.1), then we have

$${}_g^{\nu}I_{b^-}^k f(a) \leq \frac{((g(b)-g(a))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_a^b(f). \tag{3.2}$$

If we put $x=b$ in (2.1), then we have

$${}_g^{\mu}I_{a^+}^k f(b) \leq \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^b(f). \tag{3.3}$$

Adding inequalities (3.2) and (3.3), we get (3.1). □

Special cases of Theorem 3.1, are discussed in the following corollaries.

Corollary 3.2. *If we take $\mu=\nu$ in* (3.1), *then we have the following fractional integral inequality:*

$${}_g^{\mu}I_{a^+}^k f(b) +_g^{\mu} I_{b^-}^k f(a) \leq \frac{2(g(b)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^b(f). \tag{3.4}$$

Corollary 3.3. ([2]) *If we take $\mu=k=1$ and $g(x)=x$ in* (3.4), *then we get the following inequality:*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq M_a^b(f). \tag{3.5}$$

Next we apply Theorem 2.8 to obtain required results.

Theorem 3.4. *Under the assumptions of Theorem 2.8, we have the following fractional integral inequality:*

$$\begin{aligned}
&\left|{}^{\mu}_{g}I^{k}_{a^+}f(b) +^{\nu}_{g} I^{k}_{b^-}f(a) - \left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a) + \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \qquad (3.6)\\
&\leq \left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} + \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}\right)(b-a)M^b_a(|f'|).
\end{aligned}$$

Proof. If we put $x = a$ in (2.14), then we have

$$\left|{}^{\nu}_{g}I^{k}_{b^-}f(a) - \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right| \leq \frac{(g(b)-g(a))^{\frac{\nu}{k}}(b-a)}{\Gamma_k(\nu+k)}M^b_a(|f'|). \qquad (3.7)$$

If we put $x = b$ in (2.14), then we have

$$\left|{}^{\mu}_{g}I^{k}_{a^+}f(b) - \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)\right| \leq \frac{(g(b)-g(a))^{\frac{\mu}{k}}(b-a)}{\Gamma_k(\mu+k)}M^b_a(|f'|). \qquad (3.8)$$

Adding inequalities (3.7) and (3.8), we get (3.6). □

Special cases of Theorem 3.4, are discussed in the following corollaries.

Corollary 3.5. *If we take $\mu = \nu$ in* (3.6), *then we have the following fractional integral inequality:*

$$\begin{aligned}
&\left|{}^{\mu}_{g}I^{k}_{a^+}f(b) +^{\mu}_{g} I^{k}_{b^-}f(a) - \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}(f(a)+f(b))\right| \qquad (3.9)\\
&\leq \frac{2(g(b)-g(a))^{\frac{\mu}{k}}(b-a)}{\Gamma_k(\mu+k)}M^b_a(|f'|).
\end{aligned}$$

Corollary 3.6. *If we take $\mu = k = 1$ and $g(x) = x$ in* (3.9), *then we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leq (b-a)M^b_a(|f'|). \qquad (3.10)$$

By applying Theorem 2.16 similar relations can be established we leave it for the reader.

4 Concluding Remarks

The aim of this study is to explore bounds of fractional integrals in a compact form by using the concept of quasi-convexity. The authors are succeeded in the formulation of bounds of generalized fractional integrals (1.9) and (1.10). Theorem 2.1 provides upper bounds, Theorem 2.8 gives bounds in modulus form while Theorem 2.16 formulates bounds of Hadamard type. Section 3 consists of the applications of these bounds. Also some particular case of all these results are shown. Remark 1.7 includes all possible fractional integrals associated with generalized fractional integrals (1.9) and (1.10). The readers can obtain bounds for desired fractional integrals by putting the corresponding function g from Remark 1.7.

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