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HERMITE-HADAMARD TYPE INEQUALITIES FOR THE ABK-FRACTIONAL INTEGRALS

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Abstract. The author introduced the new fractional integral operator called ABK-fractional integral and proved four identities for this type. By applying the established identities, some integral inequalities connected with the right hand side of the Hermite-Hadamard type inequalities for the ABK-fractional integrals are given. Various special cases have been identified. The ideas of this paper may stimulate further research in the field of integral inequalities.

1. Introduction

The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda e_1 + (1-\lambda)e_2) \leq \lambda f(e_1) + (1-\lambda) f(e_2) \tag{1.1}$$

holds for all $e_1, e_2 \in I$ and $\lambda \in [0,1]$. Also, we say that f is concave, if the inequality in (1.1) holds in the reverse direction.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $e_1, e_2 \in I$ with $e_1 < e_2$. Then the following inequality holds:*

$$f\left(\frac{e_1+e_2}{2}\right) \leq \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx \leq \frac{f(e_1)+f(e_2)}{2}. \tag{1.2}$$

This inequality (1.2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [2],[4]-[20],[22]-[27].

In [8], Dragomir and Agarwal proved the following results connected with the right part of (1.2).

Lemma 1.3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $e_1, e_2 \in I^\circ$ with $e_1 < e_2$. If $f' \in L[e_1, e_2]$, then the following equality holds:*

$$\frac{f(e_1)+f(e_2)}{2} - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx = \frac{(e_2-e_1)}{2}\int_0^1 (1-2t)f'(te_1+(1-t)e_2)dt. \tag{1.3}$$

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Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $e_1, e_2 \in I^\circ$ with $e_1 < e_2$. If $|f'|$ is convex on $[e_1, e_2]$, then the following inequality holds:*

$$\left| \frac{f(e_1) + f(e_2)}{2} - \frac{1}{e_2 - e_1} \int_{e_1}^{e_2} f(x)dx \right| \leq \frac{(e_2 - e_1)}{8} \left(|f'(e_1)| + |f'(e_2)| \right). \tag{1.4}$$

Now, let us recall the following definitions.

Definition 1.5. $X_c^p(e_1, e_2)$ $(c \in \mathbb{R})$, $1 \leq p \leq \infty$ denotes the space of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm $\|\cdot\|_{X_c^p}$ is defined by

$$\|f\|_{X_c^p} = \left(\int_{e_1}^{e_2} \left| t^c f(t) \right|^p \frac{dt}{t} \right)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{e_1 \leq t \leq e_2} \left| t^c f(t) \right|.$$

Recently, in [12], Katugampola introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows:

Definition 1.6. Let $[e_1, e_2] \subset \mathbb{R}$ be a finite interval. Then, the left and right side Katugampola fractional integrals of order $\alpha\,(> 0)$ of $f \in X_c^p(e_1, e_2)$ are defined by

$${}^\rho I_{e_1^+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{e_1}^{x} \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt, \quad x > e_1 \tag{1.5}$$

and

$${}^\rho I_{e_2^-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{e_2} \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt, \quad x < e_2, \tag{1.6}$$

where $\rho > 0$, if the integrals exist.

In [3], Atangana and Baleanu produced two new fractional derivatives based on the Caputo and the Riemann-Liouville definitions of fractional order derivatives. They declared that their fractional derivative has a fractional integral as the antiderivative of their operators. The Atangana-Baleanu (AB) fractional order derivative is known to possess nonsingularity as well as nonlocality of the kernel, which adopts the generalized Mittag-Leffler function, see [15],[21].

Definition 1.7. The fractional AB-integral of the function $f \in H^*(e_1, e_2)$ is given by

$${}_{e_1}^{AB}I_t^\nu f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{t} (t-u)^{\nu-1} f(u)du, \quad t > e_1, \tag{1.7}$$

where $e_1 < e_2$, $0 < \nu < 1$ and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Similarly, we give the definition of the (1.7) opposite side is given by

$${}_{e_2}^{AB}I_t^\nu f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{t}^{e_2} (u-t)^{\nu-1} f(u)du, \quad t < e_2.$$

Here, $\Gamma(\nu)$ is the Gamma function. Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the fractional AB-integral of a positive function is positive. It should be noted that, when the order $\nu \to 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\nu \to 0$.

Motivated by the above literatures, the main objective of this paper is to establish some new estimates for the right hand side of Hermite-Hadamard type integral inequalities for new fractional integral operator called the ABK-fractional integral operator. Various special cases will be identified. The ideas of this paper may stimulate further research in the field of integral inequalities.

2. Hermite-Hadamard inequalities for ABK-fractional integrals

Now, we are in position to introduce the left and right side ABK-fractional integrals as follows.

Definition 2.1. Let $[e_1, e_2] \subset \mathbb{R}$ be a finite interval. Then, the left and right side ABK-fractional integrals of order $\nu \in (0,1)$ of $f \in X_c^p(e_1, e_2)$ are defined by

$$ {}^{ABK}_{e_1^+}{}^{\rho}I_t^{\nu} f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\rho^{1-\nu}\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{t} \frac{u^{\rho-1}}{(t^\rho - u^\rho)^{1-\nu}} f(u)du, \quad t > e_1 \geq 0 \tag{2.1} $$

and

$$ {}^{ABK}_{e_2^-}{}^{\rho}I_t^{\nu} f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\rho^{1-\nu}\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{t}^{e_2} \frac{u^{\rho-1}}{(u^\rho - t^\rho)^{1-\nu}} f(u)du, \quad t < e_2, \tag{2.2} $$

where $\rho > 0$ and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Remark 2.2. Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the fractional ABK-integral of a positive function is positive. It should be noted that, when the $\rho \to 1$, we recover the AB-fractional integral. Also, using the same idea as in [12], the ABK-fractional integral operators are well-defined on $X_c^p(e_1, e_2)$. Finally, using the same idea as in [1], the interested reader can find new nonlocal fractional derivative of it with Mittag-Leffler nonsingular kernel, several formulae and many applications.

Let represent Hermite-Hadamard's inequalities in the ABK-fractional integral forms as follows:

Theorem 2.3. *Let $\nu \in (0,1)$ and $\rho > 0$. Let $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a function with $0 \leq e_1 < e_2$ and $f \in X_c^p(e_1^\rho, e_2^\rho)$. If f is a convex function on $[e_1^\rho, e_2^\rho]$, then the following inequalities for the ABK-fractional integrals hold:*

$$ \begin{aligned} & \frac{2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)\rho^{2-\nu}} f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1^\rho) + f(e_2^\rho)\right] \\ \leq\ & \left[{}^{ABK}_{e_1^+}{}^{\rho}I_{e_2^\rho}^{\nu} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I_{e_1^\rho}^{\nu} f(e_1^\rho)\right] \\ \leq\ & \left(\frac{(e_2^\rho - e_1^\rho)^\nu + \rho(1-\nu)\Gamma(\nu)}{\rho\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right]. \end{aligned} \tag{2.3} $$

Proof. Let $t \in [0,1]$. Consider $x^\rho, y^\rho \in [e_1^\rho, e_2^\rho]$, defined by $x^\rho = t^\rho e_1^\rho + (1-t^\rho)e_2^\rho$, $y^\rho = (1-t^\rho)e_1^\rho + t^\rho e_2^\rho$. Since f is a convex function on $[e_1^\rho, e_2^\rho]$, we have

$$ f\left(\frac{x^\rho + y^\rho}{2}\right) \leq \frac{f(x^\rho) + f(y^\rho)}{2}. $$

Then, we get

$$ 2f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) \leq f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) + f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right). \tag{2.4} $$

Multiplying both sides of (2.4) by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} t^{\rho\nu-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$ \begin{aligned} & \frac{2}{\rho\mathbb{B}(\nu)\Gamma(\nu)} f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) \\ \leq\ & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) dt \\ =\ & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{e_2} \left(\frac{e_2^\rho - x^\rho}{e_2^\rho - e_1^\rho}\right)^{\nu-1} f(x^\rho) \frac{x^{\rho-1}}{e_2^\rho - e_1^\rho} dx \end{aligned} $$

$$+ \quad \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_{e_1}^{e_2}\left(\frac{y^\rho - e_1^\rho}{e_2^\rho - e_1^\rho}\right)^{\nu-1} f(y^\rho)\frac{y^{\rho-1}}{e_2^\rho - e_1^\rho}dy$$

Therefore, it follows that

$$\frac{2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)\rho^{2-\nu}} f\left(\frac{e_1^\rho + e_2^\rho}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1^\rho) + f(e_2^\rho)\right]$$
$$\leq \left[{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right]$$

and the left hand side inequality of (2.3) is proved.
For the proof of the right hand side inequality of (2.3) we first note that if f is a convex function, then

$$f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) \leq t^\rho f(e_1^\rho) + (1-t^\rho) f(e_2^\rho)$$

and

$$f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) \leq (1-t^\rho) f(e_1^\rho) + t^\rho f(e_2^\rho).$$

By adding these inequalities, we have

$$f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) + f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) \leq f(e_1^\rho) + f(e_2^\rho). \tag{2.5}$$

Then multiplying both sides of (2.5) by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}t^{\rho\nu-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} f\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)dt + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} f\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)dt$$

$$\leq \quad \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\left[f(e_1^\rho) + f(e_2^\rho)\right]\int_0^1 t^{\rho\nu-1}dt$$

i.e.

$$\left[{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right] \quad \leq \quad \left(\frac{(e_2^\rho - e_1^\rho)^\nu + \rho(1-\nu)\Gamma(\nu)}{\rho\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right].$$

The proof of this theorem is complete. □

Corollary 2.4. *If we take $\rho \to 1$ in Theorem 2.3, then the following Hermite-Hadamard's inequalities for the AB-fractional integrals hold:*

$$\begin{aligned}
&\frac{2\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)} f\left(\frac{e_1 + e_2}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1) + f(e_2)\right] \\
\leq &\left[{}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\right] \\
\leq &\left(\frac{(e_2 - e_1)^\nu + (1-\nu)\Gamma(\nu)}{\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1) + f(e_2)\right].
\end{aligned} \tag{2.6}$$

Remark 2.5. If in Corollary 2.4, we let $\nu \to 1$, then the inequalities (2.6) become the inequalities (1.2).

3. The ABK-fractional inequalities for convex functions

For establishing some new results regarding the right side of Hermite-Hadamard type inequalities for the ABK-fractional integrals we need to prove the following four lemmas.

Lemma 3.1. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right] - \left[{}^{ABK\,\rho}_{e_1^+}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-}I^\nu_{e_1^\rho} f(e_1^\rho)\right]$$

$$= \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 [(1-t^\rho)^\nu - t^{\rho\nu}]\, t^{\rho-1} f'(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt. \tag{3.1}$$

Proof. Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\rho)^\nu t^{\rho-1} f'(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt \\ &= \frac{(1-t^\rho)^\nu}{\rho(e_1^\rho - e_2^\rho)} f(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\Big|_0^1 - \frac{\nu}{e_1^\rho - e_2^\rho} \int_0^1 (1-t^\rho)^{\nu-1} t^{\rho-1} f(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt \\ &= \frac{f(e_2^\rho)}{\rho(e_2^\rho - e_1^\rho)} - \frac{\nu}{e_1^\rho - e_2^\rho} \int_0^1 (1-t^\rho)^{\nu-1} t^{\rho-1} f(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 t^{\rho(\nu+1)-1} f'(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt \\ &= \frac{t^{\rho(\nu+1)-1}}{\rho(e_1^\rho - e_2^\rho)} f(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\Big|_0^1 - \frac{\nu}{e_1^\rho - e_2^\rho} \int_0^1 t^{\rho(\nu+1)} f(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt \\ &= -\frac{f(e_1^\rho)}{\rho(e_2^\rho - e_1^\rho)} - \frac{\nu}{e_1^\rho - e_2^\rho} \int_0^1 t^{\rho(\nu+1)} f(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho)\, dt. \end{aligned}$$

Thus, by multiplying I_1 and I_2 with $\frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}$, using definition of the ABK-fractional integrals and subtracting them, we get the result. □

Remark 3.2. If in Lemma 3.1, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned} &\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[\, {}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\ &= \frac{(e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 [(1-t)^\nu - t^\nu]\, f'(te_1 + (1-t)e_2)\, dt. \end{aligned} \tag{3.2}$$

Remark 3.3. If in Lemma 3.1, we let $\rho, \nu \to 1$, then we obtain the equality (1.3).

Lemma 3.4. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned} &\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[\, {}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho)\right] \\ &= \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho(\nu+1)-1} \left[f'((1-t^\rho)e_1^\rho + t^\rho e_2^\rho) - f'(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)\right] dt. \end{aligned} \tag{3.3}$$

Proof. The proof is similarly as Lemma 3.1, so we omit it. □

Remark 3.5. If in Lemma 3.4, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned} &\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[\, {}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\ &= \frac{(e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^\nu \left[f'((1-t)e_1 + te_2) - f'(te_1 + (1-t)e_2)\right] dt. \end{aligned} \tag{3.4}$$

Lemma 3.6. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned}&\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right] \\ =\ & \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{ \int_0^1 \left[1 - t^{\rho(\nu+1)}\right] t^{\rho-1} f''\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) dt \right. \\ -\ & \left. \int_0^1 t^{\rho(\nu+2)-1} f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt \right\}.\end{aligned}$$

Proof. By using twice integration by parts the proof is similarly as Lemma 3.1, so we omit it. □

Remark 3.7. If in Lemma 3.6, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned}&\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\ =\ & \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \\ \times\ & \left\{ \int_0^1 \left[1 - t^{\nu+1}\right] f''\left((1-t)e_1 + te_2\right) dt - \int_0^1 t^{\nu+1} f''\left(te_1 + (1-t)e_2\right) dt \right\}.\end{aligned}$$

Lemma 3.8. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned}&\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right] \\ =\ & \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu+2)} \int_0^1 \left[1 - (1-t^\rho)^{\nu+1} - t^{\rho(\nu+1)}\right] t^{\rho-1} f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt. \quad (3.5)\end{aligned}$$

Proof. By using twice integration by parts and Lemma 3.1, we get the desired result. □

Remark 3.9. If in Lemma 3.8, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned}&\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1)\right] \\ =\ & \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \int_0^1 \left[1 - (1-t)^{\nu+1} - t^{\nu+1}\right] f''\left(te_1 + (1-t)e_2\right) dt. \quad (3.6)\end{aligned}$$

Using Lemmas 3.1, 3.4, 3.6 and 3.8, we can obtain the following the ABK-fractional integral inequalities.

Theorem 3.10. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^\rho I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^\rho I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\ \le\ & \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[p]{D(p,\rho,\nu)} \sqrt[q]{\frac{\left|f'(e_1^\rho)\right|^q + \left|f'(e_2^\rho)\right|^q}{2}}, \quad (3.7)\end{aligned}$$

where

$$D(p,\rho,\nu):=\int_0^{\frac{1}{2}}\Big[(1-t^\rho)^{p\nu}-t^{p\rho\nu}\Big]t^{\rho-1}dt+\int_{\frac{1}{2}}^1\Big[t^{p\rho\nu}-(1-t^\rho)^{p\nu}\Big]t^{\rho-1}dt$$
$$=\frac{2}{\rho(p\nu+1)}\left\{1-\left(1-\frac{1}{2^\rho}\right)^{p\nu+1}-\frac{1}{2^{\rho(p\nu+1)}}\right\}.$$

Proof. Using Lemma 3.1, convexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho}f(e_2^\rho)+\,{}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|$$
$$\le\frac{\nu(e_2^\rho-e_1^\rho)^{\nu+1}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}$$
$$\times\left(\int_0^1\Big|(1-t^\rho)^\nu-t^{\rho\nu}\Big|^pt^{\rho-1}dt\right)^{\frac{1}{p}}\left(\int_0^1t^{\rho-1}\Big|f'\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)\Big|^qdt\right)^{\frac{1}{q}}$$
$$\le\frac{\nu(e_2^\rho-e_1^\rho)^{\nu+1}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}\left(\int_0^{\frac{1}{2}}\Big[(1-t^\rho)^{p\nu}-t^{p\rho\nu}\Big]t^{\rho-1}dt+\int_{\frac{1}{2}}^1\Big[t^{p\rho\nu}-(1-t^\rho)^{p\nu}\Big]t^{\rho-1}dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1t^{\rho-1}\left(t^\rho\big|f'(e_1^\rho)\big|^q+(1-t^\rho)\big|f''(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{\nu(e_2^\rho-e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)}\times\sqrt[p]{D(p,\rho,\nu)}\sqrt[q]{\frac{\big|f'(e_1^\rho)\big|^q+\big|f'(e_2^\rho)\big|^q}{2}}.$$

The proof of this theorem is complete. □

Corollary 3.11. *With the notations in Theorem 3.10, if we take $|f'|\le K$, the following inequality for the ABK-fractional integrals holds:*

$$\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho}f(e_2^\rho)+\,{}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|$$
$$\le\frac{\nu K(e_2^\rho-e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)}\times\sqrt[p]{D(p,\rho,\nu)}. \tag{3.8}$$

Corollary 3.12. *With the notations in Theorem 3.10, if we take $\rho\to1$, the following inequality for the AB-fractional integrals holds:*

$$\left|\left(\frac{(e_2-e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)]-\left[\,{}^{AB}_{e_1}I^\nu_{e_2}f(e_2)+\,{}^{AB}_{e_2}I^\nu_{e_1}f(e_1)\right]\right|$$
$$\le\frac{\nu(e_2-e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}\times\sqrt[p]{D(p,1,\nu)}\sqrt[q]{\frac{\big|f'(e_1)\big|^q+\big|f'(e_2)\big|^q}{2}}. \tag{3.9}$$

Theorem 3.13. *Let $\nu\in(0,1)$ and $\rho>0$ and $f:[e_1^\rho,e_2^\rho]\to\mathbb{R}$ be a differentiable mapping on (e_1^ρ,e_2^ρ) with $0\le e_1<e_2$. If $|f'|^q$ is convex on $[e_1^\rho,e_2^\rho]$ for $q\ge1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho}f(e_2^\rho)+\,{}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|$$
$$\le\frac{\nu(e_2^\rho-e_1^\rho)^{\nu+1}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}\left[D(1,\rho,\nu)\right]^{1-\frac{1}{q}} \tag{3.10}$$

$$\times \quad \Big\{E(\rho,\nu)\big|f'(e_1^{\rho})\big|^q + (F(\rho,\nu)-E(\rho,\nu))\big|f'(e_2^{\rho})\big|^q$$

$$+ \quad G(\rho,\nu)\big|f'(e_1^{\rho})\big|^q + (F(\rho,\nu)-G(\rho,\nu))\big|f'(e_2^{\rho})\big|^q\Big\}^{\frac{1}{q}},$$

where

$$E(\rho,\nu) := \int_0^{\frac{1}{2}} \Big[(1-t^{\rho})^{\nu} - t^{\rho\nu}\Big] t^{2\rho-1}dt = \frac{1}{\rho}\left[\beta\left(\frac{1}{2^{\rho}};2,\nu+1\right) - \frac{1}{2^{\rho(\nu+2)}(\nu+2)}\right];$$

$$F(\rho,\nu) := \int_0^{\frac{1}{2}} \Big[(1-t^{\rho})^{\nu} - t^{\rho\nu}\Big] t^{\rho-1}dt = \int_{\frac{1}{2}}^1 \Big[t^{\rho\nu} - (1-t^{\rho})^{\nu}\Big] t^{\rho-1}dt$$

$$= \frac{1}{\rho(\nu+1)}\left[1 - \left(1-\frac{1}{2^{\rho}}\right)^{\nu+1} - \frac{1}{2^{\rho(\nu+1)}}\right];$$

$$G(\rho,\nu) := \int_{\frac{1}{2}}^1 \Big[t^{\rho\nu} - (1-t^{\rho})^{\nu}\Big] t^{2\rho-1}dt = \frac{1}{\rho}\left[\frac{1-\frac{1}{2^{\rho(\nu+2)}}}{\nu+2} + \beta\left(\frac{1}{2^{\rho}};2,\nu+1\right) - \beta(2,\nu+1)\right],$$

where $\beta(\cdot\,;\cdot,\cdot)$, $\beta(\cdot,\cdot)$ are respectively the incomplete and complete beta functions and $D(1,\rho,\nu)$ is defined as in Theorem 3.10 for value $p=1$.

Proof. Using Lemma 3.1, convexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left|\left(\frac{(e_2^{\rho}-e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^{\rho})+f(e_2^{\rho})] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho}) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\right]\right|$$

$$\leq \frac{\nu(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}\left(\int_0^1 \Big|(1-t^{\rho})^{\nu} - t^{\rho\nu}\Big|t^{\rho-1}dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \Big|(1-t^{\rho})^{\nu} - t^{\rho\nu}\Big|t^{\rho-1}\Big|f'\left(t^{\rho}e_1^{\rho} + (1-t^{\rho})e_2^{\rho}\right)\Big|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\nu(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}[D(1,\rho,\nu)]^{1-\frac{1}{q}}$$

$$\times \left\{\int_0^{\frac{1}{2}} \Big[(1-t^{\rho})^{\nu} - t^{\rho\nu}\Big]t^{\rho-1}\left(t^{\rho}\big|f'(e_1^{\rho})\big|^q + (1-t^{\rho})\big|f'(e_2^{\rho})\big|^q\right)dt\right.$$

$$\left. + \int_{\frac{1}{2}}^1 \Big[t^{\rho\nu} - (1-t^{\rho})^{\nu}\Big]t^{\rho-1}\left(t^{\rho}\big|f'(e_1^{\rho})\big|^q + (1-t^{\rho})\big|f'(e_2^{\rho})\big|^q\right)dt\right\}^{\frac{1}{q}}$$

$$= \frac{\nu(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}[D(1,\rho,\nu)]^{1-\frac{1}{q}}$$

$$\times \Big\{E(\rho,\nu)\big|f'(e_1^{\rho})\big|^q + (F(\rho,\nu)-E(\rho,\nu))\big|f'(e_2^{\rho})\big|^q$$

$$+ \quad G(\rho,\nu)\big|f'(e_1^{\rho})\big|^q + (F(\rho,\nu)-G(\rho,\nu))\big|f'(e_2^{\rho})\big|^q\Big\}^{\frac{1}{q}}.$$

The proof of this theorem is complete. □

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Corollary 3.14. *With the notations in Theorem 3.13, if we take $|f'| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\, {}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right|$$
$$\leq \frac{\nu K (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}\left[D(1,\rho,\nu)\right]. \tag{3.11}$$

Corollary 3.15. *With the notations in Theorem 3.13, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left|\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)] - \left[\, {}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\right]\right|$$
$$\leq \frac{\nu (e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left[D(1,1,\nu)\right]^{1-\frac{1}{q}}$$
$$\times \left\{E(1,\nu)\big|f'(e_1)\big|^q + (F(1,\nu)-E(1,\nu))\big|f'(e_2)\big|^q \right. \tag{3.12}$$
$$\left. + \quad G(1,\nu)\big|f'(e_1)\big|^q + (F(1,\nu)-G(1,\nu))\big|f'(e_2)\big|^q\right\}^{\frac{1}{q}}. \tag{3.13}$$

Theorem 3.16. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\, {}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right|$$
$$\leq \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}\frac{1}{\sqrt[q]{\rho+1}} \tag{3.14}$$
$$\times\left\{\sqrt[q]{|f'(e_1^\rho)|^q + \rho|f'(e_2^\rho)|^q} + \sqrt[q]{\rho|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q}\right\}.$$

Proof. Using Lemma 3.4, convexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[\, {}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right|$$
$$\leq \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{p(\rho(\nu+1)-1)}dt\right)^{\frac{1}{p}}$$
$$\times \left\{\left(\int_0^1 \left|f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|f'\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{p(\rho(\nu+1)-1)}dt\right)^{\frac{1}{p}}$$
$$\times \left\{\left(\int_0^1 \left(t^\rho\big|f'(e_1^\rho)\big|^q + (1-t^\rho)\big|f'(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 \left((1-t^\rho)\big|f'(e_1^\rho)\big|^q + t^\rho\big|f'(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}\right\}$$

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$$= \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}} \frac{1}{\sqrt[q]{\rho+1}}$$
$$\times\left\{ \sqrt[q]{|f'(e_1^\rho)|^q + \rho|f'(e_2^\rho)|^q} + \sqrt[q]{\rho|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.17. *With the notations in Theorem 3.16, if we take $|f'| \le K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{2K(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}. \tag{3.15}$$

Corollary 3.18. *With the notations in Theorem 3.16, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$
$$\le \frac{2(e_2 - e_1)^{\nu+1}}{\sqrt[p]{p\nu+1}\mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[q]{\frac{|f'(e_1)|^q + |f'(e_2)|^q}{2}}. \tag{3.16}$$

Theorem 3.19. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \ge 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \sqrt[q]{\nu+2}\mathbb{B}(\nu)\Gamma(\nu+2)} \tag{3.17}$$
$$\times\left\{ \sqrt[q]{|f'(e_1^\rho)|^q + (\nu+1)|f'(e_2^\rho)|^q} + \sqrt[q]{(\nu+1)|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

Proof. Using Lemma 3.4, convexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{e_1^+} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{e_2^-} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{\rho(\nu+1)-1} dt \right)^{1-\frac{1}{q}}$$
$$\times \left\{ \left(\int_0^1 t^{\rho(\nu+1)-1} \left| f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}} \right.$$
$$+ \left. \left(\int_0^1 t^{\rho(\nu+1)-1} \left| f'\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}} \right\}$$
$$\le \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{\rho(\nu+1)-1} dt \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\int_0^1 t^{\rho(\nu+1)-1} \left(t^\rho \left|f'(e_1^\rho)\right|^q + (1-t^\rho)\left|f'(e_2^\rho)\right|^q \right) dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 t^{\rho(\nu+1)-1} \left((1-t^\rho)\left|f'(e_1^\rho)\right|^q + t^\rho \left|f'(e_2^\rho)\right|^q \right) dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \sqrt[q]{\nu+2}\mathbb{B}(\nu)\Gamma(\nu+2)}$$

$$\times \left\{ \sqrt[q]{|f'(e_1^\rho)|^q + (\nu+1)|f'(e_2^\rho)|^q} + \sqrt[q]{(\nu+1)|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.20. *With the notations in Theorem 3.19, if we take $|f'| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{2\nu K \left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)}. \tag{3.18}$$

Corollary 3.21. *With the notations in Theorem 3.19, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$

$$\leq \frac{\nu \left(e_2 - e_1\right)^{\nu+1}}{\sqrt[q]{\nu+2}\mathbb{B}(\nu)\Gamma(\nu+2)} \tag{3.19}$$

$$\times \left\{ \sqrt[q]{|f'(e_1)|^q + (\nu+1)|f'(e_2)|^q} + \sqrt[q]{(\nu+1)|f'(e_1)|^q + |f'(e_2)|^q} \right\}.$$

Theorem 3.22. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + |f''(e_2^\rho)|^q}{2}} \right. \tag{3.20}$$

$$\left. + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + \rho|f''(e_2^\rho)|^q}{\rho+1}} \right\}.$$

Proof. Using Lemma 3.6, convexity of $|f''|^q$, Hölder inequality and properties of the modulus, we have

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu+2)}$$

$$\times \left\{ \left(\int_0^1 \left| 1 - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left| f''\left((1-t^\rho) e_1^\rho + t^\rho e_2^\rho \right) \right|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 t^{p(\rho(\nu+2)-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f''\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho \right) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho \right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu) \Gamma(\nu+2)}$$

$$\times \left\{ \left(\int_0^1 \left| 1 - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left((1-t^\rho) \left| f''(e_1^\rho) \right|^q + t^\rho \left| f''(e_2^\rho) \right|^q \right) dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 t^{p(\rho(\nu+2)-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left(t^\rho \left| f''(e_1^\rho) \right|^q + (1-t^\rho) \left| f''(e_2^\rho) \right|^q \right) dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\nu \left(e_2^\rho - e_1^\rho \right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu) \Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + |f''(e_2^\rho)|^q}{2}} \right.$$

$$\left. + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + \rho |f''(e_2^\rho)|^q}{\rho+1}} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.23. *With the notations in Theorem 3.22, if we take $|f''| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho \right)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho) \right] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu K \left(e_2^\rho - e_1^\rho \right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu) \Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \right\}. \tag{3.21}$$

Corollary 3.24. *With the notations in Theorem 3.22, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2 - e_1 \right)^\nu}{\mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1) + f(e_2) \right] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$

$$\leq \frac{\nu \left(e_2 - e_1 \right)^{\nu+2}}{\mathbb{B}(\nu) \Gamma(\nu+2)} \times \frac{\left[\sqrt[p]{p(\nu+1)} + 1 \right]}{\sqrt[p]{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1)|^q + |f''(e_2)|^q}{2}} \tag{3.22}$$

Theorem 3.25. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \leq e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \geq 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho \right)^\nu}{\rho^\nu \mathbb{B}(\nu) \Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho) \right] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho \right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu) \Gamma(\nu+2)} \tag{3.23}$$

$$\times \left\{ \left(\frac{\nu+1}{\rho(\nu+2)} \right)^{1-\frac{1}{q}} \sqrt[q]{\frac{(\nu+1)(\nu+4)}{2\rho(\nu+2)(\nu+3)} \left| f''(e_1^\rho) \right|^q + \frac{(\nu+1)}{2\rho(\nu+3)} \left| f''(e_2^\rho) \right|^q} \right.$$

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$$+ \left(\frac{1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \sqrt[q]{\frac{1}{\rho(\nu+3)}\left|f''(e_1^\rho)\right|^q + \frac{1}{\rho(\nu+2)(\nu+3)}\left|f''(e_2^\rho)\right|^q}\Bigg\}.$$

Proof. Using Lemma 3.6, convexity of $|f''|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\
\leq\ & \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times\ & \Bigg\{\left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}\left|f''\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}} \\
+\ & \left(\int_0^1 t^{\rho(\nu+2)-1}dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\rho(\nu+2)-1}\left|f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\} \\
\leq\ & \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \Bigg\{\left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^1 \left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}\left((1-t^\rho)\left|f''(e_1^\rho)\right|^q + t^\rho\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}} \\
+\ & \left(\int_0^1 t^{\rho(\nu+2)-1}dt\right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho(\nu+2)-1}\left(t^\rho\left|f''(e_1^\rho)\right|^q + (1-t^\rho)\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg\} \\
=\ & \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times\ & \Bigg\{\left(\frac{\nu+1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \sqrt[q]{\frac{(\nu+1)(\nu+4)}{2\rho(\nu+2)(\nu+3)}\left|f''(e_1^\rho)\right|^q + \frac{(\nu+1)}{2\rho(\nu+3)}\left|f''(e_2^\rho)\right|^q} \\
+\ & \left(\frac{1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \sqrt[q]{\frac{1}{\rho(\nu+3)}\left|f''(e_1^\rho)\right|^q + \frac{1}{\rho(\nu+2)(\nu+3)}\left|f''(e_2^\rho)\right|^q}\Bigg\}.
\end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.26. *With the notations in Theorem 3.25, if we take $|f''| \leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho} f(e_1^\rho)\right]\right| \\
\leq\ & \frac{\nu K(e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu+2)}. \qquad (3.24)
\end{aligned}$$

Corollary 3.27. *With the notations in Theorem 3.25, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)] - \left[{}^{AB}_{e_1}I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2}I^\nu_{e_1} f(e_1)\right]\right| \\
\leq\ & \frac{\nu(e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)}
\end{aligned}$$

$$\times \quad \left\{ \left(\frac{\nu+1}{\nu+2}\right) \sqrt[q]{\frac{(\nu+4)\left|f''(e_1)\right|^q + (\nu+2)\left|f''(e_2)\right|^q}{2(\nu+3)}} \right.$$
$$+ \quad \left. \frac{1}{(\nu+2)\sqrt[q]{\nu+3}} \sqrt[q]{(\nu+2)\left|f''(e_1)\right|^q + \left|f''(e_2)\right|^q} \right\}.$$

Theorem 3.28. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \quad \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\left|f''(e_1^\rho)\right|^q + \left|f''(e_2^\rho)\right|^q}{2}}. \tag{3.25}$$

Proof. Using Lemma 3.8, convexity of $|f''|^q$, Hölder inequality and properties of the modulus, we have

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \quad \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu+2)}$$
$$\times \quad \left(\int_0^1 \left| 1 - (1-t^\rho)^{\nu+1} - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left| f''\left(t^\rho e_1^\rho + (1-t^\rho) e_2^\rho\right) \right|^q dt \right)^{\frac{1}{q}}$$
$$\le \quad \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+\frac{1}{p}}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \times \left(\int_0^1 t^{\rho-1} \left(t^\rho \left|f''(e_1^\rho)\right|^q + (1-t^\rho)\left|f''(e_2^\rho)\right|^q \right) dt \right)^{\frac{1}{q}}$$
$$= \quad \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\left|f''(e_1^\rho)\right|^q + \left|f''(e_2^\rho)\right|^q}{2}}.$$

The proof of this theorem is complete. □

Corollary 3.29. *With the notations in Theorem 3.28, if we take $|f''| \le K$, the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho} I^\nu_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho} I^\nu_{e_1^\rho} f(e_1^\rho) \right] \right|$$
$$\le \quad \frac{\nu K (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}}. \tag{3.26}$$

Corollary 3.30. *With the notations in Theorem 3.28, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1} I^\nu_{e_2} f(e_2) + {}^{AB}_{e_2} I^\nu_{e_1} f(e_1) \right] \right|$$
$$\le \quad \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\left|f''(e_1)\right|^q + \left|f''(e_2)\right|^q}{2}}. \tag{3.27}$$

Theorem 3.31. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q \ge 1$, then the following inequality* *for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[{}_{e_1^+}^{ABK}{}^\rho I_{e_2^\rho}^\nu f(e_2^\rho)+{}_{e_2^-}^{ABK}{}^\rho I_{e_1^\rho}^\nu f(e_1^\rho)\right]\right| \\
\leq\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \\
\times\ & \sqrt[q]{C(\rho,\nu)\left|f''(e_1^\rho)\right|^q+\left(\frac{\nu}{\rho(\nu+2)}-C(\rho,\nu)\right)\left|f''(e_2^\rho)\right|^q},
\end{aligned} \tag{3.28}$$

where

$$C(\rho,\nu):=\frac{1}{\rho}\left(\frac{\nu+1}{2(\nu+3)}-\beta(2,\nu+2)\right).$$

Proof. Using Lemma 3.8, convexity of $|f''|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[{}_{e_1^+}^{ABK}{}^\rho I_{e_2^\rho}^\nu f(e_2^\rho)+{}_{e_2^-}^{ABK}{}^\rho I_{e_1^\rho}^\nu f(e_1^\rho)\right]\right| \\
\leq\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}\left|f''\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \\
\times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}\left(t^\rho\left|f''(e_1^\rho)\right|^q+(1-t^\rho)\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{\nu(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{C(\rho,\nu)\left|f''(e_1^\rho)\right|^q+\left(\frac{\nu}{\rho(\nu+2)}-C(\rho,\nu)\right)\left|f''(e_2^\rho)\right|^q}.
\end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.32. *With the notations in Theorem 3.31, if we take $|f''|\leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[{}_{e_1^+}^{ABK}{}^\rho I_{e_2^\rho}^\nu f(e_2^\rho)+{}_{e_2^-}^{ABK}{}^\rho I_{e_1^\rho}^\nu f(e_1^\rho)\right]\right| \\
\leq\ & \frac{\nu^2K(e_2^\rho-e_1^\rho)^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+3)}.
\end{aligned} \tag{3.29}$$

Corollary 3.33. *With the notations in Theorem 3.31, if we take $\rho\to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2-e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)]-\left[{}_{e_1}^{AB}I_{e_2}^\nu f(e_2)+{}_{e_2}^{AB}I_{e_1}^\nu f(e_1)\right]\right| \\
\leq\ & \frac{\nu(e_2-e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\nu+2}\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times \quad \sqrt[q]{C(1,\nu)\left|f''(e_1)\right|^q + \left(\frac{\nu}{\nu+2} - C(1,\nu)\right)\left|f''(e_2)\right|^q}.$$

Theorem 3.34. *Let $\nu \in (0,1)$ and $\rho > 0$. Let f and g be real valued, nonnegative and convex functions on $[e_1^\rho, e_2^\rho]$, where $0 \le e_1 < e_2$. Then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho)g(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)g(e_1^\rho)\right] \\
\le \quad & \left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)}\right) M(e_1^\rho, e_2^\rho) + \frac{2\nu^2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)} N(e_1^\rho, e_2^\rho), \quad (3.30)
\end{aligned}$$

where

$$M(e_1^\rho, e_2^\rho) = f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)$$

and

$$N(e_1^\rho, e_2^\rho) = f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho).$$

Proof. Since f and g are convex on $[e_1^\rho, e_2^\rho]$, then

$$f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \le t^\rho f(e_1^\rho) + (1-t^\rho) f(e_2^\rho) \quad (3.31)$$

and

$$g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \le t^\rho g(e_1^\rho) + (1-t^\rho) g(e_2^\rho). \quad (3.32)$$

From (3.31) and (3.32), we get

$$\begin{aligned}
f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \quad \le \quad & t^{2\rho} f(e_1^\rho)g(e_1^\rho) + (1-t^\rho)^2 f(e_2^\rho)g(e_2^\rho) \\
+ \quad & t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho) \quad \le \quad & (1-t^\rho)^2 f(e_1^\rho)g(e_1^\rho) + t^{2\rho} f(e_2^\rho)g(e_2^\rho) \\
+ \quad & t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].
\end{aligned}$$

By adding the above two inequalities, it follows that

$$\begin{aligned}
& f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) + f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho) \\
\le \quad & (2t^{2\rho} - 2t^\rho + 1)[f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)] + 2t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].
\end{aligned}$$

Multiplying both sides of above inequality by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} t^{\rho\nu-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
& \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)dt \\
& + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)dt \\
\le \quad & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1}(2t^{2\rho} - 2t^\rho + 1)[f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)]dt \\
+ \quad & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} 2t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)]dt \\
= \quad & \frac{\nu M(e_1^\rho, e_2^\rho)}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1}(2t^{2\rho} - 2t^\rho + 1)dt + \frac{2\nu N(e_1^\rho, e_2^\rho)}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} t^\rho(1-t^\rho)dt \\
= \quad & \frac{\nu(\nu^2+\nu+2)}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)} M(e_1^\rho, e_2^\rho) + \frac{2\nu^2}{\mathbb{B}(\nu)\Gamma(\nu+3)} N(e_1^\rho, e_2^\rho).
\end{aligned}$$

By the change of variables and with simple integral calculations, we get the desired result. □

Corollary 3.35. *With the notations in Theorem 3.34, if we choose $f = g$, the following inequality for the ABK-fractional integrals holds:*

$$\left[{}_{e_1^+}^{ABK}{}^{\rho}I_{e_2^\rho}^{\nu} f^2(e_2^\rho) + {}_{e_2^-}^{ABK}{}^{\rho}I_{e_1^\rho}^{\nu} f^2(e_1^\rho) \right] \leq \left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho \mathbb{B}(\nu)\Gamma(\nu+3)} \right) M_1(e_1^\rho, e_2^\rho) + \frac{2\nu^2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)} N_1(e_1^\rho, e_2^\rho), \quad (3.33)$$

where

$$M_1(e_1^\rho, e_2^\rho) = f^2(e_1^\rho) + f^2(e_2^\rho), \quad N_1(e_1^\rho, e_2^\rho) = 2f(e_1^\rho)f(e_2^\rho).$$

Corollary 3.36. *With the notations in Theorem 3.34, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\left[{}_{e_1}^{AB}I_{e_2}^{\nu} f(e_2)g(e_2) + {}_{e_2}^{AB}I_{e_1}^{\nu} f(e_1)g(e_1) \right] \leq \left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)} \right) M(e_1, e_2) + \frac{2\nu^2\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)} N(e_1, e_2). \quad (3.34)$$

Remark 3.37. With the notations in our theorems given in Section 3, if we take ρ, $\nu \to 1$, then we get some classical integral inequalities.

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