The interior and closure of fuzzy topologies induced by the generalized fuzzy approximation spaces

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Abstract

With respect to the Alexandrov fuzzy topologies induced by the generalized fuzzy approximation spaces, Wang defined interior of fuzzy set. In this paper, we give the closure of fuzzy set and discuss some properties of the interior and closure.

Keywords: Alexandrov fuzzy topology; the generalized fuzzy approximation spaces; interior; closure; properties

1 Introduction

In his classical paper [36], Zadeh introduced the notation of fuzzy sets and fuzzy set operation. Subsequently, Chang [2] applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. Pu etc.[18] defined a fuzzy point which took a crisp singleton, equivalently, an ordinary point, as a special case and gave the concepts of interior and closure operator w.r.t. fuzzy topology. Later, Lai and Zhang [11] modified the second axiom in Chang's definition of fuzzy topology to define an Alexandrov fuzzy topology.

The concept of Rough sets were introduced by Z. Pawlak [19] in 1982 as an powerful mathematical tool for uncertain data while modeling the problems in many fields [17,20,27]. Because the rough sets defined with equivalence relations limited the application of it. Thus many authors changed the equivalence relations into different binary relations to expand the application of it [23,35,37,38]. In recent years, the rough sets has been combined with some mathematical theories such as algebra and topology [1,5,6,8,10, 14, 16, 21, 25, 26, 28, 29]. With respect to different binary relations, the topological properties of rough sets were further investigated in [7,14,33,34].

In 1990, Dubois and Prade [3] combining fuzzy sets and rough sets proposed rough fuzzy sets and fuzzy rough sets. Afterward Morsi and Yakout [15] investigated fuzzy rough sets defined with left-continuous t-norms and R-implicators with respect to fuzzy similarity relations. Radzikowska and Kerre [24] defined a broad family of fuzzy rough sets based on t-norms and fuzzy implicators, which are called generalized fuzzy rough sets here. In recent years, the topological properties of fuzzy rough sets were further studied in many literatures [4,9,12,13,22]. Recently, with respect to the lower fuzzy rough approximation operator determined by a fuzzy implicator, Wang [30] studied various fuzzy topologies induced by different fuzzy relations and proved that I-lower fuzzy rough approximation operators were the interior operator w.r.t. some Alexandrov fuzzy topology.

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In this paper, we give closure operator w.r.t. some Alexandrov fuzzy topology given by Wang in [30]. Combined with the definition of Wang's interior, discuss some properties of the interior and closure of fuzzy set.

2 Preliminary

Definition 2.1.[36] A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, A(x)) : x \in X\}$$

where $A(x) : X \to [0, 1]$ is a mapping and $A(x)$ states the grade of belongness of x in A. The family of all fuzzy sets in X is denoted by $\mathscr{F}(X)$.

Let $\alpha \in [0, 1]$, then a fuzzy set $A \in \mathscr{F}(X)$ is a constant, while $A(x) = \alpha$ for all $x \in X$, denoted as α_X.

Deflnition 2.2.[36] Let A, B be two fuzzy sets of $\mathscr{F}(X)$
(1) A is contained in B if and only if $A(x) \leq B(x)$ for every $x \in X$.
(2) The union of A and B is a fuzzy set C, denoted by $A \cup B = C$, whose membership function $C(x) = A(x) \vee B(x)$ for every $x \in X$.
(3) The intersection of A and B is a fuzzy set C, denoted by $A \cap B = C$, whose membership function $C(x) = A(x) \wedge B(x)$ for every $x \in X$.
(4) The complement of A is a fuzzy set, denoted by A^c, whose membership function $A^c(x) = 1 - A(x)$ for every $x \in X$.

A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ (resp. $S : [0, 1] \times [0, 1] \to [0, 1]$) is called a t-norm (resp. t-conorm) on [0, 1] if it is commutative, associative, increasing in each argument and has a unit element 1 (resp. 0).

A mapping $I : [0, 1] \times [0, 1] \to [0, 1]$ is called a fuzzy implicator on [0, 1] if it satisfies the boundary conditions according to the Boolean implicator, and is decreasing in the first argument and increasing in the second argument.

Definition 2.3.[30] A fuzzy implicator I is said to satisfy
(1) the left neutrality property ((NP), for short), if $I(1, b) = b$ for all $b \in [0, 1]$;
(2) the confinement principle ((CP), for short), if $I(a, b) = 1 \Leftrightarrow a \leq b$, for all $a, b \in [0, 1]$;
(3) the regular property ((RP), for short), if N_I is an involutive negation, where N_I is defined as $N_I(a) = I(a, 0)$ for all $a \in [0, 1]$.

Definition 2.4. [11] A subset $\tau \subseteq \mathscr{F}(X)$ is called an Alexandrov fuzzy topology if it satisfies:
(1) $\alpha_X \in \tau$ for all $\alpha \in [0, 1]$,
(2) $\cap_{i \in \Lambda} A_i \in \tau$ for all $\{A_i\}_{i \in \Lambda} \subseteq \tau$,
(3) $\cup_{i \in \Lambda} A_i \in \tau$ for all $\{A_i\}_{i \in \Lambda} \subseteq \tau$.

Every member of τ is called a τ-open fuzzy set. A fuzzy set is τ-closed if and only if its complement is τ-open. In the sequel, when no confusion is likely to arise, we shall call a τ-open (τ-closed) fuzzy set simply an open (closed) set.

Definition 2.5. [18,31]. Let $\tau \subseteq \mathscr{F}(X)$ be a fuzzy topology. Then the interior of $A \in \mathscr{F}(X)$ w.r.t. fuzzy topology τ denoted as A^o is defined as follows:

$$A^o = \cup\{B \in \tau | B \subseteq A\}.$$

2

The operator A^o is called an interior operator w.r.t. fuzzy topology τ.

According to definition of the fuzzy topology, obviously A^o is an open set.

Definition 2.6. [18]. Let $\tau \subseteq \mathscr{F}(X)$ be a fuzzy topology. Then the closure of $A \in \mathscr{F}(X)$ w.r.t. fuzzy topology τ denoted as $\overline{A}$ is defined as follows:

$$\overline{A} = \cap\{B|B \supseteq A, B^c \in \tau\}$$

The operator $\overline{A}$ is called a closure operator w.r.t. fuzzy topology τ.

According to De Morgan's Law and definition of the fuzzy topology, $\overline{A}$ is a closed set.

3 Fuzzy topologies induced by the generalized fuzzy approximation spaces

A fuzzy set $R \in \mathscr{F}(X \times Y)$ is called a fuzzy relation from X to Y. If $X = Y$, then R is a fuzzy relation on X. For every fuzzy relation R on X, a fuzzy relation R^{-1} is defined as $R^{-1}(x, y) = R(y, x)$ for all $x, y \in X$. Let R be a fuzzy relation from X to Y . Then the triple (X, Y, R) is called a fuzzy approximation space. When $X = Y$ and R is a fuzzy relation on X, we also call (X, R) a fuzzy approximation space.

Definition 3.1.[30]. Let R be a fuzzy relation on X. Then R is said to be
(1) reflexive if $R(x, x) = 1$ for all $x \in X$;
(2) symmetric if $R(x, y) = R(y, x)$ for all $x, y \in X$;
(3) T-transitive if $T(R(x, y), R(y, z)) \leq R(x, z)$ for all $x, y, z \in X$.

If $T = \wedge$, then T-transitive is said to be transitive for short. A fuzzy relation R is called a fuzzy tolerance if it is reflexive and symmetric, and a fuzzy T-preorder if it is reflexive and T-transitive. Similarly, a fuzzy relation R is called a fuzzy preorder if it is reflexive and transitive.

Definition 3.2.[24,30,32]. Let (X, Y, R) be a fuzzy approximation space. Then the following mappings $\underline{R}, \overline{R} : \mathscr{F}(Y) \to \mathscr{F}(X)$ are defined as follows: for all $A \in \mathscr{F}(Y)$ and $x \in X$,

$$\underline{R}(A)(x) = \bigwedge_{y \in Y} I(R(x, y), A(y)) \text{ and } \overline{R}(A)(x) = \bigvee_{y \in Y} T(R(x, y), A(y)).$$

The mappings $\underline{R}$ and $\overline{R}$ are called $I-lower$ and $T-upper$ fuzzy rough approximation operators, respectively. The pair $(\underline{R}(A), \overline{R}(A))$ is called a generalized fuzzy rough set of A w.r.t. (X, Y, R). Also known as generalized fuzzy approximation spaces.

Let R be a fuzzy relation on X. Then a fuzzy set $A \in \mathscr{F}(X)$ is said to be
(1) I-lower definable w.r.t. fuzzy relation R if $\underline{R}(A) = A$; the family of all $I - lower$ definable fuzzy sets w.r.t. R is denoted as $\mathscr{D}_I(R)$.
(2) T-upper definable w.r.t. fuzzy relation R if $\overline{R}(A) = A$; the family of all $T - upper$ definable fuzzy sets w.r.t. R is denoted as $\mathscr{D}_T(R)$.

Proposition 3.3.[30]. Let (X, R) be a fuzzy approximation space and R be reflexive. Then
(1)$\mathscr{D}_I(R)$ is an Alexandrov fuzzy topology, if I satisfies (NP).
(2)$\mathscr{D}_T(R)$ is an Alexandrov fuzzy topology.

Let (X, R) be a fuzzy approximation space. In [30] Wang defined

$$\mathscr{R}_I(R) = \{\underline{R}(A)|A \in \mathscr{F}(X)\} \text{ and } \mathscr{R}_T(R) = \{\overline{R}(A)|A \in \mathscr{F}(X)\}.$$

To discuss the properties of generalized fuzzy rough sets, Radzikowska and Kerre [19] introduced the following auxiliary conditions: for a fuzzy implicator I and a t-norm T,
(C1) $I(a, I(b, c)) = I(T(a, b), c)$ for all $a, b, c \in [0, 1]$,
(C2) $I(a, I(b, c)) \geq I(T(a, b), c)$ for all $a, b, c \in [0, 1]$,
(C3) $I(a, I(b, c)) \leq I(T(a, b), c)$ for all $a, b, c \in [0, 1]$.
If (C1) (resp. (C2), (C3)) holds for I and T , then we say that I satisfies (C1) (resp. (C2), (C3)) for T.

Proposition 3.4.[30]. Let (X, R) be a fuzzy approximation space and R be a fuzzy T-preorder. Then
(1) $\mathscr{R}_I(R)$ is an Alexandrov fuzzy topology and $\mathscr{R}_I(R) = \mathscr{D}_I(R)$, if I satisfies (NP) and (C2) for T.
(2) $\mathscr{R}_T(R)$ is an Alexandrov fuzzy topology and $\mathscr{R}_T(R) = \mathscr{D}_T(R)$.
The above $\mathscr{D}_I(R)$, $\mathscr{D}_T(R)$, $\mathscr{R}_I(R)$ and $\mathscr{R}_T(R)$ are called fuzzy topologies induced by the generalized fuzzy approximation spaces.

4 The interior and closure of fuzzy set

Proposition 4.1.[30]. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then $\underline{R}$ is the interior operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.

Proposition 4.2. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then A is an open set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ iff $\underline{R}(A) = A^o = A$.

Proof. Suppose A is an open set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$, again $A \subseteq A$, due to definition of A^o, $A \subseteq A^o$. On the other hand,

$$\forall x \in X,\ \ \underline{R}(A)(x) = \bigwedge_{y \in X} I(R(x, y), A(y)) \leq I(R(x, x), A(x)) = I(1, A(x)) = A(x).$$

This means $\underline{R}(A) = A^o \subseteq A$. Thus $\underline{R}(A) = A^o = A$.
Conversely, suppose $\underline{R}(A) = A^o = A$, A^o is an open set, thus A is an open set.

Proposition 4.3. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then for any $A \in \mathscr{F}_{(X)}$, $[\underline{R}(A^c)]^c$ is the closure operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
Proof. For any $A \in \mathscr{F}_{(X)}$, since $\underline{R}(A^c)$ is an open set, thus $(\underline{R}(A^c))^c$ is a closed set. Again

$$\forall x \in X,\ \ \underline{R}(A^c)(x) = \bigwedge_{y \in X} I(R(x, y), A^c(y)) \leq I(R(x, x), A^c(x)) = I(1, A^c(x)) = A^c(x),$$

this means $(\underline{R}(A^c))^c \supseteq A$.
On the other hand, for any $A \subseteq B \in \mathscr{F}_{(X)}$ and $B^c \in \mathscr{D}_I(R)$. By Proposition 4.2, $\underline{R}(B^c) = B^c$, and

$$\forall x \in X,\ \ \underline{R}(A^c)(x) = \bigwedge_{y \in X} I(R(x, y), A^c(y)) \geq \bigwedge_{y \in X} I(R(x, x), B^c(x)) = \underline{R}(B^c)(x).$$

We obtain $\underline{R}(A^c) \supseteq \underline{R}(B^c) = B^c$. This means $(\underline{R}(A^c))^c \subseteq B$. By Definition of the closure, for any $A \in \mathscr{F}_{(X)}$, $[\underline{R}(A^c)]^c$ is the closure operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ i.e. $[\underline{R}(A^c)]^c = \overline{A}$.

Proposition 4.4. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then A is a closed set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ iff $(\underline{R}(A^c))^c = \overline{A} = A$.

Proof. Suppose A is a closed set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$, then A^c is an open set. Therefore $\underline{R}(A^c) = A^c$, and then $\overline{A} = (\underline{R}(A^c))^c = A$.

Conversely, suppose $\overline{A} = (\underline{R}(A^c))^c = A$, $\overline{A}$ is a closed set, thus A is a closed set.

Proposition 4.5. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A, B \in \mathscr{F}_{(X)}$ the following formula hold w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $A \subseteq \overline{A}$;
(2) $\overline{\overline{A}} = \overline{A}$;
(3) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$;
(4) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proof. (1) For all $x \in X$,

$$\begin{aligned}\overline{A}(x) &= (\underline{R}(A^c))^c(x) = 1 - \underline{R}(A^c)(x) = 1 - \bigwedge_{y\in X} I(R(x,y), A^c(y))\\ &\geq 1 - I(R(x,x), A^c(x)) = 1 - I(1, A^c(x)) = 1 - A^c(x) = A(x),\end{aligned}$$

thus $A \subseteq \overline{A}$.
(2) Since $\overline{A}$ is a closed set, By Proposition 4.4, $\overline{\overline{A}} = \overline{A}$.
(3) By $A \subseteq B$, we obtain $A^c \supseteq B^c$. According to Definition 3.2, obviously $\underline{R}(A^c) \supseteq \underline{R}(B^c)$, and then $\overline{A} = (\underline{R}(A^c))^c \subseteq (\underline{R}(B^c))^c = \overline{B}$.
(4) Since $A \subseteq A \cup B$, $B \subseteq A \cup B$, by (2) $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

On the other hand, by (1) $A \subseteq \overline{A}$, $B \subseteq \overline{B}$. Thus $A \cup B \subseteq \overline{A} \cup \overline{B}$. And then $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. Again $\overline{A} \cup \overline{B}$ is a closed set, according to Proposition 4.4 $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Thereby $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proposition 4.6. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A \in \mathscr{F}_{(X)}$, the following formula hold w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $\overline{A} = [(A^c)^o]^c$; (2) $A^o = [\overline{A^c}]^c$;
(3) $[\overline{A}]^c = [A^c]^o$; (4) $\overline{A^c} = [A^o]^c$.
Proof. (1) By Proposition 4.2, $(A^c)^o = \underline{R}(A^c)$, thus $[(A^c)^o]^c = [\underline{R}(A^c)]^c = \overline{A}$.

(2),(3),(4) can be proven in a similar way as for item (1).

Proposition 4.7. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A, B \in \mathscr{F}_{(X)}$ and $A \subseteq B$, the following holds w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $A^o \subseteq B^o$; (2) $A^{oo} = A^o$; (3) $(A \cap B)^o = A^o \cap B^o$.

Proof. (1) $\forall x \in X$, $\underline{R}(A)(x) = \bigwedge_{y\in X} I(R(x,y), A(y)) \leq \bigwedge_{y\in X} I(R(x,y), B(y)) = \underline{R}(B)(x)$. Thus $A^o \subseteq B^o$.
(2) Since A^o is a open set, by Proposition 4.2, $A^{oo} = A^o$.
(3) By Proposition 4.6 (2) and Proposition 4.5 (4),
$(A \cap B)^o = (\overline{(A \cap B)^c})^c = (\overline{A^c \cup B^c})^c = (\overline{A^c} \cup \overline{B^c})^c = (\overline{A^c})^c \cap (\overline{B^c})^c = A^o \cap B^o$.

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