Analysis of Solutions of Some Discrete Systems of Rational Difference Equations

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Abstract

The major objective of this article is to determine and formulate the analytical solutions of the following systems of rational recursive equations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(\pm 1 \mp x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\mp 1 \pm y_{n-1}x_{n-3}\right)}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are required to be arbitrary non-zero real numbers. We also introduce some graphs describing these exact solutions under a suitable choice of some initial conditions.

Keywords: difference equations, system of recursive equations, periodicity, local stability, global stability.
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1 Introduction

The global interest in exploring the qualitative behaviours of discrete systems of recursive equations has been recently emerged due to the significance of difference equations in modelling a considerable number of discrete phenomena. More specifically, recursive equations are utilized in describing some real life problems that originate in genetics in biology, queuing problems, enegineering, physics, etc. Some experts put effort to analyse dynamical systems of difference equations. Take, for instance, the following ones. Almatrafi et al. [1] studied the local stability, global attractivity, periodicity and solutions for a special case for the difference equation

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}.$$

Clark and Kulenovic [6] investigated the global attractivity of the system

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}.$$

The author in [8] explored the equilibrium points and the stability of a discrete Lotka-Volterra model shown as follows:

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \quad y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1 + \eta y_n}.$$

The positive solutions of the system

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}.$$

were obtained in [14] by Gűműş and Őcalan. Moreover, Kurbanli et al. [18] solved the dynamical systems of recursive equations given by

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

In [19] Mansour et al. presented the analytical solutions of the system

$$x_{n+1} = \frac{x_{n-1}}{\alpha - x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\beta + \gamma y_{n-1} x_n}.$$

Finally, the author in [23] demonstrated the dynamics of the system

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}.$$

To attain more information on the qualitative behaviours of dynamical difference equations, one can refer to refs [1–5, 7, 9–13, 15–17, 20–22]

In this paper, the rational solutions of the following discrete systems of difference equations will be discovered and given in four different theorems:

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1} (\pm 1 \mp x_{n-1} y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1} (\mp 1 \pm y_{n-1} x_{n-3})}, \quad n = 0, 1, ...,$$

where the initial values are as described previously.

2 Main Results

2.1 First System $x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}(1 - x_{n-1} y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}(1 - y_{n-1} x_{n-3})}$

This subsection concentrates on obtaining the solutions of a dynamical system of fourth order difference equations given by the form:

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1} (1 - x_{n-1} y_{n-3})}, \; y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1} (1 - y_{n-1} x_{n-3})}, \; n = 0, \; 1, ... \; , \tag{1}$$

where the initial values are as shown previously. The following fundamental theorem presents the solutions of system (1).

Theorem 1 *Assume that* $\{x_n, y_n\}$ *is a solution to system (1) and let* $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ *and* $y_0 = \omega$*. Then, for* $n = 0,\ 1,\ \ldots$ *we have*

$$x_{4n-3} = \frac{\gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i)\,\alpha\mu - 1]}{\alpha^{n-1}\mu^n \prod_{i=0}^{n-1} [(2i+1)\,\gamma\epsilon - 1]}, \quad x_{4n-2} = \frac{\delta^n \eta^n \prod_{i=0}^{n-1} [(2i)\,\beta\omega - 1]}{\beta^{n-1}\omega^n \prod_{i=0}^{n-1} [(2i+1)\,\delta\eta - 1]},$$

$$x_{4n-1} = \frac{\gamma^{n+1} \epsilon^n \prod_{i=0}^{n-1} [(2i+1)\,\alpha\mu - 1]}{\alpha^{n}\mu^n \prod_{i=0}^{n-1} [(2i+2)\,\gamma\epsilon - 1]}, \quad x_{4n} = \frac{\delta^{n+1} \eta^n \prod_{i=0}^{n-1} [(2i+1)\,\beta\omega - 1]}{\beta^{n}\omega^n \prod_{i=0}^{n-1} [(2i+2)\,\delta\eta - 1]}.$$

And

$$y_{4n-3} = \frac{\alpha^n \mu^n \prod_{i=0}^{n-1} [(2i)\,\gamma\epsilon - 1]}{\gamma^{n}\epsilon^{n-1} \prod_{i=0}^{n-1} [(2i+1)\,\alpha\mu - 1]}, \quad y_{4n-2} = \frac{\beta^n \omega^n \prod_{i=0}^{n-1} [(2i)\,\delta\eta - 1]}{\delta^{n}\eta^{n-1} \prod_{i=0}^{n-1} [(2i+1)\,\beta\omega - 1]},$$

$$y_{4n-1} = \frac{\alpha^{n} \mu^{n+1} \prod_{i=0}^{n-1} [(2i+1)\,\gamma\epsilon - 1]}{\gamma^{n}\epsilon^n \prod_{i=0}^{n-1} [(2i+2)\,\alpha\mu - 1]}, \quad y_{4n} = \frac{\beta^{n} \omega^{n+1} \prod_{i=0}^{n-1} [(2i+1)\,\delta\eta - 1]}{\delta^{n}\eta^n \prod_{i=0}^{n-1} [(2i+2)\,\beta\omega - 1]}.$$

Proof. For $n = 0$, our results hold. Next, let $n > 1$ and suppose that the relations hold for $n - 1$. That is

$$x_{4n-7} = \frac{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i)\,\alpha\mu - 1]}{\alpha^{n-2}\mu^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\gamma\epsilon - 1]}, \quad x_{4n-6} = \frac{\delta^{n-1} \eta^{n-1} \prod_{i=0}^{n-2} [(2i)\,\beta\omega - 1]}{\beta^{n-2}\omega^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\delta\eta - 1]},$$

$$x_{4n-5} = \frac{\gamma^{n} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\alpha\mu - 1]}{\alpha^{n-1}\mu^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\gamma\epsilon - 1]}, \quad x_{4n-4} = \frac{\delta^{n} \eta^{n-1} \prod_{i=0}^{n-2} [(2i+1)\,\beta\omega - 1]}{\beta^{n-1}\omega^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\delta\eta - 1]}.$$

And

$$y_{4n-7} = \frac{\alpha^{n-1} \mu^{n-1} \prod_{i=0}^{n-2} [(2i)\,\gamma\epsilon - 1]}{\gamma^{n-1}\epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\,\alpha\mu - 1]}, \quad y_{4n-6} = \frac{\beta^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i)\,\delta\eta - 1]}{\delta^{n-1}\eta^{n-2} \prod_{i=0}^{n-2} [(2i+1)\,\beta\omega - 1]},$$

$$y_{4n-5} = \frac{\alpha^{n-1} \mu^{n} \prod_{i=0}^{n-2} [(2i+1)\,\gamma\epsilon - 1]}{\gamma^{n-1}\epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\alpha\mu - 1]}, \quad y_{4n} = \frac{\beta^{n-1} \omega^{n} \prod_{i=0}^{n-2} [(2i+1)\,\delta\eta - 1]}{\delta^{n-1}\eta^{n-1} \prod_{i=0}^{n-2} [(2i+2)\,\beta\omega - 1]}.$$

Now, it can be obviously observed from system (1) that

$$
\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(1-x_{4n-5}y_{4n-7}\right)} \\
&= \frac{\frac{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}}{\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\left[1-\frac{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}\right]} \\
&= \frac{\frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}}{\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\left[1-\frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\right]} \\
&= \frac{\gamma^{n}\epsilon^{n}\prod_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]}{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}\left[(2i+1)\gamma\epsilon-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]-\gamma\epsilon\prod_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]\right]} \\
&= -\frac{\gamma^{n}\epsilon^{n}\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]}{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]} = \frac{\gamma^{n}\epsilon^{n}\prod_{i=0}^{n-1}\left[(2i)\alpha\mu-1\right]}{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]}.
\end{aligned}
$$

Now, system (1) gives us that

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left[1-y_{4n-5}x_{4n-7}\right]} \\
&= \frac{\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}}{\frac{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[1-\frac{\alpha^{n-1}\mu^{n}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}\right]} \\
&= \frac{\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}}{\frac{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[1-\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\right]} \\
&= \frac{\alpha^{n}\mu^{n}\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\prod_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]}{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}\left[(2i+1)\alpha\mu-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]-\alpha\mu\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\right]}
\end{aligned}
$$

$$= -\frac{\alpha^n \mu^n \prod_{i=0}^{n-2} [(2i+2)\gamma\epsilon - 1]}{\gamma^n \epsilon^{n-1} \prod_{i=0}^{n-1} [(2i+1)\alpha\mu - 1]} = \frac{\alpha^n \mu^n \prod_{i=0}^{n-1} [(2i)\gamma\epsilon - 1]}{\gamma^n \epsilon^{n-1} \prod_{i=0}^{n-1} [(2i+1)\alpha\mu - 1]}.$$

Hence, the rest of the results can be similarly proved.

2.2 Second System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

Our leading duty in this subsection is to determine the solutions of the following discrete systems:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1 + x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1 + y_{n-1}x_{n-3}\right)}. \tag{2}$$

The initial values of this system are arbitrary real numbers.

Theorem 2 *Suppose that $\{x_n, y_n\}$ is a solution to system (2) and assume that $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ ...$ we have*

$$\begin{aligned} x_{4n-3} &= \frac{\gamma^n \epsilon^n}{\alpha^{n-1}\mu^n \left(\gamma\epsilon - 1\right)^n}, \quad x_{4n-2} = \frac{\delta^n \eta^n}{\beta^{n-1}\omega^n \left(\delta\eta - 1\right)^n}, \\ x_{4n-1} &= \frac{\gamma^{n+1}\epsilon^n \left(\alpha\mu - 1\right)^n}{\alpha^n \mu^n}, \quad x_{4n} = \frac{\delta^{n+1}\eta^n \left(\beta\omega - 1\right)^n}{\beta^n \omega^n}. \end{aligned}$$

And

$$\begin{aligned} y_{4n-3} &= \frac{\alpha^n \mu^n}{\gamma^n \epsilon^{n-1} \left(\alpha\mu - 1\right)^n}, \quad y_{4n-2} = \frac{\beta^n \omega^n}{\delta^n \eta^{n-1} \left(\beta\omega - 1\right)^n}, \\ y_{4n-1} &= \frac{\alpha^n \mu^{n+1} \left(\gamma\epsilon - 1\right)^n}{\gamma^n \epsilon^n}, \quad y_{4n} = \frac{\beta^n \omega^{n+1} \left(\delta\eta - 1\right)^n}{\delta^n \eta^n}. \end{aligned}$$

Proof. It is obvious that all solutions are satisfied for $n = 0$. Next, we suppose that $n > 1$ and assume that the solutions hold for $n - 1$. That is

$$\begin{aligned} x_{4n-7} &= \frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1} \left(\gamma\epsilon - 1\right)^{n-1}}, \quad x_{4n-6} = \frac{\delta^{n-1}\eta^{n-1}}{\beta^{n-2}\omega^{n-1} \left(\delta\eta - 1\right)^{n-1}}, \\ x_{4n-5} &= \frac{\gamma^n \epsilon^{n-1} \left(\alpha\mu - 1\right)^{n-1}}{\alpha^{n-1}\mu^{n-1}}, \quad x_{4n-4} = \frac{\delta^n \eta^{n-1} \left(\beta\omega - 1\right)^{n-1}}{\beta^{n-1}\omega^{n-1}}. \end{aligned}$$

And

$$\begin{aligned} y_{4n-7} &= \frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2} \left(\alpha\mu - 1\right)^{n-1}}, \quad y_{4n-6} = \frac{\beta^{n-1}\omega^{n-1}}{\delta^{n-1}\eta^{n-2} \left(\beta\omega - 1\right)^{n-1}}, \\ y_{4n-5} &= \frac{\alpha^{n-1}\mu^n \left(\gamma\epsilon - 1\right)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}, \quad y_{4n-4} = \frac{\beta^{n-1}\omega^n \left(\delta\eta - 1\right)^{n-1}}{\delta^{n-1}\eta^{n-1}}. \end{aligned}$$

We now turn to illustrate the first result. System (2) leads to

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(-1+x_{4n-5}y_{4n-7}\right)} \\ &= \frac{\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}}{\frac{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\left[-1+\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}\right]} \\ &= \frac{\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^{n-1}\left[-1+\gamma\epsilon\right]} = \frac{\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^n}. \end{aligned}$$

Similarly, it is easy to see from system (2) that

$$\begin{aligned} y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(-1+y_{4n-5}x_{4n-7}\right)} \\ &= \frac{\frac{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}}{\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\left[-1+\frac{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}\right]} \\ &= \frac{\alpha^n\mu^n}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n-1}\left[-1+\alpha\mu\right]} = \frac{\alpha^n\mu^n}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^n}. \end{aligned}$$

The remaining solutions of system (2) can be clearly justified in a similar technique. Thus, the proof is complete.

2.3 Third System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1-x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

The central point of this subsection is to resolve a system of fourth order rational recursive equations given by the form:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1-x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1+y_{n-1}x_{n-3}\right)}, \tag{3}$$

where the initial values are as described previously.

Theorem 3 *Let $\{x_n, y_n\}$ be a solution to system (3) and suppose that $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ \ldots$ we have*

$$\begin{aligned} x_{4n-3} &= \frac{\left(-1\right)^n\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod\limits_{i=0}^{n-1}\left[\left(2i+1\right)\gamma\epsilon-1\right]}, \quad x_{4n-2} = \frac{\left(-1\right)^n\delta^n\eta^n}{\beta^{n-1}\omega^n\prod\limits_{i=0}^{n-1}\left[\left(2i+1\right)\delta\eta-1\right]}, \\ x_{4n-1} &= \frac{\left(-1\right)^n\gamma^{n+1}\epsilon^n\left(\alpha\mu-1\right)^n}{\alpha^n\mu^n\prod\limits_{i=0}^{n-1}\left[\left(2i+2\right)\gamma\epsilon-1\right]}, \quad x_{4n} = \frac{\left(-1\right)^n\delta^{n+1}\eta^n\left(\beta\omega-1\right)^n}{\beta^n\omega^n\prod\limits_{i=0}^{n-1}\left[\left(2i+2\right)\delta\eta-1\right]}. \end{aligned}$$

And

$$y_{4n-3} = \frac{(-1)^n \alpha^n \mu^n \prod_{i=0}^{n-1} [(2i)\gamma\epsilon - 1]}{\gamma^n \epsilon^{n-1} (\alpha\mu - 1)^n}, \quad y_{4n-2} = \frac{(-1)^n \beta^n \omega^n \prod_{i=0}^{n-1} [(2i)\delta\eta - 1]}{\delta^n \eta^{n-1} (\beta\omega - 1)^n},$$
$$y_{4n-1} = \frac{(-1)^n \alpha^n \mu^{n+1} \prod_{i=0}^{n-1} [(2i+1)\gamma\epsilon - 1]}{\gamma^n \epsilon^n}, \quad y_{4n} = \frac{(-1)^n \beta^n \omega^{n+1} \prod_{i=0}^{n-1} [(2i+1)\delta\eta - 1]}{\delta^n \eta^n}.$$

Proof. The results are true for $n = 0$. Next, we suppose that $n > 1$ and assume that the relations hold for $n - 1$. That is

$$x_{4n-7} = \frac{(-1)^{n-1} \gamma^{n-1} \epsilon^{n-1}}{\alpha^{n-2} \mu^{n-1} \prod_{i=0}^{n-2} [(2i+1)\gamma\epsilon - 1]}, \quad x_{4n-6} = \frac{(-1)^{n-1} \delta^{n-1} \eta^{n-1}}{\beta^{n-2} \omega^{n-1} \prod_{i=0}^{n-2} [(2i+1)\delta\eta - 1]},$$
$$x_{4n-5} = \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} (\alpha\mu - 1)^{n-1}}{\alpha^{n-1} \mu^{n-1} \prod_{i=0}^{n-2} [(2i+2)\gamma\epsilon - 1]}, \quad x_{4n-4} = \frac{(-1)^{n-1} \delta^n \eta^{n-1} (\beta\omega - 1)^{n-1}}{\beta^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i+2)\delta\eta - 1]}.$$

And

$$y_{4n-7} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1} \prod_{i=0}^{n-2} [(2i)\gamma\epsilon - 1]}{\gamma^{n-1} \epsilon^{n-2} (\alpha\mu - 1)^{n-1}}, \quad y_{4n-6} = \frac{(-1)^{n-1} \beta^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i)\delta\eta - 1]}{\delta^{n-1} \eta^{n-2} (\beta\omega - 1)^{n-1}},$$
$$y_{4n-5} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^n \prod_{i=0}^{n-2} [(2i+1)\gamma\epsilon - 1]}{\gamma^{n-1} \epsilon^{n-1}}, \quad y_{4n-4} = \frac{(-1)^{n-1} \beta^{n-1} \omega^n \prod_{i=0}^{n-2} [(2i+1)\delta\eta - 1]}{\delta^{n-1} \eta^{n-1}}.$$

Now, we establish the proofs of two relations. Firstly, system (3) gives us that

$$x_{4n-3} = \frac{x_{4n-5} y_{4n-7}}{y_{4n-5} (1 - x_{4n-5} y_{4n-7})}$$
$$= \frac{\frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]} \frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}}{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}} \left[1 - \frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]} \frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}\right]}$$
$$= \frac{\frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}}{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}} \left[1 - \frac{\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\right]}$$

$$= \frac{(-1)^{-n+1}\gamma^n\epsilon^n\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]\left[\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]-\gamma\epsilon\prod_{i=0}^{n-2}[(2i)\gamma\epsilon-1]\right]}$$

$$= \frac{-(-1)^{-n+1}\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}[(2i+1)\gamma\epsilon-1]} = \frac{(-1)^n\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}[(2i+1)\gamma\epsilon-1]}.$$

Next, it can be noticed from system (3) that

$$y_{4n-3} = \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}(-1+y_{4n-5}x_{4n-7})}$$

$$= \frac{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}}{\frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[-1+\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}\right]}$$

$$= \frac{(-1)^{-n+1}\alpha^n\mu^n\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}[-1+\alpha\mu]} = \frac{-(-1)^{n-1}\alpha^n\mu^n\prod_{i=0}^{n-1}[(2i)\gamma\epsilon-1]}{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^n}$$

$$= \frac{(-1)^n\alpha^n\mu^n\prod_{i=0}^{n-1}[(2i)\gamma\epsilon-1]}{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^n}.$$

The proofs of the remaining relations can be likewise achieved. Therefore, they are omitted.

2.4 Fourth System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})}$

Our fundamental task in this subsection is to develop fractional solutions to the system of recursive equations given by the form:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})}, \tag{4}$$

where the initial conditions are required to be non-zero real numbers.

Theorem 4 *Assume that $\{x_n, y_n\}$ is a solution to system (4) and suppose that $x_{-3} = \alpha$, $x_{-2} = \beta$, $x_{-1} = \gamma$, $x_0 = \delta$, $y_{-3} = \epsilon$, $y_{-2} = \eta$, $y_{-1} = \mu$ and $y_0 = \omega$. Then, for $n = 0, 1, \ldots$ we have*

$$x_{4n-3} = \frac{(-1)^n\gamma^n\epsilon^n\prod_{i=0}^{n-1}[(2i)\alpha\mu-1]}{\alpha^{n-1}\mu^n(\gamma\epsilon-1)^n}, \quad x_{4n-2} = \frac{(-1)^n\delta^n\eta^n\prod_{i=0}^{n-1}[(2i)\beta\omega-1]}{\beta^{n-1}\omega^n(\delta\eta-1)^n},$$

$$x_{4n-1} = \frac{(-1)^n\gamma^{n+1}\epsilon^n\prod_{i=0}^{n-1}[(2i+1)\alpha\mu-1]}{\alpha^n\mu^n}, \quad x_{4n} = \frac{(-1)^n\delta^{n+1}\eta^n\prod_{i=0}^{n-1}[(2i+1)\beta\omega-1]}{\beta^n\omega^n}.$$

And

$$y_{4n-3} = \frac{(-1)^n \alpha^n \mu^n}{\gamma^n \epsilon^{n-1} \prod_{i=0}^{n-1} [(2i+1)\alpha\mu - 1]}, \quad y_{4n-2} = \frac{(-1)^n \beta^n \omega^n}{\delta^n \eta^{n-1} \prod_{i=0}^{n-1} [(2i+1)\beta\omega - 1]},$$

$$y_{4n-1} = \frac{(-1)^n \alpha^n \mu^{n+1} (\gamma\epsilon - 1)^n}{\gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i+2)\alpha\mu - 1]}, \quad y_{4n} = \frac{(-1)^n \beta^n \omega^{n+1} (\delta\eta - 1)^n}{\delta^n \eta^n \prod_{i=0}^{n-1} [(2i+2)\beta\omega - 1]}.$$

Proof. The relations hold for $n = 0$. Next, we let $n > 1$ and assume that the formulas hold for $n - 1$. That is

$$x_{4n-7} = \frac{(-1)^{n-1} \gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i)\alpha\mu - 1]}{\alpha^{n-2} \mu^{n-1} (\gamma\epsilon - 1)^{n-1}}, \quad x_{4n-6} = \frac{(-1)^{n-1} \delta^{n-1} \eta^{n-1} \prod_{i=0}^{n-2} [(2i)\beta\omega - 1]}{\beta^{n-2} \omega^{n-1} (\delta\eta - 1)^{n-1}},$$

$$x_{4n-5} = \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}}, \quad x_{4n-4} = \frac{(-1)^{n-1} \delta^n \eta^{n-1} \prod_{i=0}^{n-2} [(2i+1)\beta\omega - 1]}{\beta^{n-1} \omega^{n-1}}.$$

And

$$y_{4n-7} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}, \quad y_{4n-6} = \frac{(-1)^{n-1} \beta^{n-1} \omega^{n-1}}{\delta^{n-1} \eta^{n-2} \prod_{i=0}^{n-2} [(2i+1)\beta\omega - 1]},$$

$$y_{4n-5} = \frac{(-1)^{n-1} \alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1}}{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]}, \quad y_{4n-4} = \frac{(-1)^{n-1} \beta^{n-1} \omega^n (\delta\eta - 1)^{n-1}}{\delta^{n-1} \eta^{n-1} \prod_{i=0}^{n-2} [(2i+2)\beta\omega - 1]}.$$

We now turn to verify the proof of two relations. It can be obviously seen from system (4) that

$$\begin{aligned} x_{4n-3} &= \frac{x_{4n-5} y_{4n-7}}{y_{4n-5} (-1 + x_{4n-5} y_{4n-7})} \\ &= \frac{\frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}} \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}}{\frac{(-1)^{n-1} \alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1}}{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]} \left[-1 + \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}} \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]} \right]} \\ &= \frac{(-1)^{-n+1} \gamma^n \epsilon^n \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1} [-1 + \gamma\epsilon]} = \frac{-(-1)^{n-1} \gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i)\alpha\mu - 1]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^n} \\ &= \frac{(-1)^n \gamma^n \epsilon^n \prod_{i=0}^{n-1} [(2i)\alpha\mu - 1]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^n}. \end{aligned}$$

Further, it can be attained from system (4) that

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(1-y_{4n-5}x_{4n-7}\right)} \\
&= \frac{\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}}{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}}\left[1-\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}\right]} \\
&= \frac{\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}}{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}}\left[1-\frac{\alpha\mu\prod_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\right]} \\
&= \frac{(-1)^{-n+1}\alpha^{n}\mu^{n}\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]}{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-2}\left[(2i+1)\alpha\mu-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]-\alpha\mu\prod_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\right]} \\
&= \frac{(-1)^{n}\alpha^{n}\mu^{n}}{\gamma^{n}\epsilon^{n-1}\prod_{i=0}^{n-1}\left[(2i+1)\alpha\mu-1\right]}.
\end{aligned}
$$

Other results can be proved in a similar way. Thus, the remaining proofs are omitted.

2.5 Numerical Examples

This subsection aims to present graphical confirmations to the whole solutions obtained in the previous subsections. Here, we plot the solutions (by using MATLAB software) under specific selections of some initial conditions.

Example 1. This example shows the paths of the solutions of system (1). The initial conditions of this example are given as follows: $x_{-3}=3,\ x_{-2}=1,\ x_{-1}=5,\ x_{0}=2,\ y_{-3}=1,\ y_{-2}=3,\ y_{-1}=5$ and $y_{0}=5$. See Figure 1.

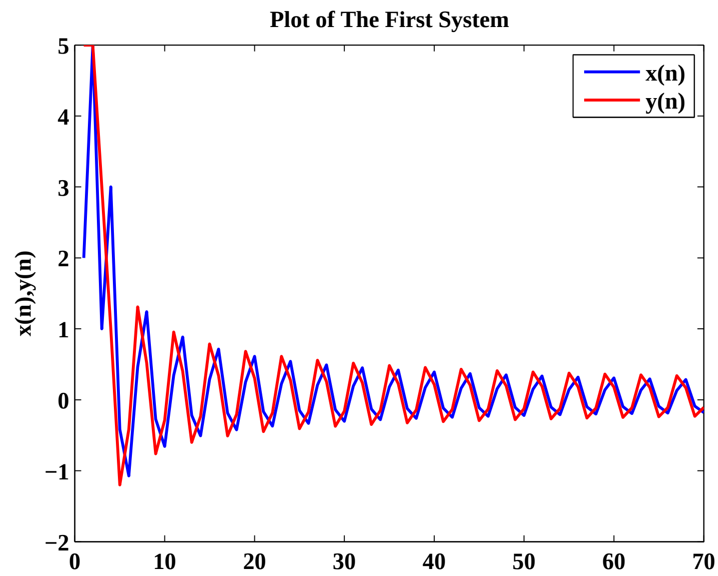

Figure 1: The behaviour of the solution of system (1).

Example 2. In Figure 2, we illustrate the behaviour of the solution of system (2) under the following selection of initial conditions: $x_{-3} = 3.4,\ x_{-2} = 0.7,\ x_{-1} = 2,\ x_0 = 3,\ y_{-3} = 1.5,\ y_{-2} = 1.5,\ y_{-1} = 0.5$ and $y_0 = 1.22$.

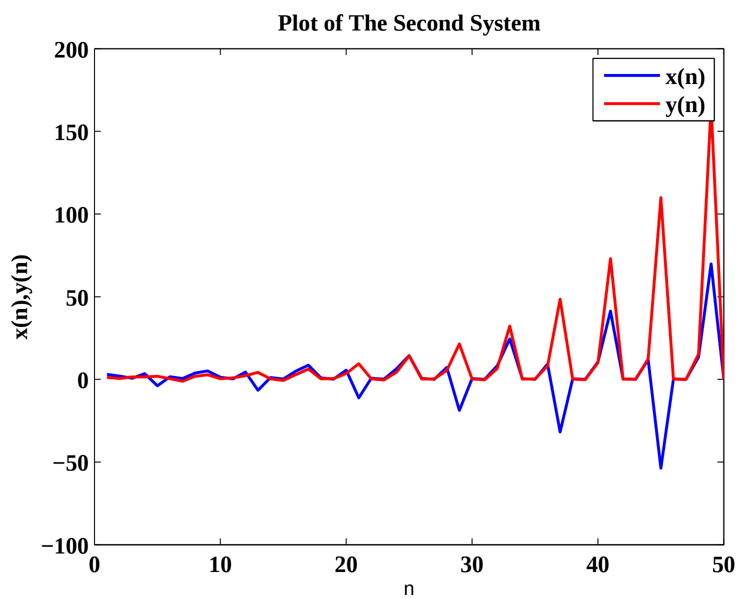

Figure 2: The behaviour of the solution of system (2).

Example 3. Figure 3 illustrates the curves of the solutions of system (3) when we assume that $x_{-3} = 0.7,\ x_{-2} = 2.1,\ x_{-1} = 1,\ x_0 = 0.5,\ y_{-3} = 0.1,\ y_{-2} = 0.2,\ y_{-1} = 2.2$ and $y_0 = 0.5$.

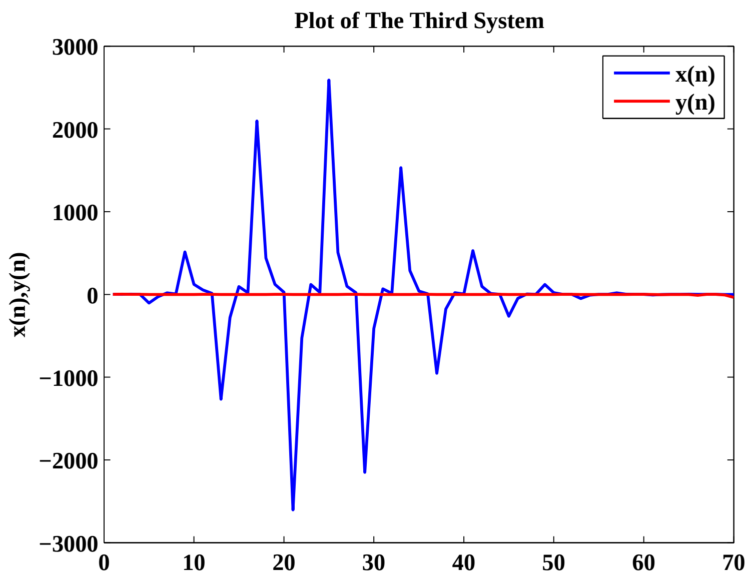

Figure 3: The behaviour of the solution of system (3).

Example 4. The solutions of system (4) are depicted in Figure 4 under the following initial data: $x_{-3} = 0.2,\ x_{-2} = 1,\ x_{-1} = 0.3,\ x_0 = 0.2,\ y_{-3} = 3,\ y_{-2} = 1,\ y_{-1} = 2$ and $y_0 = 0.3$.

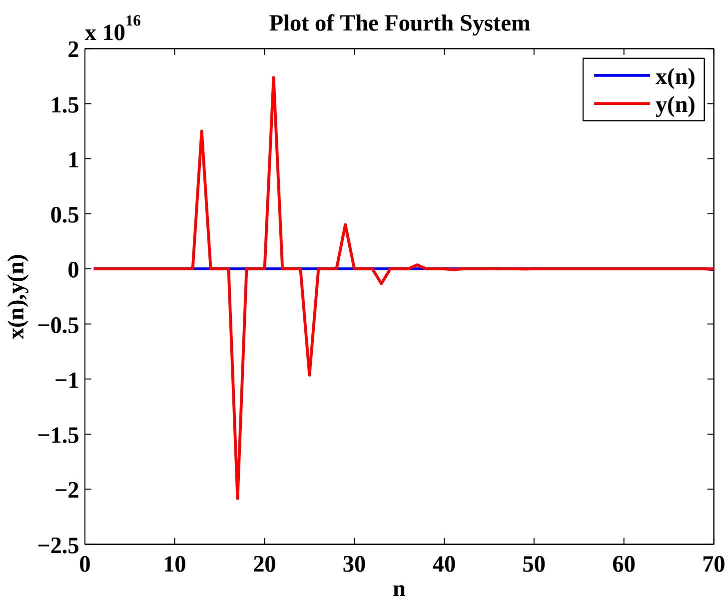

Figure 4: The behaviour of the solution of system (4).

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