Differential Transform Method for Solving Fuzzy Fractional Wave Equation†

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Abstract: In this letter, the differential transform method (DTM) is applied to solve fuzzy fractional wave equation. The elemental properties of this method are investigated based on the two-dimensional differential transform method (DTM), generalized Taylor's formula and fuzzy Coputo's derivative. The proposed method is also illustrated by using some examples. The results reveal that DTM is a highly effective scheme for obtaining analytical solutions of the fuzzy fractional wave equation.

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1 Introduction

In 1965, the fuzzy sets were introduced for the first time by Zadeh in [28]. hundreds of examples have been supplied where the nature of uncertainty in the behavior of given system processes are fuzzy rather than stochastic nature. In the last few years, many authors have interested in the study of the theoretical framework of fuzzy initial value problems. Chang and Zadeh in [6] have introduced the concept of fuzzy derivative. Kandel and Byatt in [12] have initially presented the concept of the fuzzy differential equation. Bede and Gal in [4] have studied the concept of strongly generalized differentiable of fuzzy valued functions, which enlarged the class of differentiable fuzzy valued functions.

In 1695, the fractional calculus was first studied. The subject of fractional calculus has gained importance during the past three decades due mainly to its demonstrated applications in different area of physics and engineering in [16]. Fuzzy fractional differential equations (FFDE) play an important role in modelling of science and engineering problems. Padmapriya and Kaliyappan in [22] established analytical and numerical methods to solve fuzzy fractional differential equations. the concept of differential of fuzzy function with two variables and fuzzy wave equations studied in [26]. In the last years many authors have developed and introduced some variant methods for solving fuzzy wave equation. Kermani in [15] used finite difference method to solve the fuzzy wave equation numerically. Also, Martin and Radek in [25] used f-transforms to solve the fuzzy wave equation.

Zhou in [29] has presented the concept of the differential transform method (DTM), this method constructs an analytical solution inform of a polynomial, which is different from the tradition higher order Taylor formula method. Recently some researchers used differential transform method (DTM) to solve fuzzy fractional differential equations and fuzzy differential equations in [9, 23, 1, 19, 20].

This paper is structured as follows. In Section 2, we call some definitions on fuzzy numbers, fuzzy functions and fuzzy Caputo's derivative. In Section 3, The generalization of Taylor's formula is presented. In Section 4, the generalized two-dimensional differential transform method (DTM) for

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the solution of the fuzzy wave equation with space and time-fractional derivatives are developed and derived. Examples are shown in Section 5. Finely, conclusion is given in section 6.

2 Basic concepts

The results about fuzzy numbers space E^1, we recall that $E^1 = \{\tilde{u} : R \to [0,1] : u$ satisfies (1)(4) below } (refer to [6])

1. $\tilde{u}$ is normal, i.e., there exists $x_0 \in R$ such that $\tilde{u}(x_0) = 1$;
2. $\tilde{u}$ is convex, i.e., for all and $\lambda \in [0,1], x, y \in R$,
$$\tilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\},$$
holds;
3. $\tilde{u}$ is upper semicontinuous, i.e., for any $x_0 \in R$,
$$\tilde{u}(x_0) \geq \lim_{x \longrightarrow x_0^{\pm}} \tilde{u}(x);$$
4. supp $\tilde{u} = \{x \in R | \tilde{u}(x) > 0\}$ is the support of $\tilde{u}$, and its closure cl (supp $\tilde{u}$) is compact.

For $0 < r \leq 1$, denote $[\tilde{u}]_r = \{x : \tilde{u}(x) \geq r\}$. Then from (1)-(4), follows that the r-level set $[\tilde{u}]_r$ is a closed and bounded interval for all $r \in [0,1]$.

For $\tilde{u}, \tilde{v} \in E^1,\ k \in R$, the addition and scalar multiplication are defined using the equations
$$[\tilde{u} + \tilde{v}]_r = [\tilde{u}]_r + [\tilde{v}]_r,$$
$$[k\tilde{u}]_r = k[\tilde{u}]_r,$$
respectively.

Define $D : E^1 \times E^1 \to R^+ \cup \{0\}$ using the equation
$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} d([\tilde{u}]_r [\tilde{v}]_r),$$
where d is Hausdorff metric space as
$$\begin{aligned} d([\tilde{u}]_r, [\tilde{v}]_r) &= \inf\{\varepsilon : [\tilde{u}]_r \subset N([\tilde{v}]_r, \varepsilon), [\tilde{v}]_r \subset N([\tilde{u}]_r, \varepsilon)\} \\ &= \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}, \end{aligned}$$
where $N([\tilde{u}]_r, \varepsilon), N([\tilde{v}]_r, \varepsilon)$ is the ε-neighborhood of $[\tilde{u}]_r, [\tilde{v}]_r$, respectively, and $\underline{u}_r, \underline{v}_r, \overline{u}_r, \overline{v}_r$ are endpoints of $[\tilde{u}]_r, [\tilde{v}]_r$, respectively.
By using the results of [13], we see that

- (E^1, D) is complete metric space,
- $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$ for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(k\tilde{u}, k\tilde{v}) = |k| D(\tilde{u}, \tilde{v})$.

In addition, we can introduce a partial order in E^1 by $\tilde{u} \leq \tilde{v}$ if and only if $[\tilde{u}]_r \leq [\tilde{v}]_r,\ r \in [0,1]$ if and only if $\underline{u}_r \leq \underline{v}_r, \overline{u}_r \leq \overline{v}_r, r \in [0,1]$. For applications of the partial order on E^1 (refer to [27]).

As the fuzzy number is resolved by using the interval $\tilde{u}_r = [\underline{u}_r, \overline{u}_r]$, see [8] defined another statements, parametrically, of fuzzy numbers as in following.

Definition 2.1.[31, 32] For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1, \tilde{u} = [\underline{u}_r, \overline{u}_r], \tilde{v} = [\underline{v}_r, \overline{v}_r]$, the quantity $D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$ is the distance between $\tilde{u}$ and $\overline{v}$ and also the following properties hold:

- (E^1, D) is a complete metric space,
- $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{0}) \leq D(\tilde{u}, \tilde{0}) + D(\tilde{v}, \tilde{0}), \forall \tilde{u}, \tilde{v} \in E^1$,
- $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v} \in E^1, k \in R$,
- $D(k_1 \odot \tilde{u}, k_2 \odot \tilde{u}) = |k_1 - k_2| D(\tilde{u}, \tilde{0}), \forall \tilde{u} \in E^1, k_1, k_2 \in R$, with $k_1 \cdot k_2 \geq 0$.

Let us recall the definition of the Hukuhara difference (H-difference) in [33]. Suppose that $\tilde{u}, \tilde{v} \in E^1$. The Hukuhara H-difference has been presented as a set $\tilde{w}$ for which $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \tilde{u} = \tilde{v} \oplus \tilde{w}$. The H-difference is unique, but it does not always exist (a necessary condition for $\tilde{u} \ominus_{gH} \tilde{v}$ to exist is that $\tilde{u}$ contains a translate $\{c\} \oplus \tilde{v}$ of $\tilde{v}$). A generalization of the Hukuhara difference aims to overcome this situation.

Definition 2.2.[33, 31] The generalized Hukuhara difference between two fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$ is defined as following:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \begin{cases} \quad \text{(i) } \tilde{u} = \tilde{v} \oplus \tilde{w}, \\ or \text{ (ii) } \tilde{v} = \tilde{u} \oplus (-\tilde{w}). \end{cases} \tag{2.1}$$

In terms of the $r-$levels, we get $[\tilde{u} \ominus_{gH} \tilde{v}] = [\min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}, \max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}]$ and if the H-difference exists, then $\tilde{u} \ominus \tilde{v} = \tilde{u} \ominus_{gH} \tilde{v}$; the conditions for existence of $\tilde{w} = \tilde{u} \ominus_{gH} \tilde{v} \in E^1$ are

$$\text{Case (i)} \quad \begin{cases} \underline{w}_r = \underline{u}_r - \underline{v}_r \text{ and } \overline{w}_r = \overline{u}_r - \overline{v}_r, \forall_r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.2}$$

$$\text{Case (ii)} \quad \begin{cases} \underline{w}_r = \overline{u}_r - \overline{v}_r \text{ and } \overline{w}_r = \underline{u}_r - \underline{v}_r, \forall_r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.3}$$

It is easy to show that (i) and (ii) are both valid if and only if $\tilde{w}$ is a crisp number. In the case, it is possible that the gH-difference of two fuzzy numbers does not exist. To address this shortcoming, a new difference between fuzzy numbers was introduced in [33].

Lemma 2.1.[10, 24] A fuzzy number $\tilde{u}$ in parametric form is a pair $[\underline{u}_r, \overline{u}_r]$ of function $\underline{u}_r$ and $\overline{u}_r$ for any $r \in [0,1]$, which satisfies the following requirements.

- $\underline{u}_r$ is a bounded non-decreasing left continuous function in (0,1];
- $\overline{u}_r$ is a bounded non-increasing left continuous function in (0,1];
- $\underline{u}_r \leq \overline{u}_r$.

Some the author of the classified fuzzy numbers into several types of fuzzy membership function. To the deepest of our study, triangular fuzzy membership function or also often referred to as triangular fuzzy numbers are the most widely used membership function.

In order to describe the fuzzy numbers and real numbers clearly, in convenience, the fuzzy numbers and fuzzy-valued functions in the whole paper are added with a tilde sign at the top, while the real-value function and interval-value functions are written directly.

A fuzzy valued function $\tilde{f}$ of two variables is a rule that assigns to each ordered pair of real numbers, (x, t), in a set D, a unique fuzzy numbers denoted by $\tilde{f}(x, t)$. The set D is the domain of $\tilde{f}$ and its range is the set of values taken by f, i.e., $\{\tilde{f}(x, t) | (x, t) \in D\}$.

The parametric representation of the fuzzy valued function $f : D \to E^1$ is expressed by $f(x, t)(r) = [\underline{f}(x, t)(r), \overline{f}(x, t)(r)]$, for all $(x, t) \in D$ and $r \in [0, 1]$.

Suppose $f : D \to E^1$ be a fuzzy valued function of two variable. Then, we say that the fuzzy limit of $f(x, t)$ as (x, t) approaches to (a, b) is $L \in E^1$, and we write $\lim_{(x,t)\to(a,b)} f(x, t) = L$ if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x,t) \in D, \| (x,t) - (a,b) \| < \delta \Rightarrow D(f(x,t), L) < \varepsilon$, where $\| \cdot \|$ denotes the Euclidean norm in R^n (ref. to [3])

A fuzzy valued function $f : D \to E^1$ is said to be fuzzy continuous at $(x_0, t_0) \in D$ if $\lim_{(x,t)\to(x_0,t_0)} f(x,t) = f(x_0, t_0)$. We say that f is fuzzy continuous on D if f is fuzzy continuous at every point (x_0, t_0) in D (ref. to [3, 30]).

Definition 2.3.[11] Suppose that $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We say that $\tilde{u}$ is strongly generalized differentiable on (x_0, t) if there exists an element $\frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)} \in E^1$ such that

i. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h\to 0+} = \frac{(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

ii. for all $h > 0$ sufficiently small, $\exists_{gH} \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

iii. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

iv. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)}.$$

Definition 2.4.[4] Suppose that $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We define the n th-order derivative of $\tilde{u}$ as follows: we say that $\tilde{u}$ is strongly generalized differentiable of the n th-order at (x_0, t) if there exists an element $\frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)} \in E^1$, $\forall s = 1, 2, \cdots, n$ such that

i. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t), \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

ii. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t), \tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

iii. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t), \tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

iv. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t), \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)}.$$

2.1 Fuzzy Coputo's derivative

We denote $C^F[a,b]$ as a space of all fuzzy valued functions which are continuous on $[a,b]$, and the space of all Kaleva integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $K^F[a,b]$, we denote the space of fuzzy value functions $\tilde{f}(x)$ which have continuous H-derivative up to order $n-1$ on $[a,b]$ such that $\tilde{f}^{(n-1)}(x) \in AC^F([a,b])$ by $AC^{(n)F}([a,b])$, where $AC^F([a,b])$ denote the set of all fuzzy-valued functions which are absolutely continuous (ref. to [13, 9]).

Definition 2.5.[2] Suppose $\tilde{f}(x) \in C^F[a,b] \cap K^F[a,b]$, the fuzzy Riemann Liouville integral of fuzzy valued function $\tilde{f}$ is defined as following:

$$(I^{\alpha}_{a+}\tilde{f})(x,r) = [(I^{\alpha}_{a+}\underline{f})(x,r), (I^{\alpha}_{a+}\overline{f})(x,r)],$$

where $0 \le r \le 1$

$$(I^{\alpha}_{a+}\underline{f})(x,r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\underline{f}(t)(r)\mathrm{d}t}{(x-t)^{1-\alpha}},\ 0 \le r \le 1,$$

$$(I^{\alpha}_{a+}\overline{f})(x,r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\overline{f}(t)(r)\mathrm{d}t}{(x-t)^{1-\alpha}},\ 0 \le r \le 1.$$

Suppose $\tilde{f}(x) \in C^F((0,a]) \cap K^F(0,a)$, be a given function such that $\tilde{f}(t,r) = [\underline{f}(t,r), \overline{f}(t,r)]$ for all $t \in (0,a]$ and $0 \le r \le 1$. We define $D^{\alpha}_{*a}\tilde{f}(t;r)$ the fuzzy fractional Riemann-Liouville derivative of order $0 < \alpha < 1$ of $\tilde{f}$ in the parametric from,

$$D^{\alpha}_{*a}\tilde{f}(t;r) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{d}{dt} \int_0^t (t-s)^{-\alpha} \underline{f}(s,r)ds, \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \overline{f}(s,r)ds \right],$$

provided that equation defines a fuzzy number $D^{\alpha}_{*a}\tilde{f}(t) \in E^1$. In fact,

$$D^{\alpha}_{*a}\tilde{f}(t,r) = [D^{\alpha}_{*a}\underline{f}(t,r), D^{\alpha}_{*a}\overline{f}(t,r)].$$

Obviously, $D^{\alpha}_{*a}\tilde{f}(t) = \frac{d}{dt} I^{1-\alpha}\tilde{f}(t)\ for\ t \in (0,a]$.

3 Generalized Taylor's formula

In this section, we present the generalized Taylor's formula that involves Caputo fractional derivative.

Theorem 3.1.[21] Let that $(D^{\alpha}_{*a})^j f(x) \in C(a,b]$ for $j = 0, 1, \cdots, n+1$, where $0 < \alpha \le 1$, that we get

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} ((D^{\alpha}_{*a})^i f)(a+) + \frac{((D^{\alpha}_{*a})^{n+1} f)(\zeta)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha}, \tag{3.4}$$

with $a \leq \zeta \leq x$, $\forall x \in (a, b]$ and D^{α}_{*a} is the Caputo fractional derivative of order α, where $(D^{\alpha}_{*a})^j = D^{\alpha}_{*a} D^{\alpha}_{*a} \cdots D^{\alpha}_{*a}$. In case of $\alpha = 1$, the generalized Taylor's formula (3.4) reduces to the classical Taylor's formula.

Theorem 3.2.[17] Let that $(D^{\alpha}_{*a})^j f(x) \in C(a, b]$ for $j = 0, 1, \cdots\cdots, N + 1$, where $0 < \alpha \leq 1$. If $x \in [a, b]$, then

$$f(x) \simeq \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} ((D^{\alpha}_{*a})^i f)(a+). \tag{3.5}$$

Furthermore, there is a value ζ with $a \leq \zeta \leq x$ so that the error term $R^{\alpha}_N(x)$ has the from

$$R^{\alpha}_N(x) = \frac{((D^{\alpha}_{*a})^{N+1} f)(\zeta)}{\Gamma((N+1)\alpha+1)} (x-a)^{(N+1)\alpha}. \tag{3.6}$$

The accuracy of $R^{\alpha}_N(x)$ increases when we choose large N and decreases as value of x moves away from the center a. Hence, we must choose N large enough so that the error does not exceed a specified bound. In the following theorem, we find precise condition under which the exponents hold for arbitrary fractional operators.

Theorem 3.3.[18] Let that $f(x) = x^{\lambda^*} g(x)$, where $\lambda^* > -1$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$ with radius of convergence $R > 0, where\ \ 0 < \alpha \leq 1$. Then

$$D^{\gamma}_{*a} D^{\beta}_{*a} f(x) = D^{\gamma+\beta}_{*a} f(x) \tag{3.7}$$

for all $x \in (0, R)$ if one of the following conditions is satisfied:

1. $\beta < \lambda^* + 1$, and γ arbitrary,
2. $\beta \geq \lambda^* + 1, \gamma$ arbitrary,, and $a_j = 0$ for $j = 0, 1, \cdots\cdots, m-1$, where $m - 1 < \beta \leq m$.

4 Differential transform method and fuzzy fractional wave equation

4.1 Generalized two-dimensional differential transform method

In this section, we will derive the generalized two-dimensional differential transform method (DTM) that we get developed for the solution of the wave equation with space and time-fractional derivatives. The proposed method is based on Taylor's formula. Consider a function of two variables $u(x, t)$, and Let that it can be represented as a product of two single variable functions, $u(x, t) = f(x)g(t)$. Based on the properties of generalized two dimensional differential transform method, function $u(x, t)$ can be represented as.

$$u(x,t) = \sum_{j=0}^{\infty} F_{\alpha}(j) \cdot (x-x_0)^{j\alpha} \sum_{h=0}^{\infty} G_{\beta}(h) \cdot (t-t_0)^{h\beta} = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h)(x-x_0)^{j\alpha}(t-t_0)^{h\beta}, \tag{4.8}$$

where $0 < \alpha, \beta \leq 1, U_{\alpha,\beta}(j, h) = F_{\alpha}(j)G_{\beta}(h)$ is called the spectrum of $u(x, t)$. If function $u(x, t)$ is analytical and differentiated continuously with respect to time t^* in the domain of interest, then we define the generalized two-dimensional differential transform method (DTM) of the function $u(x, t)$ as follows:

$$U_{\alpha,\beta}(j,h) = \frac{1}{\Gamma(\alpha j+1)\Gamma(\beta h+1)} [(D^{\alpha}_{x_0})^j (D^{\beta}_{t_0})^h u(x,t)]_{(x_0,t_0)}, \tag{4.9}$$

where $(D^{\alpha}_{x_0})^j = D^{\alpha}_{x_0} \cdot D^{\alpha}_{x_0} \cdots\cdot D^{\alpha}_{x_0}$. In this work, the lowercase $u(x, t)$ represents the original function while the uppercase $U_{\alpha,\beta}(j, h)$ stands for the transformed function. The generalized differential transform method (DTM) inverse of $U_{\alpha,\beta}(j, h)$ is defined as follows

$$u(x,t) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h) \cdot (x-x_0)^{j\alpha}(t-t_0)^{h\beta} \tag{4.10}$$

In case of $\alpha = 1$ and $\beta = 1$. then generalized two-dimensional differential transform (DTM) (4.9) reduces to the classical two-dimensional DTM [5]. From equation (4.9) and (4.10), some basic properties of the generalized two-dimensional differential transform (DTM) are introduced below (ref. to [17]).

Theorem 4.1 If $u(x,t) = v(x,t) \pm w(x,t),$ *then* $U_{\alpha,\beta}(j,h) = V_{\alpha,\beta}(j,h) \pm W_{\alpha,\beta}(j,h).$

Theorem 4.2 If $u(x,t) = cv(x,t),$ *then* $U_{\alpha,\beta}(j,h) = cV_{\alpha,\beta}(j,h).$

Theorem 4.3 If $u(x,t) = v(x,t)w(x,t)$, then

$$U_{\alpha,\beta}(j,h) = \sum_{r=0}^{j}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(j-r,s). \tag{4.11}$$

Theorem 4.4 If $u(x,t) = D_{x_0}^{\alpha} v(x,t)$ and $0 < \alpha \leq 1$, then we get

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)}{\Gamma(\alpha j+1)} V_{\alpha,\beta}(j+1,h). \tag{4.12}$$

Theorem 4.5 If $u(x,t) = D_{x_0}^{\alpha} D_{t_0}^{\beta} v(x,t)$ *and* $0 < \alpha, \beta \leq 1$, then we get

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)\Gamma(\beta(h+1)+1)}{\Gamma(\alpha j+1)\Gamma(\beta h+1)} V_{\alpha,\beta}(j+1,h+1). \tag{4.13}$$

Theorem 4.6 If $u(x,t) = (x-x_0)^{n\alpha}(t-t_0)^{m\alpha},$ then $U_{\alpha,\beta}(j,h) = \delta(j-n)(h-m).$

Theorem 4.7 If $u(x,t) = D_{x_0}^{\gamma} v(x,t), m-1 < \gamma \leq m$ *and* $v(x,t) = f(x)g(t)$, where $f(x)$ satisfies the conditions in Theorem 3.3, then

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)} U_{\alpha,\beta}(j+\gamma/\alpha,h). \tag{4.14}$$

Theorem 4.8 If $u(x,t) = D_{x_0}^{\gamma} D_{t_0}^{\eta} v(x,t)$, where $m-1 < \gamma \leq m, n-1 < \eta \leq n$ and $v(x,t) = f(x)g(t)$, where the functions $f(x)$ and $g(x)$ satisfy the conditions given in Theorem 3.3, then

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)} \frac{\Gamma(\beta h+\eta+1)}{\Gamma(\beta h+1)} U_{\alpha,\beta}(j+\gamma/\alpha,h+\eta/\beta). \tag{4.15}$$

4.2 Fuzzy fractional wave equation

Consider the fuzzy fractional wave equation with the indicated initial conditions and boundary conditions.

$$\frac{\partial^{\alpha}\tilde{u}}{\partial t^{\alpha}} = c^2 \odot \frac{\partial^2 \tilde{u}}{\partial x^2}, \qquad 0 < \alpha \leq 2, \qquad 0 < x < L, \qquad t > 0, \tag{4.16}$$

subject to the boundary conditions

$$\tilde{u}(0,t) = 0, \quad and \quad \tilde{u}(L,t) = 0, \tag{4.17}$$

and initial conditions.

$$\tilde{u}(x,0) = \tilde{f}(x), \quad and \quad \tilde{u}_t(x,0) = \tilde{g}(x). \tag{4.18}$$

We note that the case (i) of Definition 2.3 is coincident with the Hukuhara derivative [14]. We say that a function is (i) differentiable if it is differentiable as in (i) of Definition 2.3, a function is (ii)

differentiable if it is differentiable as in (ii) of Definition 2.3. In this paper we consider the two cases (i) and (ii). In Ref. [4] the authors consider four cases: the case (i) in [14] is coincident with (i); the case (iii) of Definition 2.1 is equivalent to (ii); in the other cases, the derivative is trivial because it is reduced to crisp element. For details see Theorem 7 in [4]. Thus, we only consider the cases (i) and (ii).

Lemma 4.2. [7]. Let $\tilde{u}(x,t) : D \to E^1$. Then the following statements hold.
(i) If $\tilde{u}(x,t)$ is (i)-partial differentiable for x (i.e. $\tilde{u}$ is partial differentiable for x under the meaning of Definition 2.1 (i), similarly to t), then

$$\left[\frac{\partial \tilde{u}}{\partial x}\right]_r = \left[\frac{\partial \underline{u}(x,t)(r)}{\partial x}, \frac{\partial \bar{u}(x,t)(r)}{\partial x}\right]; \tag{4.19}$$

(ii) If $\tilde{u}(x,t)$ is (ii)-partial differentiable for x (i.e. $\tilde{u}$ is partial differentiable for x under the meaning of Definition 2.1 (ii), similarly to t), then

$$\left[\frac{\partial \tilde{u}}{\partial x}\right]_r = \left[\frac{\partial \bar{u}(x,t)(r)}{\partial x}, \frac{\partial \underline{u}(x,t)(r)}{\partial x}\right]. \tag{4.20}$$

Remark 4.1. For $\tilde{u}(x,t) : D \to E^1$, the following results hold.

$$\left[\frac{\partial^2 \tilde{u}}{\partial x^2}\right]_r = \left[\frac{\partial^2 \underline{u}(x,t)(r)}{\partial x^2}, \frac{\partial^2 \bar{u}(x,t)(r)}{\partial x^2}\right], \tag{4.21}$$

in cases for that $(i,i),(ii,ii)$-$\frac{\partial^2 \tilde{u}}{\partial x^2}$ exist;

$$\left[\frac{\partial^2 \tilde{u}}{\partial x^2}\right]_r = \left[\frac{\partial^2 \bar{u}(x,t)(r)}{\partial x^2}, \frac{\partial^2 \underline{u}(x,t)(r)}{\partial x^2}\right]. \tag{4.22}$$

in cases for that $(i,ii),(ii,i)$-$\frac{\partial^2 \tilde{u}}{\partial t^2}$ exist.

Remark 4.2. In this paper, we only consider that the cases of $(i-ii)^n$-$\frac{\partial^n \tilde{u}}{\partial t^n}$ such that

$$\left[\frac{\partial^n \tilde{u}}{\partial x^n}\right]_r = \left[\frac{\partial^n \underline{u}(x,t)(r)}{\partial x^n}, \frac{\partial^n \bar{u}(x,t)(r)}{\partial x^n}\right], \tag{4.23}$$

where $(i-ii)^n$-$\frac{\partial^n \tilde{u}}{\partial t^n}$ stands for n time derivative in the cases (i) or (ii).

5 Examples

Example 5.1. Consider the following fuzzy fractional wave equation

(A)

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = 4 \odot \frac{\partial^2 \tilde{u}}{\partial x^2} \qquad 0 \le x \le 1, \quad 0 < t, \tag{5.24}$$

subject to the boundary conditions

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \qquad 0 < t, \tag{5.25}$$

and initial conditions

$$\begin{aligned} \tilde{u}(x,0) &= \tilde{f}(x) = \tilde{k}^n \odot \sin(\pi x), \quad && 0 \le x \le 1, \\ \frac{\partial \tilde{u}(x,0)}{\partial t} &= \tilde{g}(x) = 0, && 0 \le x \le 1. \end{aligned} \tag{5.26}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} s, & s \in [0,1], \\ 2-s & s \in (1,2], \\ 0 & s \notin [0,2], \end{cases} \tag{5.27}$$

and $[\underline{\tilde{k}^n}](r) = r^n, [\overline{\tilde{k}^n}](r) = (2-r)^n$.
The parametric form of (5.24) is

$$\frac{\partial^2 \underline{u}}{\partial t^2} = 4\frac{\partial^2 \underline{u}}{\partial x^2} \qquad 0 \le x \le 1, \qquad 0 < t, \tag{5.28}$$

$$\frac{\partial^2 \overline{u}}{\partial t^2} = 4\frac{\partial^2 \overline{u}}{\partial x^2}, \qquad 0 \le x \le 1, \qquad 0 < t, \tag{5.29}$$

for $r \in [0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.
Taking the differential transform of equations (5.28) and (5.29), we get

$$(j+2)(j+1)\underline{U}(i,j+2)(r) = 4(i+2)(i+1)\underline{U}(i+2,j)(r), \tag{5.30}$$

$$(j+2)(j+1)\overline{U}(i,j+2)(r) = 4(i+2)(i+1)\overline{U}(i+2,j)(r). \tag{5.31}$$

From the initial given by equation (5.26), we get

$$\underline{u}(x,0)(r) = \sum_{i=0}^{\infty} \underline{U}(i,0)(r)x^i = \underline{k}(r)\sin(\pi x) = r^n \sum_{i=1,3,\ldots.}^{\infty} \frac{(-1)^{\frac{(i-1)}{2}}}{i!}\pi^i x^i, \tag{5.32}$$

$$\overline{u}(x,0)(r) = \sum_{i=0}^{\infty} \overline{U}(i,0)(r)x^i = \overline{k}(r)\sin(\pi x) = (2-r)^n \sum_{i=1,3,\ldots.}^{\infty} \frac{(-1)^{\frac{(i-1)}{2}}}{i!}\pi^i x^i. \tag{5.33}$$

The corresponding spectra can be obtained as follows,

$$\underline{U}(i,0)(r) = \begin{cases} 0, & \text{for i is even,} \\ \dfrac{(-1)^{\frac{(i-1)}{2}}}{i!} r^n \pi^i, & \text{for i is odd} \end{cases} \tag{5.34}$$

$$\overline{U}(i,0)(r) = \begin{cases} 0, & \text{for i is even,} \\ \dfrac{(-1)^{\frac{(i-1)}{2}}}{i!} (2-r)^n \pi^i, & \text{for i is odd} \end{cases} \tag{5.35}$$

and from equation (5.26) it can be obtained that,

$$\frac{\partial \underline{u}(x,0)(r)}{\partial t} = \sum_{i=0}^{\infty} \underline{U}(i,1)(r)x^i = 0, \tag{5.36}$$

$$\frac{\partial \overline{u}(x,0)(r)}{\partial t} = \sum_{i=0}^{\infty} \overline{U}(i,1)(r)x^i = 0. \tag{5.37}$$

Hence,

$$\underline{u}(i,1)(r) = 0, \tag{5.38}$$

$$\overline{u}(i,1)(r) = 0. \tag{5.39}$$

Substituting equations (5.34) -(5.39) to equations (5.30) and (5.31), all spectra can be found as,

$$\underline{U}(i,j)(r)=\begin{cases}0, & \text{for } i \text{ is even } \textit{or } j \text{ is odd}\\ \dfrac{2^j(-1)^{\frac{(i+j-1)}{2}}}{i!j!}\,r^n\pi^{i+j}, & \text{for } i \text{ is odd } \textit{or } j \text{ is even}\end{cases} \tag{5.40}$$

$$\overline{U}(i,j)(r)=\begin{cases}0, & \text{for } i \text{ is even } \textit{or } j \text{ is odd}\\ \dfrac{2^j(-1)^{\frac{(i+j-1)}{2}}}{i!j!}\,(2-r)^n\pi^{i+j}, & \text{for } i \text{ is odd } \textit{or } j \text{ is even}\end{cases} \tag{5.41}$$

So, the closed from of the solution can be easily written as

$$\begin{aligned}\underline{u}(x,t)(r)&=\underline{k}^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\underline{U}(i,j)(r)x^it^j=r^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=r^n\left[\left(\sum_{i=1,3,\dots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\dots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=r^n\sin(\pi x)\cos(2\pi t),\end{aligned} \tag{5.42}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=\overline{k}^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\overline{U}(i,j)(r)x^it^j=(2-r)^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=(2-r)^n\left[\left(\sum_{i=1,3,\dots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\dots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=(2-r)^n\sin(\pi x)\cos(2\pi t).\end{aligned} \tag{5.43}$$

(B) Consider the following fuzzy fractional wave equation (5.24) with the boundary conditions:

$$\tilde{u}(0,t)=\tilde{u}(1,t)=0, \qquad 0<t, \tag{5.44}$$

and initial conditions

$$\begin{aligned}\tilde{u}(x,0)&=\tilde{f}(x)=\tilde{k}^n\oplus\sin(\pi x), && 0\le x\le 1,\\ \frac{\partial\tilde{u}(x,0)}{\partial t}&=\tilde{g}(x)=0, && 0\le x\le 1.\end{aligned} \tag{5.45}$$

By following the same steps, we will find that the solution. So, the closed from of the solution can be easily written as

$$\begin{aligned}\underline{u}(x,t)(r)&=\underline{k}^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\underline{U}(i,j)(r)x^it^j=r^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=r^n+\left[\left(\sum_{i=1,3,\dots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\dots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=r^n+(\sin(\pi x)\cos(2\pi t)),\end{aligned} \tag{5.46}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=\overline{k}^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\overline{U}(i,j)(r)x^it^j=(2-r)^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=(2-r)^n+\left[\left(\sum_{i=1,3,\dots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\dots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=(2-r)^n+(\sin(\pi x)\cos(2\pi t)).\end{aligned} \tag{5.47}$$

(C) Consider the following fuzzy fractional wave equation (5.24) with the boundary conditions:

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \qquad 0 < t, \tag{5.48}$$

and initial conditions

$$\begin{aligned} \tilde{u}(x,0) &= \tilde{f}(x) = \tilde{k}^n \ominus_{gH} \sin(\pi x), \quad 0 \le x \le 1, \\ \frac{\partial \tilde{u}(x,0)}{\partial t} &= \tilde{g}(x) = 0, \qquad\qquad 0 \le x \le 1. \end{aligned} \tag{5.49}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... , fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} 2(s-0.5), & s \in [0.5, 1], \\ 2(1.5-s), & s \in (1, 1.5], \\ 0 & s \notin [0.5, 1.5], \end{cases} \tag{5.50}$$

and$\{\underline{\tilde{k}^n}\}(r) = (0.5+0.5r)^n$, $\{\overline{\tilde{k}^n}\}(r) = (1.5-0.5r)^n$.
By following the same steps, we will find that the solution. So, the closed from of the solution can be easily written as

$$\begin{aligned} \underline{u}(x,t)(r) &= \underline{k}^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \underline{U}(i,j)(r) x^i t^j = (0.5+0.5r)^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}} \pi^{i+j} x^i t^j \\ &= (0.5+0.5r)^n - \left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= (0.5+0.5r)^n - (\sin(\pi x)\cos(2\pi t)), \end{aligned} \tag{5.51}$$

$$\begin{aligned} \overline{u}(x,t)(r) &= \overline{k}^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overline{U}(i,j)(r) x^i t^j = (1.5-0.5r)^n - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}} \pi^{i+j} x^i t^j \\ &= (1.5-0.5r)^n - \left[\left(\sum_{i=1,3,\ldots}^{\infty} \frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty} \frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right] \\ &= (1.5-0.5r)^n - (\sin(\pi x)\cos(2\pi t)). \end{aligned} \tag{5.52}$$

Example 5.2. Consider the following fuzzy time-fractional wave equation.

(A)

$$\frac{\partial^{1.5}\tilde{u}}{\partial t^{1.5}} = \frac{\partial^2 \tilde{u}}{\partial x^2}, \qquad t > 0, \tag{5.53}$$

subject to the initial conditions

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \odot \sin(x), \quad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \odot (-\sin(x)). \tag{5.54}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,..., fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} 2(s-0.5), & s \in [0.5, 1], \\ 2(1.5-s), & s \in (1, 1.5], \\ 0 & s \notin [0.5, 1.5], \end{cases} \tag{5.55}$$

and$\{\underline{\tilde{k}^n}\}(r) = (0.5+0.5r)^n, \{\overline{\tilde{k}^n}\}(r) = (1.5-0.5r)^n$.
The parametric form of (5.53) is

$$\frac{\partial^{1.5}\underline{u}}{\partial t^{1.5}} = \frac{\partial^2\underline{u}}{\partial x^2}, \qquad t>0, \tag{5.56}$$

$$\frac{\partial^{1.5}\overline{u}}{\partial t^{1.5}} = \frac{\partial^2\overline{u}}{\partial x^2}, \qquad t>0. \tag{5.57}$$

for $r \in [0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.

Let the solution $u(x,t) = f(x)g(t)$ where the function $g(t)$ satisfies the conditions given in Theorem 3.3. Then selecting $\alpha = 0.5, \beta = 1$ and applying the generalized two-dimensional differential transform method (DTM) to both sides of equations (5.56) and (5.57) by Theorem 4.7, equations (5.56) and (5.57) Transforms to

$$\underline{U}_{0.5,1}(j,h+3)(r) = \frac{(j+1)(j+2)\Gamma(\frac{h}{2}+1)}{\Gamma(\frac{h}{2}+\frac{5}{2})}\underline{U}_{0.5,1}(j+2,h)(r), \tag{5.58}$$

$$\overline{U}_{0.5,1}(j,h+3)(r) = \frac{(j+1)(j+2)\Gamma(\frac{h}{2}+1)}{\Gamma(\frac{h}{2}+\frac{5}{2})}\overline{U}_{0.5,1}(j+2,h)(r). \tag{5.59}$$

The generalized two-dimensional differential transform of the initial conditions (5.54) are given by

$$\underline{U}_{0.5,1}(j,0)(r) = (0.5+0.5r)^n\frac{1}{j!}\sin(\frac{\pi j}{2}), \tag{5.60}$$

$$\underline{U}_{0.5,1}(j,1)(r) = 0, \tag{5.61}$$

$$\underline{U}_{0.5,1}(j,2)(r) = (0.5+0.5r)^n\frac{-1}{j!}\sin(\frac{\pi j}{2}), \tag{5.62}$$

$$\overline{U}_{0.5,1}(j,0)(r) = (1.5-0.5r)^n\frac{1}{j!}\sin(\frac{\pi j}{2}), \tag{5.63}$$

$$\overline{U}_{0.5,1}(j,1)(r) = 0, \tag{5.64}$$

$$\overline{U}_{0.5,1}(j,2)(r) = (1.5-0.5r)^n\frac{-1}{j!}\sin(\frac{\pi j}{2}). \tag{5.65}$$

Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.
So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)x \\
&+ (0.5+0.5r)^n\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)x^3 \\
&+ (0.5+0.5r)^n\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)x^5 \\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (0.5+0.5r)^n\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right),
\end{aligned} \tag{5.66}$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)\cdot x\\
&+(1.5-0.5r)^n\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)\cdot x^3\\
&+(1.5-0.5r)^n\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)\cdot x^5\\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (1.5-0.5r)^n\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.67)
\end{aligned}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z)=\tilde{k}^n\odot\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.68)$$

(B) Consider the following fuzzy time-fractional wave equation (5.53) with the initial conditions:

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\oplus\sin(x),\quad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\oplus(-\sin(x)). \qquad (5.69)$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.
So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n+\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)x\\
&+(0.5+0.5r)^n+\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)x^3\\
&+(0.5+0.5r)^n+\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)x^5\\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n+\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (0.5+0.5r)^n+\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right), \qquad (5.70)
\end{aligned}$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n + \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)\cdot x \\
&+ (1.5-0.5r)^n + \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)\cdot x^3 \\
&+ (1.5-0.5r)^n + \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)\cdot x^5 \\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n + \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (1.5-0.5r)^n + \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.71)
\end{aligned}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z) = \tilde{k}^n \oplus \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.72)$$

(C) Consider the following fuzzy time fractional wave equation (5.53) with initial conditions:

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \ominus_{gH} \sin(x), \quad \frac{\partial\tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \ominus_{gH} (-\sin(x)). \qquad (5.73)$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.
So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n - \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right) x \\
&+ (0.5+0.5r)^n - \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right) x^3 \\
&+ (0.5+0.5r)^n - \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right) x^5 \\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n - \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (0.5+0.5r)^n - \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right), \qquad (5.74)
\end{aligned}$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n - \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)\cdot x \\
&+ (1.5-0.5r)^n - \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)\cdot x^3 \\
&+ (1.5-0.5r)^n - \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)\cdot x^5 \\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n - \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (1.5-0.5r)^n - \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.75)
\end{aligned}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z) = \tilde{k}^n \ominus_{gH} \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.76)$$

Example 5.3. Consider the following fuzzy linear space time fractional wave equation

(A)

$$\frac{\partial^{1.5}\tilde{u}}{\partial t^{1.5}} = \frac{1}{2}x^2 \odot \frac{\partial^{1.25}\tilde{u}}{\partial x^{1.25}} \quad x>0, \quad t>0, \qquad (5.77)$$

subject to the initial conditions

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \odot \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \odot \sum_{n=0}^{\infty} b_n x^n. \qquad (5.78)$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} s, & s\in[0,1], \\ 2-s & s\in(1,2], \\ 0 & s\notin[0,2], \end{cases} \qquad (5.79)$$

and $[\underline{\tilde{k}^n}](r) = r^n, [\overline{\tilde{k}^n}](r) = (2-r)^n$.
The parametric form of (5.77) is

$$\frac{\partial^{1.5}\underline{u}}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}\underline{u}}{\partial x^{1.25}} \quad x>0, \quad t>0 \qquad (5.80)$$

$$\frac{\partial^{1.5}\overline{u}}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}\overline{u}}{\partial x^{1.25}} \quad x>0, \quad t>0 \qquad (5.81)$$

for $r\in[0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.

Let the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = f(x)g(t)$ where the functions $f(x)$ and $g(t)$ satisfy the conditions given in Theorem 3.3. Selecting $\alpha = 0.5, \beta = 0.25$ and applying the generalized two-dimensional differential transform to both sides of equations (5.80) and (5.81), the fuzzy linear space-time fractional wave equations (5.80) and (5.81) transform to

$$\underline{U}_{1/2,1/4}(j,h+3)(r)=\begin{cases}\frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(j/4+7/4)}{\Gamma(h/2+5/2)\Gamma(j/4+2/4)}\underline{U}_{1/2,1/4}(j+3,h)(r), & j\geq 2\\ 0, & j<2.\end{cases} \tag{5.82}$$

$$\overline{U}_{1/2,1/4}(j,h+3)(r)=\begin{cases}\frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(j/4+7/4)}{\Gamma(h/2+5/2)\Gamma(j/4+2/4)}\overline{U}_{1/2,1/4}(j+3,h)(r), & j\geq 2\\ 0, & j<2.\end{cases} \tag{5.83}$$

The generalized two-dimensional transforms of the initial conditions (5.78) are given by

$$\underline{U}_{1/2,1/4}(j,0)(r)=r^n a_j, \tag{5.84}$$
$$\underline{U}_{1/2,1/4}(j,1)(r)=0, \tag{5.85}$$
$$\underline{U}_{1/2,1/4}(j,2)(r)=r^n b_j, \tag{5.86}$$
$$\overline{U}_{1/2,1/4}(j,0)(r)=(2-r)^n a_j, \tag{5.87}$$
$$\overline{U}_{1/2,1/4}(j,1)(r)=0, \tag{5.88}$$
$$\overline{U}_{1/2,1/4}(j,2)(r)=(2-r)^n b_j. \tag{5.89}$$

Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}\underline{u}(x,t)(r)=&\,r^n\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+r^n\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+r^n\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+r^n\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+r^n\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots,\end{aligned} \tag{5.90}$$

$$\begin{aligned}\overline{u}(x,t)(r)=&\,(2-r)^n\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+(2-r)^n\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+(2-r)^n\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+(2-r)^n\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+(2-r)^n\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots.\end{aligned} \tag{5.91}$$

(B) Consider the following fuzzy linear-space-time-fractional wave equation (5.77) with the initial conditions:

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\oplus\sum_{n=0}^{\infty}a_nx^n,\qquad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\oplus\sum_{n=0}^{\infty}b_nx^n. \tag{5.92}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}\underline{u}(x,t)(r)=r^n&+\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+r^n+\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+r^n+\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+r^n+\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+r^n+\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots,\end{aligned} \tag{5.93}$$

$$\begin{aligned}\overline{u}(x,t)(r)=(2-r)^n&+\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+(2-r)^n+\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+(2-r)^n+\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+(2-r)^n+\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+(2-r)^n+\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots.\end{aligned} \tag{5.94}$$

(C) Consider the following fuzzy linear space-time-fractional wave equation (5.77) with the initial conditions:

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\ominus_{gH}\sum_{n=0}^{\infty}a_nx^n,\qquad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\ominus_{gH}\sum_{n=0}^{\infty}b_nx^n. \tag{5.95}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}\underline{u}(x,t)(r) = r^n &- \left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right) \\ + r^n &- \left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4} \\ + r^n &- \left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4} \\ + r^n &- \left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4} \\ + r^n &- \left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots, \end{aligned} \tag{5.96}$$

$$\begin{aligned}\overline{u}(x,t)(r) = (2-r)^n &- \left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right) \\ + (2-r)^n &- \left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4} \\ + (2-r)^n &- \left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4} \\ + (2-r)^n &- \left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4} \\ + (2-r)^n &- \left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots. \end{aligned} \tag{5.97}$$

6 Conclusions

In this paper, the differential transform method (DTM) has been successfully applied for solving fuzzy fractional wave equation. The proposed method is also illustrated by three examples. The new method is investigated based on the two-dimensional differential transform method, generalized Taylor's formula and fuzzy Caputo,s derivative. The results reveal that DTM is a highly effective scheme for obtaining analytical solutions of the fuzzy fractional wave equation.

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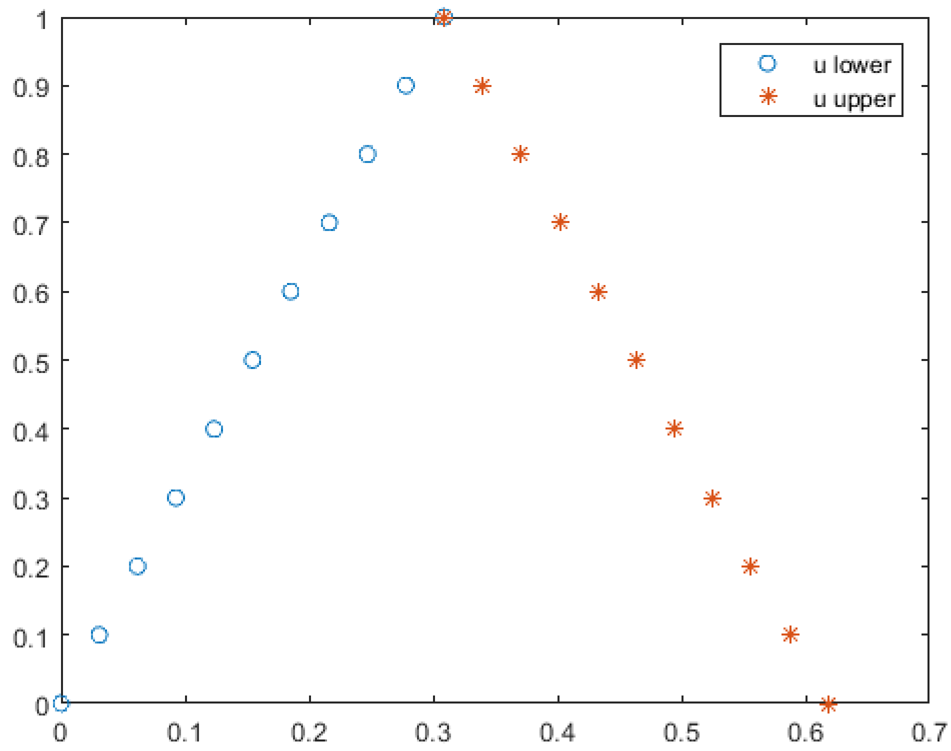

Figure 1: Example (5.1), Case (A), $t = 0.000001, x = 0.1, n = 1$.

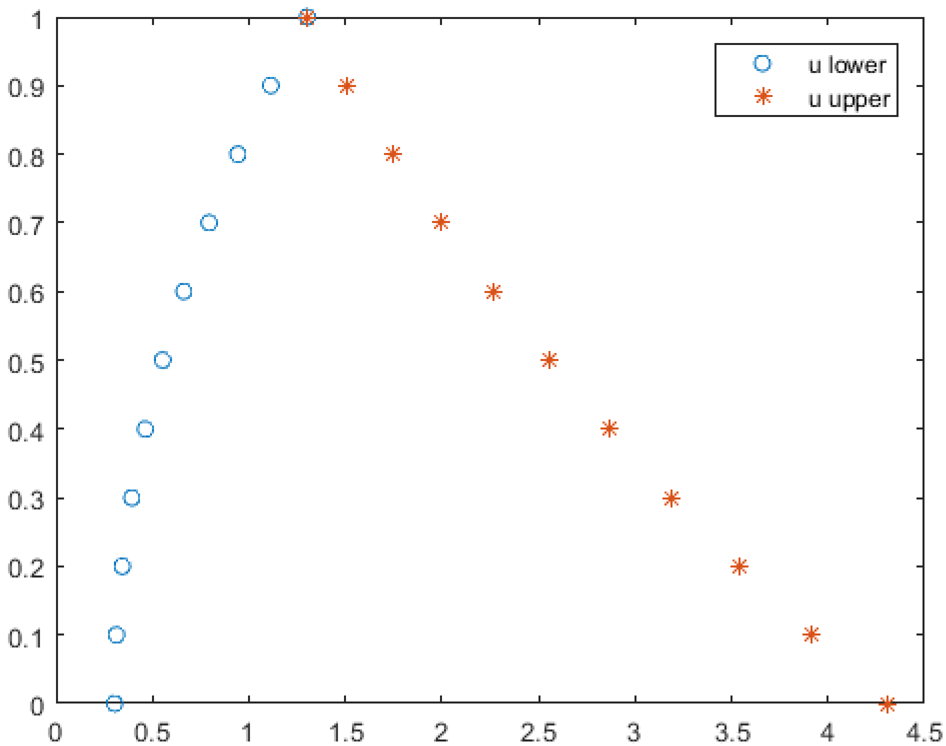

Figure 2: Example (5.1), Case (B), $t = 0.03, x = 0.1, n = 2$.

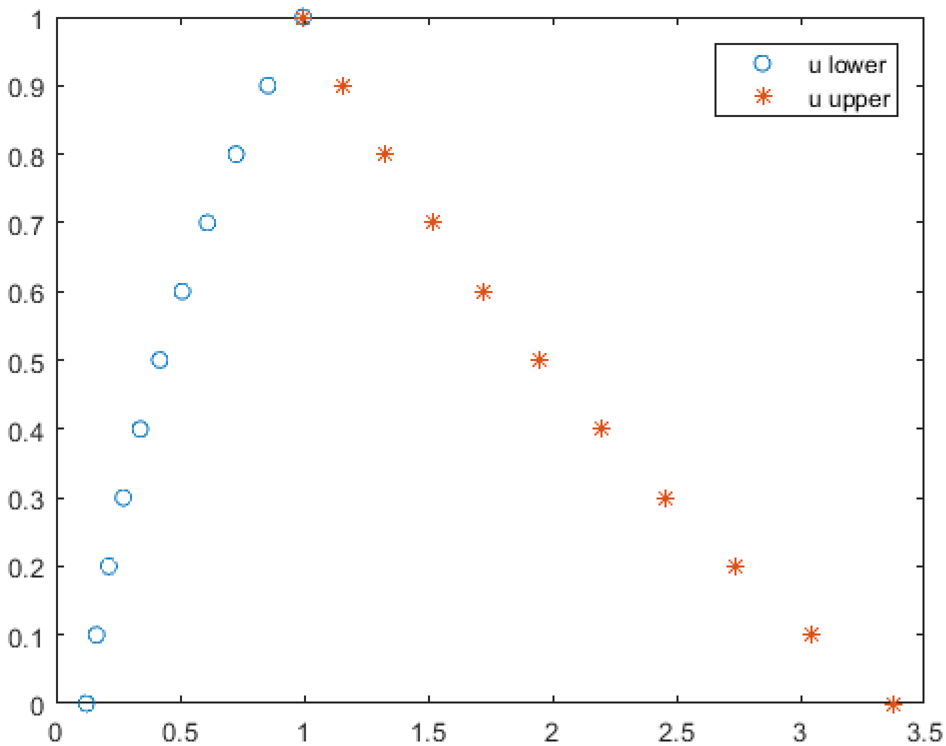

Figure 3: Example (5.1), Case (C), $t = 0.0001, x = 0.001, n = 3$.

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