A Numerical Technique for Solving Fuzzy Fractional Optimal Control Problems†

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Abstract In this paper, the fuzzy fractional optimal control problem with both fixed and free final state conditions has been considered. Our problem is defined in the sense of Riemann-Liouville fractional derivative based on Hukuhara difference, and the dynamic constraint is described by a fractional differential equation of order less than 1. Using fuzzy variational approach, a necessary conditions of our problem has been derived. A numerical technique based on Grünwald-Letnikov definition of fractional derivative and the relation between right Riemann-Liouville fractional derivative and right Caputo fractional derivative is proposed. Finally, some numerical examples are given to illustrate our main results.
Keywords: Fuzzy fractional calculus;Grünwald-Letnikov fractional derivative;Fuzzy fractional optimal control problem;Fixed final state problem;Free final state problem;Fuzzy variational approach;Necessary conditions.

1. Introduction

Optimal control is the standard method for solving dynamic optimization problems, which deal with finding a control law for a given system such that a certain optimality criterion is achieved. It's playing an increasingly important role in modern system design, and considered to be a powerful mathematical tool that can be used to make decisions in real life. On the other hand, accurate modeling of some real problems in scientific fields and engineering, sometimes lead to a set of fractional differential and integral equations. Fractional optimal control problem is an optimal control problem whose dynamic system is described by fractional differential equations. We can define the fractional optimal control problem in sense of different definitions of fractional derivative, for example Riemann-Liouville fractional derivative, Caputo fractional derivative and so on.

Due to, uncertainty in the input, output and manner of many dynamical systems, meanwhile, fuzziness is a way to express an uncertain phenomena in real world. Thus, importing fuzziness in the optimal control theory, give a better display of the problems with control parameters in real world such as physical models and dynamical systems.

In the last decade, fuzzy fractional optimal control problems have attracted a great deal of attention and the interest in the filed of fuzzy fractional optimal control problems has increased. In [1], Fard and Soolaki, prove the necessary optimality conditions of pontryagin type for a class of fuzzy fractional optimal control problems with the fuzzy fractional derivative described in the Caputo sense. In [2], Fard and Salehi studied the constrained and unconstrained fuzzy fractional variational problems containing the Caputo-type fractional derivatives using the approach of the generalized differentiability. In [3], Karimyar and Fakharzadeh introduced the solution of fuzzy fractional optimal control problems by using Mittag-Leffler function.

In this paper, we will study a fixed and free final state fuzzy fractional optimal control problems with the fuzzy fractional derivative described in Riemann-Liouville type in sense of Hukuhara difference.

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Then, we derive the necessary conditions of that problems based on fuzzy variational approach. A numerical algorithm is proposed to solve the necessary conditions to find the optimal fuzzy control and optimal fuzzy state as a solutions of our problems. The definitions of a strong and weak solutions of our problems are given, to guarantee the optimal solutions are a fuzzy functions.

This paper is organized as follows. In Section 2 we introduce and generalize some basic concepts and notations that are key to our discussion. In Section 3 we present basic elements of fuzzy fractional calculus and fuzzy calculus of variations. In Section 4 we establish our main results, Theorem(4.1), that provides the necessary conditions of fuzzy fractional optimal control problems with both fixed and free final state conditions. In Section 5 we propose a numerical technique to solve the necessary conditions. Finally, we discuss the applicability of the main theorem and the numerical algorithm through an examples.

2. Definitions and preliminaries

Here, we start with basic definitions and lemmas needed in the other sections for a better understanding of this work. The details of this concepts are clearly found in [7, 9, 10, 11, 12, 17].

Definition 2.1 A fuzzy set $\tilde{A}: R \to [0,1]$ is called a fuzzy number if $\tilde{A}$ is normal, convex fuzzy set, upper semi-continuous and supp$A = \overline{\{x \in R | \tilde{A}(x) > 0\}}$ is compact, where $\overline{M}$ denotes the closure of M. In the rest of this paper we use E^1 to denote the fuzzy number space.

Where it is $\alpha-$level set $\tilde{a}[\alpha] = \{x \in R : \tilde{a}(x) \geq \alpha\} = [a^l(\alpha), a^r(\alpha)], \forall \alpha \in (0,1]$, and $0-$level set $\tilde{a}[0]$ is defined as $\overline{\{x \in R | \tilde{a}(x) > 0\}}$. Obviously, the α-level set $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ is bounded closed interval in R for all $\alpha \in [0,1]$, where $a^l(\alpha)$ and $a^r(\alpha)$ denote the left-hand and right-hand end points of $\tilde{a}[\alpha]$, respectively. $\tilde{a}$ is a crisp number with value k if its membership function is defined by,

$$\tilde{a}(x) = \begin{cases} 1 & , x = k \\ 0 & , x \neq k \end{cases}$$

Thus,

$$\tilde{0}(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0. \end{cases}$$

Let $\tilde{u}, \tilde{v} \in E^1, k \in R$, we can define the addition and scalar multiplication by using α-level set respectively as

$$(\tilde{a} + \tilde{b})[\alpha] = \tilde{a}[\alpha] + \tilde{b}[\alpha],\ (k\tilde{a})[\alpha] = k\tilde{a}[\alpha],$$

where $\tilde{a}[\alpha] + \tilde{b}[\alpha]$ means the usual addition of two intervals of R, and $k\tilde{a}[\alpha]$ means the usual product between a scalar and interval of R. Furthermore, the opposite of the fuzzy number $\tilde{a}$ is $-\tilde{a}$, i.e., $-\tilde{a}(x) = \tilde{a}(-x)$, it means, $-\tilde{a}[\alpha] = [-a^r(\alpha), -a^l(\alpha)]$.

The binary operation "." in R can be extended to the binary operation "$\odot$" of two fuzzy numbers by using the extension principle. Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers, then

$$(\tilde{a} \odot \tilde{b})(z) = \sup_{x \cdot y = z} \min\{\tilde{a}(x), \tilde{b}(x)\}.$$

Using α-level set the product $(\tilde{a} \odot \tilde{b})$ is defined by

$$\begin{aligned}(\tilde{a} \odot \tilde{b})[\alpha] &= \Big[\min\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}, \\ &\quad \max\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}\Big].\end{aligned}$$

The metric structure is given by the Hausdorff distance $\mathbb{D}: E^1 \times E^1 \times R \to R_+ \cup \{0\}$,

$$\mathbb{D}(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{| a^l(\alpha) - b^l(\alpha) |, | a^r(\alpha) - b^r(\alpha) |\}.$$

A special class of fuzzy numbers is the class of triangular fuzzy numbers. For $a_1 < a_2 < a_3$ and $a_1, a_2, a_3 \in R$, the triangular fuzzy number $\tilde{a}$ is generally denoted by $\tilde{a} = (a_1, a_2, a_3)$ is determined by a_1, a_2, a_3 such that $a^l(\alpha) = a_1 + (a_2 - a_1)\alpha$ and $a^r(\alpha) = a_3 - (a_3 - a_2)\alpha$, when $\alpha = 0$ then $\tilde{a}[0] = [a_1, a_3]$ and when $\alpha = 1$ then $\tilde{a}[1] = [a_2, a_2] = a_2$.

We know that, we can identify a fuzzy number $\tilde{a} \in E^1$ by the left and right hand functions of its $\alpha-$level set, the following lemma introduce the properties of this functions.

Lemma 2.1 Suppose that $a^l : [0,1] \to R$ and $a^r : [0,1] \to R$ satisfy the conditions:

C1: a^l is bounded increasing function,

C2: a^r is bounded decreasing function,

C3: $a^l(1) \leq a^r(1)$,

C4: $\lim_{\alpha \to k^-} a^l(\alpha) = a^l(k)$ and $\lim_{\alpha \to k^-} a^r(\alpha) = a^r(k)$, for all $0 < k \leq 1$,

C5: $\lim_{\alpha \to 0^+} a^l(\alpha) = a^l(0)$ and $\lim_{\alpha \to 0^+} a^r(\alpha) = a^r(0)$.

Then $\tilde{a} : R \to [0,1]$ defined by $\tilde{a}(x) = \sup\{\alpha | a^l(\alpha) \leq x \leq a^r(\alpha)\}$ is a fuzzy number with $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$. Moreover, if $\tilde{a} : R \to [0,1]$ is a fuzzy number with $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$, then the functions $a^l(\alpha)$ and $a^r(\alpha)$ satisfy conditions **C1- C5**.

Definition 2.2 (H-difference). Let $\tilde{a}, \tilde{b} \in E^1$, where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$ for all $\alpha \in [0,1]$, the H-difference is defined by

$$\tilde{a} \ominus \tilde{b} = \tilde{c} \iff \tilde{a} = \tilde{b} + \tilde{c}.$$

Obviously, $\tilde{a} \ominus \tilde{a} = \tilde{0}$, and the α-level set of H-difference is

$$(\tilde{a} \ominus \tilde{b})[\alpha] = [a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha)], \forall \alpha \in [0,1].$$

Definition 2.3 (Partial ordering). Let $\tilde{a}, \tilde{b} \in E^1$, we write $\tilde{a} \preceq \tilde{b}$, if $a^l(\alpha) \leq b^l(\alpha)$ and $a^r(\alpha) \leq b^r(\alpha)$ for all $\alpha \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists $\alpha_0 \in [0,1]$ such that $a^l(\alpha_0) < b^l(\alpha_0)$ or $a^r(\alpha_0) < b^r(\alpha_0)$. Furthermore, $\tilde{a} = \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$. In other words, $\tilde{a} = \tilde{b}$, if $\tilde{a}[\alpha] = \tilde{b}[\alpha]$ for all $\alpha \in [0,1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in E^1$ are *comparable* if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$, and *non-comparable* otherwise.

From now we consider S as a subset of R.

Definition 2.4 (Fuzzy valued function). The function $\tilde{f} : S \to E^1$ is called a fuzzy-valued function if $\tilde{f}(t)$ is assign a fuzzy number for any $e \in S$. We also denote $\tilde{f}(t)[\alpha] = [f^l(t,\alpha), f^r(t,\alpha)]$, where $f^l(t,\alpha) = (\tilde{f}(t))^l(\alpha) = \min\{\tilde{f}(t)[\alpha]\}$ and $f^r(t,\alpha) = (\tilde{f}(t))^r(\alpha) = \max\{\tilde{f}(t)[\alpha]\}$. Therefore any fuzzy-valued function $\tilde{f}$ may be understood by $f^l(t,\alpha)$ and $f^r(t,\alpha)$ being respectively a bounded increasing function of α and a bounded decreasing function of α for $\alpha \in [0,1]$. And also it holds $f^l(t,\alpha) \leq f^r(t,\alpha)$ for any $\alpha \in [0,1]$.

Definition 2.5 (Continuity of a fuzzy valued function). We say that $\tilde{f} : S \to E^1$ is continuous at $t \in S$, if both $f^l(t,\alpha)$ and $f^r(t,\alpha)$ are continuous functions at $t \in S$ for all $\alpha \in [0,1]$.

If $\tilde{f}(t)$ is continuous in the metric $\mathbb{D}$, then its definite integral exists and defined by

$$\int_a^b \tilde{f}(t)[\alpha]dt = \left[\int_a^b f^l(t,\alpha)dt, \int_a^b f^r(t,\alpha)dt\right].$$

Definition 2.6 (Distance measure between fuzzy valued functions). Suppose that $\tilde{f}, \tilde{g} : S \to E^1$ are two fuzzy functions. We define the distance measure between $\tilde{f}$ and $\tilde{g}$ by

$$\begin{aligned} \mathbb{D}_{E^1}(\tilde{f}(x), \tilde{g}(x)) &= \sup_{0 \leq \alpha \leq 1} \mathbb{H}(\tilde{f}(x)[\alpha], \tilde{g}(x)[\alpha]) \\ &= \max\{\sup_{z \in \tilde{f}(x)[\alpha]} d(z, \tilde{g}(x)[\alpha]), \sup_{y \in \tilde{g}(x)[\alpha]} d(\tilde{f}(x)[\alpha], y)\}, \ \ \forall x \in S. \end{aligned}$$

Where $\mathbb{H}$ is the Hausdorff metric on the family of all nonempty compact subsets of R, and

$$d(a, B) = \inf_{b \in B} d(a, b).$$

Moreover, we can define

$$\| \tilde{f}(x) \|_{E^1}^2 = \mathbb{D}_{E^1}(\tilde{f}(x), \tilde{f}(x)), \quad \forall x \in S,$$

for any $\tilde{f} : S \to E^1$.

3. Elements of fuzzy fractional calculus and fuzzy calculus of variations

Several definitions of a fractional derivative have been studied, such as Riemann-Liouville, Grünwald-Letnikov, Caputo and so on. In this paper, we deal with the problems defined by Riemann-Liouville fractional derivative. In this section, we first introduce the definition of fuzzy Riemann-Liouville integrals and derivatives in sense of Hukuhara difference.

Definition 3.1(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a, b] \in R$ and $0 < \beta \leq 1$, then the fuzzy Riemann-Liouville integral of $\tilde{f}(x)$ of order β is defined by

$${}_aI_x^\beta \tilde{f}(x) = \frac{1}{\Gamma(\beta)} \int_a^x \tilde{f}(t)(x-t)^{\beta-1} dt,$$

where $\Gamma(\beta)$ is the Gamma function and $x > a$.

Theorem 3.1(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a, b] \in R$. The fuzzy Riemann-Liouville integral of $\tilde{f}(x)$ can be expressed as follows

$${}_aI_x^\beta \tilde{f}(x)\,[\alpha] = \left[{}_aI_x^\beta f^l(x, \alpha), {}_aI_x^\beta f^r(x, \alpha) \right], \quad 0 \leq \alpha \leq 1,$$

where

$$\begin{aligned} {}_aI_x^\beta f^l(x, \alpha) &= \frac{1}{\Gamma(\beta)} \int_a^x f^l(t, \alpha)(x-t)^{\beta-1} dt, \\ {}_aI_x^\beta f^r(x, \alpha) &= \frac{1}{\Gamma(\beta)} \int_a^x f^r(t, \alpha)(x-t)^{\beta-1} dt. \end{aligned}$$

In the next definition, we define the fuzzy Riemann-Liouville fractional derivative of order $0 < \beta < 1$ of a fuzzy valued function $\tilde{f}(x)$.

Definition 3.2(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a, b] \in R$. $x_0 \in (a, b)$ and then: $G(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{\tilde{f}(t)dt}{(x-t)^\beta}$. We say that $\tilde{f}$ is Riemann-Liouville H-differentiable of order $0 < \beta < 1$ at x_0, if there exist an element ${}_aD_x^\beta \tilde{f}(x_0) \in E^1$ such that for $h > 0$ sufficiently small

(1) ${}_aD_x^\beta \tilde{f}(x_0) = \lim\limits_{h \to 0^+} \frac{G(x_0+h) \ominus G(x_0)}{h} = \lim\limits_{h \to 0^+} \frac{G(x_0) \ominus G(x_0-h)}{h}$,
or

(2) ${}_aD_x^\beta \tilde{f}(x_0) = \lim\limits_{h \to 0^+} \frac{G(x_0) \ominus G(x_0+h)}{-h} = \lim\limits_{h \to 0^+} \frac{G(x_0-h) \ominus G(x_0)}{-h}$,
or

(3) ${}_aD_x^\beta \tilde{f}(x_0) = \lim\limits_{h \to 0^+} \frac{G(x_0+h) \ominus G(x_0)}{h} = \lim\limits_{h \to 0^+} \frac{G(x_0-h) \ominus G(x_0)}{-h}$,
or

(4) ${}_aD_x^\beta \tilde{f}(x_0) = \lim\limits_{h \to 0^+} \frac{G(x_0) \ominus G(x_0+h)}{-h} = \lim\limits_{h \to 0^+} \frac{G(x_0) \ominus G(x_0-h)}{h}$.

For sake of simplicity, we say that the fuzzy valued function $\tilde{f}(x)$ is Riemann-Liouville $[(i)-\beta]-$differentiable if it is differentiable as in the Definition(3.2) case(i), $i=1,2,3,4$ respectively.

Theorem 3.2(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$ and $\tilde{f}(x)[\alpha]=[f^l(x,\alpha),f^r(x,\alpha)]$,then for $\alpha\in[0,1]$, $x\in(a,b)$ and $\beta\in(0,1)$

(i) Let us consider $\tilde{f}$ is Riemann-Liouville $[(1)-\beta]-$differentiable fuzzy-valued function, then:

$$ {}_aD_x^\beta\tilde{f}(x_0)[\alpha]=\left[{}_aD_x^\beta f^l(x_0,\alpha),{}_aD_x^\beta f^r(x_0,\alpha)\right]. $$

(ii) Let us consider $\tilde{f}$ is Riemann-Liouville $[(2)-\beta]-$differentiable fuzzy-valued function, then:

$$ {}_aD_x^\beta\tilde{f}(x_0)[\alpha]=\left[{}_aD_x^\beta f^r(x_0,\alpha),{}_aD_x^\beta f^l(x_0,\alpha)\right]. $$

Where

$$ \begin{aligned} {}_aD_x^\beta f^l(x_0,\alpha) &= \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x\frac{f^l(t,\alpha)dt}{(x-t)^\beta}\right]\Bigg|_{x=x_0}, \\ {}_aD_x^\beta f^r(x_0,\alpha) &= \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x\frac{f^r(t,\alpha)dt}{(x-t)^\beta}\right]\Bigg|_{x=x_0}. \end{aligned} $$

Theorem 3.3(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]$ is a Riemann-Liouville H-differentiable of order $0<\beta<1$ on each point $x\in(a,b)$ in the sense of Definition(3.2) case(3) or case(4), then ${}_aD_x^\beta\tilde{f}(x)\in R$ for all $x\in(a,b)$.

Now we state some elements of fuzzy calculus of variations.

Definition 3.3(Fuzzy increment[10]). Suppose that $\tilde{x}(.)$ and $\tilde{x}(.)+\delta\tilde{x}(.)$ are fuzzy functions for which the fuzzy functional $\tilde{J}$ is defined. The increment of $\tilde{J}$, denoted by $\Delta\tilde{J}$, is

$$ \Delta\tilde{J}:=\tilde{J}(\tilde{x}+\delta\tilde{x})\ominus\tilde{J}(x), \tag{3.1} $$

Where $\delta\tilde{x}(.)$ is the variation of $\tilde{x}(.)$.

Because the increment $\Delta\tilde{J}$ depends on the fuzzy functions $\tilde{x}$ and $\delta\tilde{x}$, we denote $\Delta\tilde{J}$ by $\Delta\tilde{J}(\tilde{x},\delta\tilde{x})$.

Definition 3.4(Differentiability of a fuzzy functional[10, 15]). Suppose that $\Delta\tilde{J}$ can be written as

$$ \Delta\tilde{J}(\tilde{x},\delta\tilde{x}):=\delta\tilde{J}(\tilde{x},\delta\tilde{x})+\tilde{j}(\tilde{x},\delta\tilde{x})\cdot\parallel\delta\tilde{x}\parallel_{E^1}, \tag{3.2} $$

Where $\delta\tilde{J}$ is linear in $\delta\tilde{x}$. We say that $\tilde{J}$ is differentiable with respect to $\tilde{x}$ if for any $\epsilon>0$,

$$ D_{E^1}(\tilde{j}(\tilde{x},\delta\tilde{x}),0)<\epsilon,\ \ as\parallel\delta\tilde{x}(.)\parallel_{E^1}\to 0. $$

From now $\tilde{C}[t_0,t_1]$ represent the class of all fuzzy continuous functions on $[t_0,t_1]$.

Definition 3.5(Fuzzy relative minimum[10]) A fuzzy functional $\tilde{J}$ with domain $\tilde{C}[t_0,t_1]$, has a fuzzy relative minimizer $\tilde{x}^*=\tilde{x}^*(t)$, if

$$ \tilde{J}(\tilde{x})\succeq\tilde{J}(\tilde{x}^*), \tag{3.3} $$

for all fuzzy functions $\tilde{x}\in\tilde{C}[t_0,t_1]$.

It is clear that the inequality (3.3) holds iff

$$ J^l(\tilde{x},\alpha)\geq J^l(\tilde{x}^*,\alpha),\ \text{ and }\ J^r(\tilde{x},\alpha)\geq J^r(\tilde{x}^*,\alpha), \tag{3.4} $$

for all $\alpha\in[0,1]$ and all $\tilde{x}\in\tilde{C}[t_0,t_1]$.

The following theorem is the fundamental theorem of the calculus of variations in fuzzy environment.

Theorem 3.4 Let $\tilde{x},\delta\tilde{x}\in\tilde{C}[t_0,t_1]$ be two fuzzy functions of $t\in[t_0,t_1]$, and $\tilde{J}(\tilde{x})$ differentiable fuzzy functional of $\tilde{x}$. If $\tilde{x}^*$ is a fuzzy minimizer of $\tilde{J}$, then the variation of $\tilde{J}$ regardless of any boundary conditions must vanish on $\tilde{x}^*$, that is,

$$ \delta\tilde{J}(\tilde{x}^*,\delta\tilde{x})=0, \tag{3.5} $$

for all admissible $\delta\tilde{x}$ having the property $\tilde{x} + \delta\tilde{x} \in \tilde{C}[t_0, t_1]$.

It is obviously that the equality (3.5) holds if and only if

$$\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) = 0, \tag{3.6}$$

$$\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) = 0, \tag{3.7}$$

for all $\alpha \in [0,1], t \in [t_0, t_1]$ and all admissible $\delta\tilde{x}$ where,

$$\delta\tilde{x}(t)[\alpha] = [\delta x^l(t,\alpha), \delta x^r(t,\alpha)].$$

Proof. See [10]

4. Fuzzy fractional optimal control problem

In this section, we first define fuzzy fractional optimal control problem with fixed and free final state conditions, and then we derive necessary conditions for optimality by applying fuzzy variational approaches to our problem.

We define fuzzy fractional optimal control problem as:

$$\begin{aligned} \min_{\tilde{u}} \tilde{J}(\tilde{u}) &= \tilde{\phi}(\tilde{x}(t_1), t_1) + \int_{t_0}^{t_1} \tilde{f}(\tilde{x}(t), \tilde{u}(t), t)dt, \\ \text{subject to:} \quad {}_{t_0}D_t^{\beta}\tilde{x} &= \tilde{g}(\tilde{x}(t), \tilde{u}(t), t) \\ \tilde{x}(t_0) &= \tilde{x}_0. \end{aligned} \tag{4.1}$$

For fixed final state problem we have additional condition $\tilde{x}(t_1) = \tilde{x}_1$. Where $\tilde{f}, \tilde{g} : E^1 \times E^1 \times R \to E^1$ are assumed to be continuous first and second partial derivatives on $t \in I = [t_0, t_1] \subseteq R$ with respect to all their arguments and Riemann integrable, the fuzzy state $\tilde{x}(t)$ and the fuzzy control $\tilde{u}(t)$ are functions of $t \in I$, and the fuzzy state function $\tilde{x}(t)$ is Riemann-Liouville $[(1) - \beta]-$differentiable fuzzy-valued function and satisfies appropriate boundary conditions, and $\beta \in (0,1)$.

Definition 4.1 We say that an admissible fuzzy curve $(\tilde{x}^*, \tilde{u}^*)$ is solution of (4.1), if for all admissible fuzzy curve $(\tilde{x}, \tilde{u})$ of (4.1),

$$\tilde{J}(\tilde{x}^*, \tilde{u}^*) \preceq \tilde{J}(\tilde{x}, \tilde{u}).$$

Note that, we consider an admissible fuzzy control $\tilde{u}$ is not bounded.

Remark 4.1 If we choose $\beta = 1$, problem (4.1) is reduced to classical fuzzy optimal control problem.

Definition 4.2(Fuzzy Hamiltonian Function). We define fuzzy Hamiltonian function as,

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) = \tilde{f}(\tilde{x}(t), \tilde{u}(t), t) + \tilde{\lambda}(t)\tilde{g}(\tilde{x}(t), \tilde{u}(t), t). \tag{4.2}$$

It means that,

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)[\alpha] = [H^l(x^l, u^l, \lambda^l, t, \alpha), H^r(x^r, u^r, \lambda^r, t, \alpha)]. \tag{4.3}$$

for any $\alpha \in [0,1]$, and where $H^l(x^l, u^l, \lambda^l, t, \alpha)$ and $H^r(x^r, u^r, \lambda^r, t, \alpha)$ are classical Hamiltonian functions.

Remark 4.2 In the following theorem, we assume that $J^l(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)$ (or $J^r(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)$) is stated in terms containing only $x^l(t,\alpha)$, $u^l(t,\alpha)$ and $\lambda^l(t,\alpha)$ (or only $x^r(t,\alpha)$, $u^r(t,\alpha)$ and $\lambda^r(t,\alpha)$) in order to simplify the result presentations.

4.1 Derivation of Necessary Conditions

Now we are in the position to state a fundamental result of this work in the following theorem.
Theorem 4.1(Necessary Conditions) Assume that $\tilde{x}^*(t)$ be an admissible fuzzy state and $\tilde{u}^*(t)$ be an admissible fuzzy control. Then the necessary conditions for $\tilde{u}^*$ to be an optimal control for (4.1) and for all $\alpha \in [0,1]$, $t \in [t_0, t_1]$ are:

$$ {}_{t_0}D_t^{\beta} x^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.4} $$

$$ {}_{t_0}D_t^{\beta} x^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.5} $$

$$ {}_{t}^{C}D_{t_1}^{\beta} \lambda^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.6} $$

$$ {}_{t}^{C}D_{t_1}^{\beta} \lambda^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.7} $$

$$ \frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha) = 0, \tag{4.8} $$

$$ \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha) = 0. \tag{4.9} $$

with

$$ \lambda^l(t_1,\alpha) = \left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=t_1}, \tag{4.10} $$

$$ \lambda^r(t_1,\alpha) = \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=t_1}. \tag{4.11} $$

for free final state problems.

Proof. First we adopt fuzzy lagrange multiplier to form an augmented functional incorporating the constraints, then we modify the performance index as,

$$ \tilde{J}_a(\tilde{u}) = \int_{t_0}^{t_1} \left[\tilde{f}(\tilde{x}(t), \tilde{u}(t), t) + \frac{d\tilde{\phi}}{dt} + \tilde{\lambda} \left(\tilde{g}(\tilde{x}(t), \tilde{u}(t), t) \ominus_{t_0} D_t^{\beta} \tilde{x} \right) \right] dt, \tag{4.12} $$

It means that,

$$ \left[J_a^l(u^l,\alpha), J_a^r(u^r,\alpha) \right] = \left[\int_{t_0}^{t_1} \left[f^l(x^l,u^l,t,\alpha) + \frac{d\phi^l}{dt} + \lambda^l(t,\alpha) \left(g^l(x^l,u^l,t,\alpha) - {}_{t_0}D_t^{\beta} x^l \right) \right] dt, \right. $$
$$ \left. \int_{t_0}^{t_1} \left[f^r(x^r,u^r,t,\alpha) + \frac{d\phi^r}{dt} + \lambda^r \left(g^r(x^r,u^r,t,\alpha) - {}_{t_0}D_t^{\beta} x^r \right) \right] dt \right]. $$

In the remaining of the proof we will ignore the similar arguments and only we consider the left hand of all functions of its α-level set.

$$ J_a^l(u^l,\alpha) = \int_{t_0}^{t_1} \left[f^l(x^l(t), u^l(t), t, \alpha) + \lambda^l(t,\alpha) g^l(x^l(t), u^l(t), t, \alpha) - \lambda^l(t,\alpha)_{t_0} D_t^{\beta} x^l(t,\alpha) + \frac{d\phi^l}{dt} \right] dt. \tag{4.13} $$

Using the definition of fuzzy Hamiltonian function, then we can rewrite equation (4.13) as,

$$J_a^l(u^l,\alpha)=\int_{t_0}^{t_1}\left[H^l(x^l(t),u^l(t),\lambda^l(t),t,\alpha)+\frac{d\phi^l}{dt}-\lambda^l(t,\alpha)_{t_0}D_t^{\beta}x^l(t,\alpha)\right]. \tag{4.14}$$

Taking variation of equation (4.14), we obtain

$$\delta J_a^l(u^l,\alpha)=\int_{t_0}^{t_1}\frac{\partial H^l}{\partial x^l}\delta x^l+\frac{\partial H^l}{\partial u^l}\delta u^l+\frac{\partial H^l}{\partial \lambda^l}\delta\lambda^l+\frac{\partial \phi^l}{\partial x^l}\delta x^l-\delta\lambda^l{}_{t_0}D_t^{\beta}x^l-\lambda^l\delta_{t_0}D_t^{\beta}x^l, \tag{4.15}$$

where δx^l, $\delta\lambda^l$ and δu^l are the variations of x^l, λ^l and u^l respectively.

Using the formula for fractional integration by parts, integrate the last term on the RHS of (4.15), then we obtain

$$\delta J_a^l(u^l,\alpha)=\int_{t_0}^{t_1}\left(\frac{\partial H^l}{\partial x^l}-{}_t^C D_{t_1}^{\beta}\lambda^l\right)\delta x^l+\frac{\partial H^l}{\partial u^l}\delta u^l+\left(\frac{\partial H^l}{\partial \lambda^l}-{}_{t_0}D_t^{\beta}x^l\right)\delta\lambda^l dt+\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\Big|_{t=t_1}\delta x^l(t_1). \tag{4.16}$$

where ${}_t^C D_{t_1}^{\beta}$ represent the classical right Caputo fractional derivative.

u^{*^l} is an extremal if the variation of J_a^l is zero, that is, for all $\alpha\in[0,1]$ we require

$$\int_{t_0}^{t_1}\left(\frac{\partial H^l}{\partial x^l}-{}_t^C D_{t_1}^{\beta}\lambda^l\right)\delta x^l+\frac{\partial H^l}{\partial u^l}\delta u^l+\left(\frac{\partial H^l}{\partial \lambda^l}-{}_{t_0}D_t^{\beta}x^l\right)\delta\lambda^l dt+\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\Big|_{t=t_1}\delta x^l(t_1)=0. \tag{4.17}$$

It is convenient to choose the coefficients of $\delta x^l,\delta u^l$, and $\delta\lambda^l$ in (4.17) to be zero. This leads to

$${}_{t_0}D_t^{\beta}x^{*^l}(t,\alpha)=\frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha),u^{*^l}(t,\alpha),\lambda^{*^l}(t,\alpha),t,\alpha), \tag{4.18}$$

$${}_t^C D_{t_1}^{\beta}\lambda^{*^l}(t,\alpha)=\frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha),u^{*^l}(t,\alpha),\lambda^{*^l}(t,\alpha),t,\alpha), \tag{4.19}$$

$$\frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha),u^{*^l}(t,\alpha),\lambda^{*^l}(t,\alpha),t,\alpha)=0, \tag{4.20}$$

Finally, we have

$$\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\Big|_{t=t_1}\delta x^l(t_1)=0, \tag{4.21}$$

1. For the fixed final state problem

$$\delta x^l(t_1)=0, \tag{4.22}$$

2. For the free final state problem

$$\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\Big|_{t=t_1}=0. \tag{4.23}$$

Equations $(4.18)-(4.20)$ represents the necessary conditions for u^{*^l} to be an optimal with the condition (4.22) for the fixed final state problem and (4.23) for the free final state problem.

By following the same steps(using the right hand of all functions of its α-level set) for $\delta J_a^r(u^{*^r},\alpha)=0$, for all $\alpha\in[0,1]$ and $t\in[0,1]$, we will obtain

$${}_{t_0}D_t^{\beta}x^{*^l}(t,\alpha)=\frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha),u^{*^r}(t,\alpha),\lambda^{*^r}(t,\alpha),t,\alpha), \tag{4.24}$$

$$ {}_{t}^{C}D_{t_1}^{\beta}\lambda^{*^r}(t,\alpha) = \frac{\partial H^l}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.25} $$

$$ \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha) = 0. \tag{4.26} $$

Equations (4.24) − (4.26) represents the necessary conditions for u^{*^r} to be an extremal with the conditions $\delta x^r(t_1) = 0$ for the fixed final state problem and $\left(\frac{\partial \phi^r}{\partial x^r} - \lambda^l\right)\Big|_{t=t_1} = 0$ for the free final state problem.

The above equations form a set of necessary conditions that the left and right hand functions of its $\alpha-$level set of the fuzzy optimal control $\tilde{u}^*$ and fuzzy optimal state $\tilde{x}^*$ must satisfy. □

We know that, $\tilde{u}^*(t)$ and $\tilde{x}^*(t)$ are a fuzzy numbers with $\tilde{u}^*(t)[\alpha] = \left[u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)\right]$ and $\tilde{x}^*(t)[\alpha] = \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right]$ if $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ satisfy are related properties in **C1-C5** of Lemma(2.1). In the following definition, based on the conditions **C1** and **C2** of Lemma(2.1), we introduce the definition of strong and weak solutions of our problem.

Definition 4.3(Strong and Weak Solutions).

1. (Strong Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are strong solutions of (4.1) if $u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)$,$x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ obtained from (4.4) − (4.11) satisfy the conditions **C1-C2** of Lemma(2.1), for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.

2. (Weak Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are weak solutions of (4.1) if $u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)$,$x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ obtained from (4.4) − (4.11) do not satisfy the conditions **C1-C2** of Lemma(2.1), then we define $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ as:

   $\tilde{u}^*(t)[\alpha] =$

$$ \begin{cases} [2u^{r^*}(t,1) - u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)], \text{ if } u^{l^*}, u^{r^*} \text{are decreasing functions of } \alpha, \\ [u^{l^*}(t,\alpha), 2u^{l^*}(t,1) - u^{r^*}(t,\alpha)], \text{ if } u^{l^*}, u^{r^*} \text{are increasing functions of } \alpha, \\ [u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)], \text{ if } u^{l^*} \text{is decreasing and } u^{r^*} \text{is increasing of } \alpha \end{cases} $$

   and,

   $\tilde{x}^*(t)[\alpha] =$

$$ \begin{cases} [2x^{r^*}(t,1) - x^{l^*}(t,\alpha), x^{r^*}(t,\alpha)], \text{ if } x^{l^*}, x^{r^*} \text{are decreasing functions of } \alpha, \\ [x^{l^*}(t,\alpha), 2x^{l^*}(t,1) - x^{r^*}(t,\alpha)], \text{ if } x^{l^*}, x^{r^*} \text{are increasing functions of } \alpha, \\ [x^{r^*}(t,\alpha), x^{l^*}(t,\alpha)], \text{ if } x^{l^*} \text{is decreasing and } x^{r^*} \text{is increasing of } \alpha \end{cases} $$

for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.

Now, we consider fixed and free final state problems with a quadratic performance index.

4.2 Fixed Final State Problem

We can define fuzzy fractional optimal control problem with fixed final state as

$$ \begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \frac{1}{2}\int_{t_0}^{t_1} \left[q(t)\tilde{x}^2 + r(t)\tilde{u}^2\right] dt, \\ &\text{subject to:} \quad {}_0D_t^{\beta}\tilde{x} = a(t)\tilde{x} + b(t)\tilde{u}, \\ &\qquad \tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t_1) = \tilde{x}_1. \end{aligned} \tag{4.27} $$

where $q(t) \geq 0$ and $r(t) > 0$.

Theorem(4.1), give the necessary conditions for u^{*^l} to be an optimal as

$$ {}_{t_0}D_t^{\beta} x^l = a(t)x^l + b(t)u^l, \tag{4.28} $$

$$ {}_t^C D_{t_1}^{\beta} \lambda^l = q(t)x^l + a(t)\lambda^l, \tag{4.29} $$

$$ r(t)u^l + b(t)\lambda^l = 0. \tag{4.30} $$

Equations (4.28) and (4.30) gives

$$ {}_{t_0}D_t^{\beta} x^l = a(t)x^l - r^{-1}(t)b^2(t)\lambda^l. \tag{4.31} $$

We will obtain $x^l(t,\alpha)$ and $u^l(t,\alpha)$ by solving Equations (4.29) – (4.31) with the boundary conditions $x^l(t_0) = x_0^l$ and $x^l(t_1) = x_1^l$.

Similarly Theorem(4.1), give the necessary conditions for u^{*^r} to be an optimal as

$$ {}_{t_0}D_t^{\beta} x^r = a(t)x^r + b(t)u^r, \tag{4.32} $$

$$ {}_t^C D_{t_1}^{\beta} \lambda^r = q(t)x^r + a(t)\lambda^r, \tag{4.33} $$

$$ r(t)u^r + b(t)\lambda^r = 0. \tag{4.34} $$

Equations (4.32) and (4.34) gives

$$ {}_{t_0}D_t^{\beta} x^r = a(t)x^r - r^{-1}(t)b^2(t)\lambda^r. \tag{4.35} $$

We will obtain $x^r(t,\alpha)$ and $u^r(t,\alpha)$ by solving Equations (4.33) – (4.35) with the boundary conditions $x^r(t_0) = x_0^r$ and $x^r(t_1) = x_1^r$.

4.3 Free Final State Problem

We can define fuzzy fractional optimal control problem with free final state as

$$ \begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \tilde{\phi}(\tilde{x}(t_1), t_1) + \frac{1}{2}\int_{t_0}^{t_1} \left[q(t)\tilde{x}^2 + r(t)\tilde{u}^2\right] dt, \\ &\text{subject to:} \quad {}_{t_0}D_t^{\beta}\tilde{x} = a(t)\tilde{x} + b(t)\tilde{u}, \\ &\qquad\qquad\quad \tilde{x}(t_0) = \tilde{x}_0. \end{aligned} \tag{4.36} $$

where $q(t) \geq 0$ and $r(t) > 0$.

Following the same steps, we will obtain $x^l(t,\alpha)$ and $u^l(t,\alpha)$ by solving Equations (4.29) – (4.31) with respect to the conditions

$$ x^l(t_0) = x_0^l \quad \text{and} \quad \lambda^l(t_1,\alpha) = \left.\left(\frac{\partial \phi^l}{\partial x^l}\right)\right|_{t=t_1}. \tag{4.37} $$

Also we will obtain $x^r(t,\alpha)$ and $u^r(t,\alpha)$ by solving Equations (4.33) – (4.35) with respect to the conditions

$$ x^r(t_0) = x_0^r \quad \text{and} \quad \lambda^r(t_1,\alpha) = \left.\left(\frac{\partial \phi^r}{\partial x^r}\right)\right|_{t=t_1}. \tag{4.38} $$

In the next section we propose an algorithm used to find the solution of both cases numerically, the details of this algorithm in [4, 5].

5. Numerical technique

Considering the both cases of fixed and free final state problems defined above, in order to find the solution of our problems, we use the Grünwald-Letnikov(GL-for short) approximation of the left Riemann-Liouville fractional derivative and using the relation between right Riemann-Liouville fractional derivative and right Caputo fractional derivative and then use GL-approximation, we can approximate (4.31) and (4.29) as

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} x_{m-j}^l = a(mh)x_m^l - r^{-1}(mh)b^2(mh)\lambda_m^l, \tag{5.1}$$

for $m = 1, 2, ..., N$, and

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} \lambda_{m+j}^l = q(mh)x_m^l + a(mh)\lambda_m^l + \frac{\lambda_N^l (t_1 - mh)^{-\beta}}{\gamma(1-\beta)}, \tag{5.2}$$

for $m = N-1, N-2, ..., 0$, respectively. Where N is the number of equal divisions of the interval $[0, t_1]$, the nodes are labeled as $0, 1, ..., N$. The size of each division is given as $h = \frac{t_1}{N}$, and $t_j = jh$ represent the time at node j. The coefficients are defined as

$$w_j^{\beta} = (-1)^j \binom{\beta}{j}. \tag{5.3}$$

Where x_i^l and λ_i^l represent the numerical approximations of $x^l(t,\alpha)$ and $\lambda^l(t,\alpha)$ at node i.

Similarly, we can approximate (4.35) and (4.33) as

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} x_{m-j}^r = a(mh)x_m^r - r^{-1}(mh)b^2(mh)\lambda_m^r, \tag{5.4}$$

for $m = 1, 2, ..., N$, and

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} \lambda_{m+j}^r = q(mh)x_m^r + a(mh)\lambda_m^l + \frac{\lambda_N^r (t_1 - mh)^{-\beta}}{\gamma(1-\beta)}, \tag{5.5}$$

for $m = N-1, N-2, ..., 0$, respectively.

Also x_i^r and λ_i^r represent the numerical approximations of $x^r(t,\alpha)$ and $\lambda^r(t,\alpha)$ at node i. In general, Equations (5.1) and (5.2) or Equations (5.4) and (5.5) give a set of $2N$ equations in terms of $2N$ variables, i.e., $Ax = b$, it means that, we can use any linear equation solver to find the solution. Regardless the left and right bounds of the fuzzy numbers $\tilde{x}$ and $\tilde{\lambda}$, the vector x is constructed as follows

- For fixed final state problem

$$x = [x_1 \ \ x_2 \ \ ... \ \ x_{N-1} \ \ \lambda_0 \ \ \lambda_1 \ \ ... \ \ \lambda_N]^T .$$

- For free final state problem

$$x = [x_1 \ \ x_2 \ \ ... \ \ x_N \ \ \lambda_0 \ \ \lambda_1 \ \ ... \ \ \lambda_{N-1}]^T .$$

In the next section, we will give four examples can serve to illustrate our main results.

6. Numerical examples

Example 6.1 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_0^1 \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= t\tilde{x} + \tilde{u}, \\ \tilde{x}(0) &= (0,1,2), \quad \tilde{x}(1) = (-2,-1,1). \end{aligned}$$

Solution. We have,

$$q(t) = r(t) = b(t) = t_1 = 1, \text{ and } a(t) = t,$$

Then for the left bound of state and control Theorem(4.1) gives,

$${}_0D_t^{\beta}x^l = tx^l - \lambda^l, \tag{6.1}$$

$${}_t^CD_1^{\beta}\lambda^l = x^l + t\lambda^l, \tag{6.2}$$

$$u^l + \lambda^l = 0. \tag{6.3}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= \alpha, \\ x^l(1,\alpha) &= -2+\alpha. \end{aligned}$$

For the right bound of state and control, Theorem(4.1) gives,

$${}_0D_t^{\beta}x^r = tx^r - \lambda^r, \tag{6.4}$$

$${}_t^CD_1^{\beta}\lambda^r = x^r + t\lambda^r, \tag{6.5}$$

$$u^r + \lambda^r = 0. \tag{6.6}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 2-\alpha, \\ x^r(1,\alpha) &= 1-2\alpha. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(1(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(1(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is an increasing function of α, $u^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1), furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a strong fuzzy solution of this problem.

Example 6.2 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_1^2 \tilde{u}^2 dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= (2t-1)\tilde{x} \ominus \sin(t)\tilde{u}, \\ \tilde{x}(1) &= (0,1,2), \quad \tilde{x}(2) = (-2,-1,1). \end{aligned}$$

Solution. We have, $q(t) = 0$, $r(t) = t_0 = 1$, $b(t) = -\sin(t)$, and $a(t) = (2t-1)$, then for the left bound of the state and control, Theorem(4.1) gives,

$$ {}_1D_t^{\beta}x^l = (2t-1)x^l - \sin^2(t)\lambda^l, \tag{6.7}$$

$$ {}_t^CD_2^{\beta}\lambda^l = (2t-1)\lambda^l, \tag{6.8}$$

$$ u^l - \sin(t)\lambda^l = 0. \tag{6.9}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= \alpha, \\ x^l(1,\alpha) &= -2+\alpha. \end{aligned}$$

For the right bound of state and control Theorem(4.1) gives,

$$ {}_1D_t^{\beta}x^r = (2t-1)x^r - \sin^2(t)\lambda^r, \tag{6.10}$$

$$ {}_t^CD_2^{\beta}\lambda^r = (2t-1)\lambda^r, \tag{6.11}$$

$$ u^r - \sin(t)\lambda^r = 0. \tag{6.12}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 2-\alpha, \\ x^r(1,\alpha) &= 1-2\alpha. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(2(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(2(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

Example 6.3 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_0^1 \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= -(0,1,3)\tilde{x} + \tilde{u}, \\ \tilde{x}(0) &= (1,1,1), \quad \tilde{x}(1) = (0,0,0). \end{aligned}$$

Solution. We know that,

$$\left[{}_0D_t^\beta x^l, {}_0D_t^\beta x^r\right] = \left[-(3-2\alpha)x^l + u^l, -\alpha x^r + u^r\right],$$

then we have,

$$q(t) = r(t) = b(t) = x_0 = t_1 = 1,$$

$a(t) = -(3-2\alpha)$ and $a(t) = -\alpha$ for the left and right derivatives respectively, then for the left bound of the state and control Theorem(4.1) gives,

$$ {}_0D_t^\beta x^l = -(3-2\alpha)x^l - \lambda^l, \tag{6.13}$$

$$ {}_t^C D_1^\beta \lambda^l = x^l - (3-2\alpha)\lambda^l, \tag{6.14}$$

$$ u^l + \lambda^l = 0. \tag{6.15}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= 1, \\ x^l(1,\alpha) &= 0. \end{aligned}$$

For the right bound of the state and control Theorem(4.1) gives,

$$ {}_1D_t^\beta x^r = -\alpha x^r - \lambda^r, \tag{6.16}$$

$$ {}_t^C D_2^\beta \lambda^r = x^r - \alpha\lambda^r, \tag{6.17}$$

$$ u^r + \lambda^r = 0. \tag{6.18}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 1, \\ x^r(1,\alpha) &= 0. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(3(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(3(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

Example 6.4 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\tilde{x}^2(1) + \frac{1}{2}\int_0^1 \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{aligned} {}_0D_t^\beta \tilde{x} &= -(0,1,3)\tilde{x} + \tilde{u}, \\ \tilde{x}(0) &= (1,1,1). \end{aligned}$$

Solution. We have,

$$q(t) = r(t) = b(t) = x_0 = t_1 = 1,$$

$a(t) = -(3-2\alpha)$ and $a(t) = -\alpha$ for the left and right derivatives respectively, then Theorem(4.1) gives,

$$_{t_0}D_t^{\beta} x^l = -(3-2\alpha)x^l - \lambda^l, \tag{6.19}$$

$$_t^C D_{t_1}^{\beta} \lambda^l = x^l - (3-2\alpha)\lambda^l, \tag{6.20}$$

$$u^l + \lambda^l = 0. \tag{6.21}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= 1, \\ \lambda^l(0,\alpha) &= x^l(1,\alpha). \end{aligned}$$

For the right bound of the state and control Theorem(4.1) gives,

$$_1D_t^{\beta} x^r = -\alpha x^r - \lambda^r, \tag{6.22}$$

$$_t^C D_2^{\beta} \lambda^r = x^r - \alpha\lambda^r, \tag{6.23}$$

$$u^r + \lambda^r = 0. \tag{6.24}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 1, \\ \lambda^r(0,\alpha) &= x^r(1,\alpha). \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(4(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(4(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1**-**C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

7. Conclusion

In this paper, the necessary conditions of fuzzy fractional optimal control problem with both fixed and free final state conditions at the final time has been derived using fuzzy variational approach. Our problems is defined in the sense of Riemann-Liouville fractional derivative based on Hukuhara difference. A numerical technique is proposed based on Grünwald-Letnikov definition of fractional derivative. The concepts of strong and weak solutions of our problems are given. lastly, four examples are provided to show the effectiveness of Theorem(4.1) and the numerical algorithm.

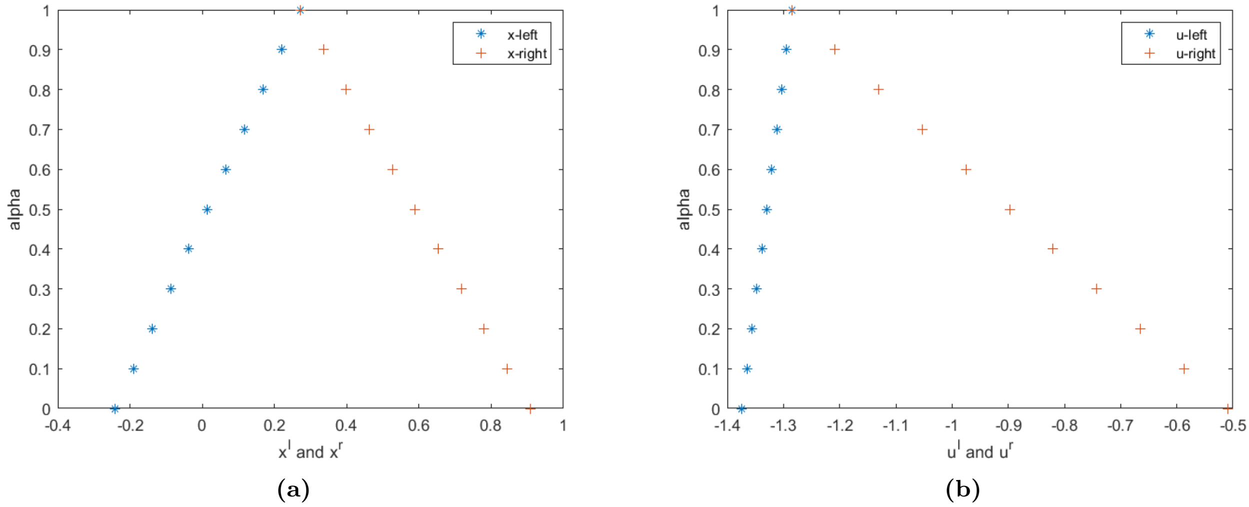

Figure 1: Example(6.1) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.

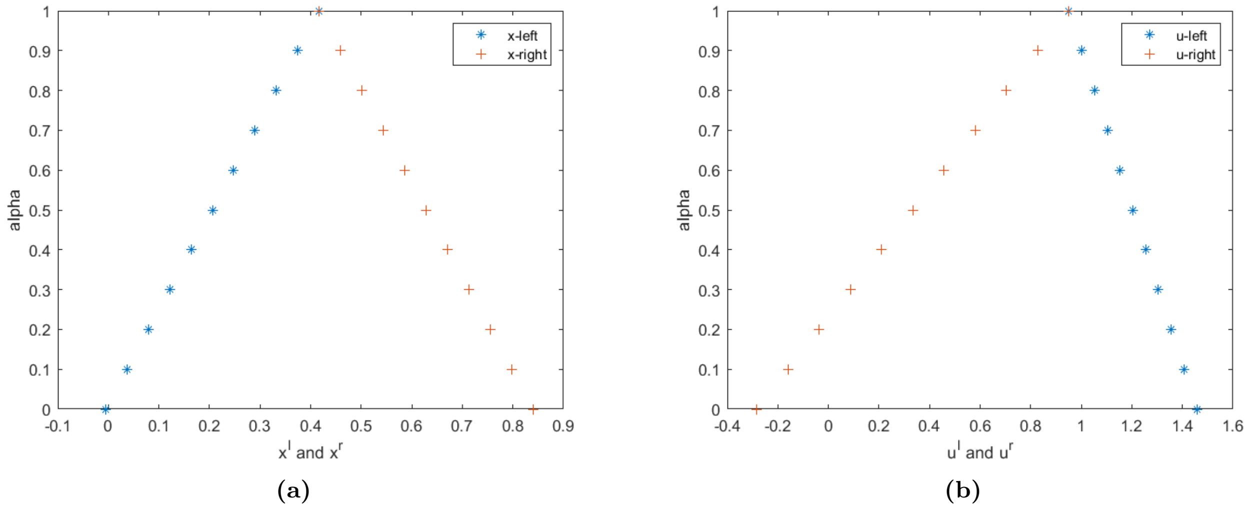

Figure 2: Example(6.2) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.

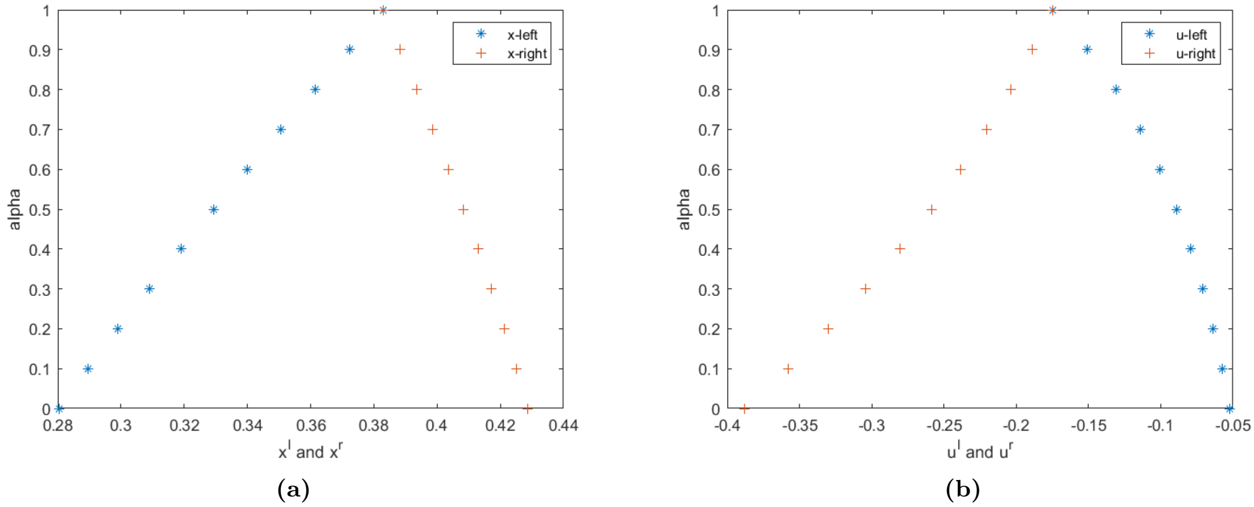

Figure 3: Example(6.3) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.

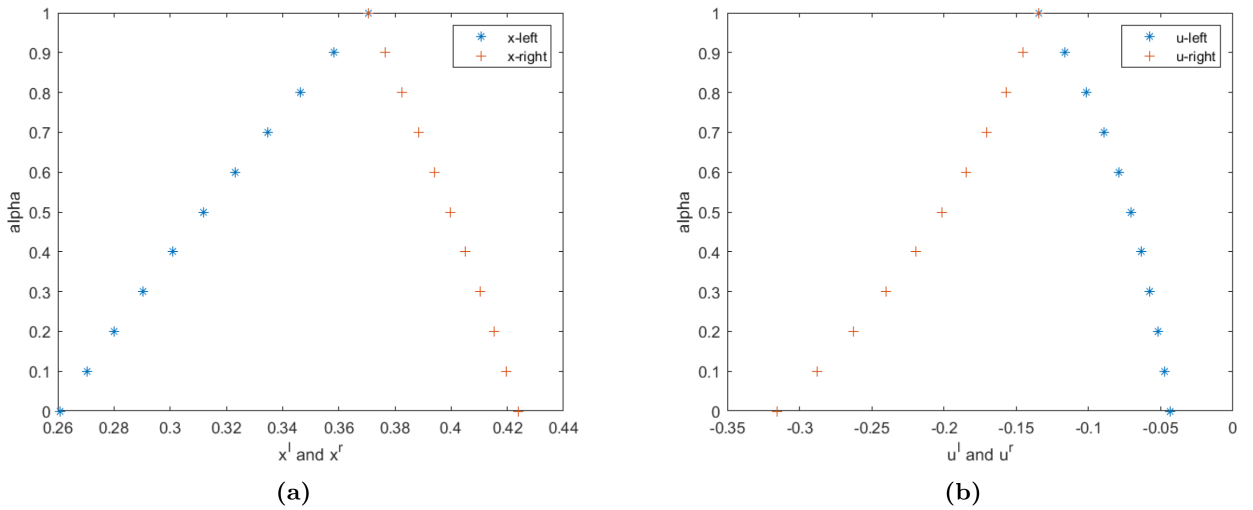

Figure 4: Example(6.4) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control $t = 0.1, \beta = 0.77$.

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