J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Weighted composition operator acting between some classes of analytic function spaces

A. El-Sayed Ahmed [1,2]
[1]Taif University, Faculty of Science, Math. Dept. Taif, Saudi Arabia
[2] Sohag University, Faculty of Science, Math. Dept. Egypt
e-mail: ahsayed80@hotmail.com
and
Aydah Al-Ahmadi[3]
[3]Jouf University
Colleague of Sciences and arts at Al-Qurayyat- Mathematics Department,
Jouf- Saudi Arabia

Abstract

In this paper, we define some general classes of weighted analytic function spaces in the unit disc. For the new classes, we investigate boundedness and compactness of the weighted composition operator uC_ϕ under some mild conditions on the weighted functions of the classes.

1 Introduction

Let $\mathbb{H}(\mathbb{D})$ denote the class of analytic functions in the unit disk $\mathbb{D}$. As usual, two quantities L_f and M_f, both depending on analytic function f on the unit disk $\mathbb{D}$, are said to be equivalent, and written in the form $L_f \approx M_f$, if there exists a positive constant C such that

$$\frac{1}{C} M_f \leq L_f \leq C\, M_f.$$

The notation $A \lesssim B$ means that there exists a positive constant C_1 such that $A \leq C_1 B$.
For $0 < \alpha < \infty$. The weighted type space H_α^∞ is the space of all $f \in \mathbb{H}(\mathbb{D})$ such that

$$\|f\|_{H_\alpha^\infty} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f(z)| < \infty.$$

and $H_{\alpha,\,0}^\infty$ denotes the closed subspace of H_α^∞ such that $f \in H_\alpha^\infty$ satisfies

$$(1-|z|^2)^\alpha |f(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

Let the Green's function $g(z,a) = \ln |\frac{1-\bar{a}z}{a-z}| = \ln \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{1-\bar{a}z}{a-z}$ stands for Möbius transformation. The following classes of weighted function spaces are defined in [7]:

Definition 1.1 *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and let f be an analytic function in $\mathbb{D}$ then $f \in \mathcal{N}_K$ if*

$$\|f\|_{\mathcal{N}_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))\, dA(z) < \infty,$$

AMS 2010 classification: 30H30, 30C45, 46E15.
Key words and phrases: analytic classes, weighted composition operators.

where $dA(z)$ defines the normalized area measure on $\mathbb{D}$, so that $A(\mathbb{D}) \equiv 1$.
Now, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z) = 0,$$

then f is said to belong to the class $\mathcal{N}_{K,0}$.

Clearly, if $K(t) = t^p$, then $\mathcal{N}_K = \mathcal{N}_p$ (see [19]) , since $g(z,a) \approx (1 - |\varphi_a(z)|^2)$. For $K(t) = 1$ it gives the Bergman space $\mathcal{A}^2$ (see [17]).
It is easy to check that $\|\cdot\|_{\mathcal{N}_K}$ is a complete semi-norm on $\mathcal{N}_K$ and it is Möbius invariant in the sense that

$$\|f \circ \varphi_a\|_{\mathcal{N}_K} = \|f\|_{\mathcal{N}_K}, \ \ a \in \mathbb{D},$$

whenever $f \in \mathcal{N}_K$ and $\varphi_a \in \mathrm{Aut}(\mathbb{D})$ is the group of all Möbius maps of $\mathbb{D}$. If $\mathcal{N}_K$ consists of just the constant functions, we say that it is trivial.

We assume from now that all $K : [0, \infty) \to [0, \infty)$ to appear in this paper is right-continuous and nondecreasing function such that the integral

$$\int_0^{1/e} K(\log(1/\rho))\rho \, d\rho \ = \int_1^{\infty} K(t)e^{-2t} \, dt < \infty.$$

From a change of variables we see that the coordinate function z belongs to $\mathcal{N}_K$ space if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \ K\big(\log(1/|z|)\big) \, d\, A(z) < \infty.$$

Simplifying the above integral in polar coordinates, we conclude that $\mathcal{N}_K$ space is nontrivial if and only if

$$\sup_{t\in(0,1)} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K\big(\log(1/r)\big) r dr < \infty. \tag{1}$$

An important tool in the study of $\mathcal{N}_K$ space is the auxiliary function ϕ_K defined by

$$\phi_K(s) = \sup_{0<t<1} \frac{K(st)}{K(t)}, \ \ 0 < s < \infty.$$

The following condition has played a crucial role in the study of $\mathcal{N}_K$ space:

$$\int_1^{\infty} \phi_K(s) \frac{ds}{s^2} < \infty, \tag{2}$$

and

$$\int_0^1 \phi_K(s) \frac{ds}{s} < \infty. \tag{3}$$

The test function in $\mathcal{N}_K$ can be stated as follows (see [7]):

Lemma 1.1 *For $w \in \mathbb{D}$ we define*

$$h_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^2}.$$

Suppose that condition (1) is satisfied. Then $h_w \in \mathcal{N}_K$ and

$$\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_K} \le 1.$$

2 Analytic $\mathcal{N}_{K,\,\omega}$ and $H^\infty_{\alpha,\,\omega}$-spaces

Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$ be any reasonable and right continuous nondecreasing function and $K : [0,\ \infty) \longrightarrow [0,\ \infty)$ be right continuous nondecreasing function too. Then, we give the following definitions.

Definition 2.1 *Let $0 < \alpha < \infty$. The weighted type space $H^\infty_{\alpha,\,\omega}$ is defined by*

$$H^\infty_{\alpha,\omega} := \{f \in \mathbb{H}(\mathbb{D})\ :\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)}|f(z)| < \infty\}.$$

If

$$\lim_{|z|\longrightarrow 1} \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)}|f(z)| = 0,$$

we say that f belongs to $H^\infty_{\alpha,\,\omega,0}$.

Definition 2.2 *The analytic $\mathcal{N}_{K,\omega}$ -space is defined by*

$$\mathcal{N}_{K,\omega} := \{f \in \mathbb{H}(\mathbb{D})\ :\ \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)} dA(z) < \infty\}.$$

If

$$\lim_{|a|\longrightarrow 1} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)} dA(z) = 0,$$

we say that f belongs to the class $\mathcal{N}_{K,\omega,0}$.

Clearly, if $K(t) = t^p$ and $\omega \equiv 1$, then $\mathcal{N}_{K,1} = \mathcal{N}_p$.
For $K(t) = 1$ and $\omega \equiv 1$, it gives the Bergman space $\mathcal{A}^2$.

In the study of the space $\mathcal{N}_{K,\omega}$, we assume the following condition holds:

$$\int_0^1 \frac{K(\log\frac{1}{r})}{\omega^2(1-r^2)}\, r\, dr < \infty. \tag{4}$$

Throughout this paper, we always assume that condition (4) is satisfied, so that the $\mathcal{N}_{K,\omega}$ space we study is not trivial.

Remark 2.1 *It should be remarked that the weight function $\omega(1-|z|)$ is used to define and study some general classes of function spaces, see [10, 15, 21] and others.*

For a point $a \in \mathbb{D}$ and $0 < r < 1$, let $D(a,r)$ denote an Euclidean disk with center $\frac{(1-r^2)a}{1-r^2|a|^2}$ and radius $\frac{(1-|a|^2)r}{1-r^2|a|^2}$ (see [20]). Suppose also that $E(a,r) = \{z \in \mathbb{D}\ :\ |z-a| < r(1-|a|)\ \}$.
Now, we will prove the following lemma:

Lemma 2.1 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$ be any reasonable and right continuous nondecreasing function and let $K : [0,\infty) \to [0,\infty)$ be right continuous nondecreasing function. Then*

$$\mathcal{N}_{K,\omega} \subset H^\infty_{1,\,\omega}.$$

Proof: Suppose that $f \in \mathcal{N}_{K,\,\omega}$, and let C be a constant such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)}\, dA(z) = C < \infty.$$

By the fact that K is nondecreasing, for all $r, 0 < r < 1$, we have

$$\begin{aligned} C &\geq \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \int_{D(a,r)} |f(z)|^2 \frac{K\big(\log \frac{1}{|\varphi_a(z)|}\big)}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \int_{E(a,r)} |f(z)|^2 \frac{K\big(\log \frac{1}{|\varphi_a(z)|}\big)}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} \int_{E(a,r)} |f(z)|^2\, dA(z). \end{aligned}$$

Since $|f(z)|^2$ is subharmonic it follows that,

$$\begin{aligned} C &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} \int_{E(a,r)} |f(z)|^2\, dA(z) \\ &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} 2r^2\pi(1-|a|^2)^2|f(a)|^2. \end{aligned}$$

For $r_0 \in (0,1)$, there exists a constant λ such that

$$\begin{aligned} \frac{|f(a)|^2(1-|a|^2)^2}{\omega^2(1-|a|^2)} &\leq \lambda \int_{D(a,r_0)} |f(z)|^2 dA(z) \\ &\leq \frac{C\,\lambda}{K(\log 1/r_0)}. \end{aligned}$$

Since r_0 is fixed, then

$$\sup_{a\in\mathbb{D}} \frac{|f(a)|(1-|a|^2)}{\omega(1-|a|^2)} \leq \sqrt{\frac{C\,\lambda}{K(\log 1/r_0)}}\ .$$

Thus $f \in H^\infty_{1,\,\omega}$ in $\mathbb{D}$. Hence, $\mathcal{N}_{K,\,\omega} \subset H^\infty_{1,\,\omega}$.

We will prove the following lemmas on $\mathcal{N}_{K,\,\omega}$- spaces:

Lemma 2.2 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ and $X, Y \in \{H^\infty_{\alpha,\,\omega}, \mathcal{N}_{K,\,\omega}\}$. Suppose that $uC_\phi(X) \subset Y$. Then $uC_\phi : X \to Y$ is compact if and only if for every bounded sequence $\{f_j\} \in X$ which converges to 0 uniformly on compact subset of $\mathbb{D}$, we have*

$$\lim_{j\to\infty} \|uC_\phi f_j\|_Y = 0.$$

Proof: This is an extension of a well-known result on the compactness of the composition operator on the Hardy spaces (see [9], Proposition 3.11). We see that any bounded sequence in $H^\infty_{\alpha,\,\omega}$ forms a normal family. Also by Lemma 2.1 we have the relation

$$\|f\|_{H^\infty_{1,\,\omega}} \leq \|f\|_{\mathcal{N}_{K,\,\omega}}$$

and the growth estimate for $f \in H^\infty_{1,\,\omega}$ imply that any bounded sequence in $\mathcal{N}_{K,\,\omega}$ forms a normal family. Hence a similar argument by using Montel's theorem also proves this lemma, and so we omit its proof.

Now, for $\alpha \in (0,\infty)$, $\theta \in [0, 2\pi)$ and $r \in (0,1]$, we put

$$f_{\theta,r}(z) := \sum_{k=0}^{\infty} 2^{\alpha k} (re^{i\theta})^{2^k} z^{2^k} \quad (z \in \mathbb{D}).$$

Using the function $f_{\theta,r}$, we have the following result:

Lemma 2.3 *The function $f_{\theta,r}(z)$ belongs to $H^\infty_{\alpha,\,\omega}$ and $\|f_{\theta,r}\|_{H^\infty_{\alpha,\,\omega}} \lesssim 1$ which is independent of θ and r. In particular, $f_{\theta,r} \in H^\infty_{\alpha,\,\omega.\,0}$ if $r \in (0,1)$.*

Proof: The proof is similar to the corresponding results in [27], with some simple modifications, so it will be omitted.

Lemma 2.4 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ with $\omega(kt) = k\omega(t)$, $k > 0$. Suppose that condition (4) is satisfied. For all $z, w \in \mathbb{D}$, we define the function $h_w(z)$ by*

$$h_w(z) = \frac{(1-|w|^2)(1-|z|)}{(1-\overline{w}z)^2}.$$

Then $h_w(z) \in \mathcal{N}_{K,\omega}$ and $\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_{K,\omega}} \lesssim 1$.
Proof: First, we have that

$$\|h_w\|_{\mathcal{N}_{K,\,\omega}} = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{(1-|w|^2)}{(1-\overline{w}z)^2}\right|^2 \frac{(1-|z|)^2}{\omega^2(1-|z|^2)} K(g(z,w)dA(z).$$

Since, $1-|w| \le |1-\bar{w}z| \le 1+|w| < 2$ and $1-|z| \le |1-\bar{w}z| \le 1+|z| < 2$ where $z, w \in \mathbb{D}$, then

$$\|h_w\|_{\mathcal{N}_{K,\,\omega}} \le 4 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(g(z,w)}{\omega^2(1-|z|^2)} dA(z) = 4 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|\varphi_w(z)|})}{\omega^2(1-|z|^2)} dA(z).$$

Now, let $z = \varphi_w(z)$, then

$$\begin{aligned} \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\le 4 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{\omega^2(1-|\varphi_w(z)|^2)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, dA(z) \\ &\le 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2\omega^2(1-|\varphi_w(z)|^2)}\, dA(z). \end{aligned}$$

Since,

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}.$$

Then, we obtain that

$$\begin{aligned} \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\le 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2\, \omega^2\big(\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}\big)}\, dA(z) \\ &= 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\, \omega^2\big(\frac{(1-|z|^2)}{|1-\overline{w}z|^2}\big)}\, dA(z) \\ &\le 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\, \omega^2\big(\frac{(1-|z|^2)}{(1-|w|)^2}\big)}\, dA(z) \\ &= 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|)^4 K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\, \omega^2(1-|z|^2)}\, dA(z) \\ &\le c \int_0^1 \frac{K(\log\frac{1}{r})}{\omega^2(1-r^2)}\, r\, dr < \infty, \end{aligned}$$

where c is a positive constant. Then,

$$\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_{K,\omega}} \lesssim 1.$$

This completes the proof.

3 Weighted composition operator on $H^{\infty}_{\alpha,\,\omega}$ and $\mathcal{N}_{K,\,\omega}$ spaces

Let ϕ be an analytic self-map of the unit disk $\mathbb{D}$. For any $u \in \mathbb{H}(\mathbb{D})$ and $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator $uC_\phi : \mathbb{H}(\mathbb{D}) \to \mathbb{H}(\mathbb{D})$ is defined by $uC_\phi f = u.(f \circ \phi)$. This class of operators has been appeared in the studies of isometries of many holomorphic function spaces. In fact, many isometries of holomorphic function spaces are described as weighted composition operators. For more information and various studies on weighted composition operators, we refer to [8, 12, 13, 16, 18, 25, 27, 28, 29] and others.

In this section we study weighted composition operators acting on $\mathcal{N}_{K,\,\omega}$-space.
Let $\phi \in \mathbb{H}(\mathbb{D})$ to denoted a non-constant function satisfying $\phi(\mathbb{D}) \subset \mathbb{D}$. First, in the following result, we describe boundedness for the $\mathcal{N}_{K,\,\omega}$-class. The results in this section generalizing some results in [19].

Theorem 3.1 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2))} < \infty. \tag{5}$$

Proof: First assume that (5) holds. Then

$$\begin{aligned} \|uC_\phi f\|_{H^{\infty}_{\alpha,\,\omega}} &= \sup_{z\in\mathbb{D}} |u(z)|f(\phi(z))\left|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)}\right| \\ &\lesssim \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi_a(z)|^2)\omega(1-|z|^2)} \sup_{z\in\mathbb{D}} |f(\phi(z))| \frac{(1-|\phi_a(z)|^2)}{\omega(1-|\phi(z)|^2)} \\ &\lesssim \|f\|_{H^{\infty}_{1,\,\omega}} \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi_a(z)|^2)\omega(1-|\phi_a(z)|^2)} \\ &\leq \lambda\|f\|_{\mathcal{N}_{K,\,\omega}}, \end{aligned}$$

where λ is a positive constant.

Conversely, assume that $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is bounded, then

$$\|uC_\phi f\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim \|f\|_{\mathcal{N}_{K,\,\omega}}.$$

Fix a $z_0 \in \mathbb{D}$, and let h_w be the test function in Lemma 2.4 with $w = \phi(z_0)$. Then

$$\begin{aligned} 1 \gtrsim \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\geq \lambda_1\|uC_\phi h_w\|_{H^{\infty}_{\alpha,\,\omega}} \\ &\geq \frac{|u(z_0)|(1-|w|^2)}{|1-\overline{w}\phi_a(z_0)|^2)\omega(1-|z_0|^2)}(1-|z_0|^2)^\alpha \\ &= \frac{|u(z_0)|(1-|z_0|^2)^\alpha}{(1-|\phi_a(z_0|^2))\omega(1-|z_0|^2)}, \end{aligned}$$

where λ_1 is a positive constant. The proof of Theorem 3.1 is therefore established.

Theorem 3.2 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then the weighted composition operator $uC_\phi : H^{\infty}_{\alpha,\,\omega} \to \mathcal{N}_{K,\,\omega}$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}(\omega^2(1-|z|^2))} K(g(z,a))dA(z) < \infty. \tag{6}$$

Proof: First we assume that condition (6) holds and let

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}(\omega^2(1-|z|^2))} K(g(z,a))dA(z) < C,$$

where C is a positive constant. If $f \in H^{\infty}_{\alpha,\,\omega}$, then for all $a \in \mathbb{D}$ we have

$$\begin{aligned}
\|uC_\phi f\|_{\mathcal{N}_{K,\,\omega}} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |u(z)|^2|f(\phi(z)|^2\frac{K(g(z,a)}{\omega^2(1-|z|^2)}dA(z)\\
&= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|u(z)|^2}{\omega^2(1-|z|^2)}\frac{(1-|\phi(z)|^2)^{2\alpha}|f(\phi(z)|^2}{\omega^2(1-|\phi(z)|^2)}.\frac{\omega^2(1-|\phi(z)|^2)K(g(z,a))}{(1-|\phi(z)|^2)^{2\alpha}}dA(z)\\
&\le \|f\|^2_{H^{\infty}_{\alpha,\,\omega}}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}\,\omega^2(1-|z|^2)}K(g(z,a))dA(z)\\
&\le C\|f\|^2_{H^{\infty}_{\alpha,\,\omega}}.
\end{aligned}$$

Conversely, assume that $uC_\phi : H^{\infty}_{\alpha,\,\omega} \to \mathcal{N}_{K,\,\omega}$ is bounded, then

$$\|uC_\phi f\|^2_{\mathcal{N}_{K,\,\omega}} \lesssim \|f\|^2_{H^{\infty}_{\alpha,\,\omega}}.$$

fixing a point $z_0 \in \mathbb{D}$, with $w = \phi(z_0)$ then we set that

$$f_w(z) = \frac{\omega\big(1-\overline{w}\phi(z_0)\big)}{(1-\overline{w}z)^{\alpha}},$$

it is easy to check that $\|f_w\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim 1$. Then,

$$\begin{aligned}
\|uC_\phi f_w\|^2_{\mathcal{N}_{K,\,\omega}} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z_0)|^2\omega^2(1-|\phi(z_0)|^2)K(g(z_0,a))}{\big(1-\overline{\phi(z_0)}\phi(z_0)\big)^{2\alpha}\omega^2(1-|z_0|^2)}dA(z_0)\\
&= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z_0)|^2\omega^2(1-|\phi(z_0)|^2)K(g(z_0,a))}{\big(1-|\phi(z_0)|^2\big)^{2\alpha}\omega^2(1-|z_0|^2)}dA(z_0)\\
&\lesssim \|f_w\|^2_{H^{\infty}_{\alpha,\,\omega}}.
\end{aligned}$$

Theorem 3.3 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then, the operator $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is compact if and only if*

$$\lim_{r\to 1}\sup_{|\phi(z)|>r}\frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\phi(z)|^2)\omega(1-|z|^2)} = 0. \tag{7}$$

Proof: First assume that $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is compact and suppose that there exists $\varepsilon_0 > 0$ a sequence $(z_n) \subset \mathbb{D}$ such that

$$\frac{|u(z_n)|(1-|z_n|^2)^{\alpha}}{(1-|\phi(z_n)|^2)\omega(1-|z_n|^2)} \ge \varepsilon_0 \quad \text{whenever} \quad |\phi(z_n)| > 1-\frac{1}{n}.$$

Clearly, we can assume that

$$w_n = \phi(z_n) \longrightarrow w_0 \in \partial\mathbb{D} \ \text{ as } \ n \to \infty.$$

Let $h_{w_n} = \dfrac{(1-|w_n|^2)}{(1-\overline{w_n}z)^2}$ be the test function in Lemma 2.4. Then $h_{w_n} \to h_{w_0}$ with respect to the compact open topology. Define $f_n = h_{w_n} - h_{w_0}$. Then $\|f_n\|_{\mathcal{N}_{K,\,\omega}} \le 1$ (see Lemma 2.4) and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus, $uf_n \circ \phi \to 0$ in $H^{\infty}_{\alpha,\,\omega}$ by assumption. But, for n big enough, we obtain

$$\begin{aligned}
\|uC_\phi f_n\|_{H^{\infty}_{\alpha,\,\omega}} &\ge |u(z_n)|\big|h_{w_n}(\phi(z_n)) - h_{w_0}(\phi(z_n))\big|\frac{(1-|z_n|^2)^{\alpha}}{\omega(1-|z_n|^2)}\\
&\ge \underbrace{\frac{|u(z_n)|(1-|z_n|^2)^{\alpha}}{(1-|\phi(z_n)|^2)\omega(1-|z_n|^2)}}_{\ge\,\varepsilon_0}\ \underbrace{\left|1-\frac{(1-|w_n|^2)(1-|w_0|^2)}{\big|1-\overline{w_0}\,w_n\big|}\right|}_{=\,1},
\end{aligned}$$

which is a contradiction.

Conversely, assume that for all $\varepsilon > 0$ there exists $r \in (0,1)$ such that

$$\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2)} < \varepsilon \quad \text{whenever} \quad |\phi(z)| > r.$$

Let $(f_n)_n$ be a bounded sequence in $\mathcal{N}_{K,\,\omega}$ norm which converges to zero on compact subsets of $\mathbb{D}$. Clearly, we may assume that $|\phi(z)| > r$. Then

$$\begin{aligned} \|uC_\phi f_n\|_{H^\infty_{\alpha,\,\omega}} &= \sup_{z\in\mathbb{D}} |u(z)||f_n(\phi(z))| \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)} \\ &= \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2)} \ \big|f_n(\phi(z))\big|(1-|\phi(z)|^2). \end{aligned}$$

It is not hard to show that

$$||.||_{H^\infty_{1,\,\omega}} \lesssim ||.||_{\mathcal{N}_{K,\omega}}.$$

Thus, we obtain that

$$\|uC_\phi f_n\|_{H^\infty_{\alpha,\,\omega}} \ \le \ \varepsilon \ \|f_n\|_{H^\infty_{1,\,\omega}} \ \le \ \varepsilon \ \|f_n\|_{\mathcal{N}_{K,\,\omega}} \ \le \ \varepsilon.$$

It follows that uC_ϕ is a compact operator. This completes the proof of the theorem.

Remark 3.1 *It is still an open problem to extend the results of this paper in Clifford analysis, for several studies of function spaces in Clifford analysis, we refer to [1, 2, 3, 4, 5, 6] and others.*

Remark 3.2 *It is still an open problem to study properties for differences of weighted composition operators between $\mathcal{N}_{K,\,\omega}$ and $H^\infty_{\alpha,\,\omega}$ classes. For more information of studying differences of weighted composition operators, we refer to [14, 22, 23, 26] and others.*

References

[1] A. El-Sayed Ahmed, On weighted α-Besov spaces and α-Bloch spaces of quaternion-valued functions, Numer. Func. Anal. Optim.29(2008), 1064-1081.

[2] A. El-Sayed Ahmed, Lacunary series in quaternion $B^{p,\,q}$- spaces, Complex Var. Elliptic Equ, 54(7)(2009), 705-723.

[3] A. El-Sayed Ahmed, Lacunary series in weighted hyperholomorphic $B^{p;\,q}(G)$ spaces, Numer. Funct. Anal. Optim, 32(1)(2011), 41-58.

[4] A. El-Sayed Ahmed, Hyperholomorphic $\mathcal{Q}$-classes, Math. Comput. Modelling, 55(2012) 1428-1435.

[5] A. El-Sayed Ahmed and A. Ahmadi, On weighted Bloch spaces of quaternion-valued functions, AIP Conference Proceedings, 1389(2011), 272-275.

[6] A. El-Sayed Ahmed, K. Gur̈lebeck, L. F.Resńdis and L.M. Tovar, Characterizations for the Bloch space by $B^{p;\,q}$ spaces in Clifford analysis, Complex. var. Elliptic. Equ., l51(2)(2006),119- 136.

[7] A. El-Sayed Ahmed and M. A. Bakhit, Holomorphic $\mathcal{N}_K$ and Bergman-type spaces, Birkhuser Series on Oper Theo Adva.Appl. (2009), Birkhuser Verlag Publisher Basel-Switzerland, 195 (2009), 121-138.

[8] F. Colonna, Weighted composition operators between and BMOA, Bull. Korean Math. Soc. 50(1)(2013), 185-200.

[9] C. C. Cowen and B. D. Maccluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[10] K. M. Dyakonov, Weighted Bloch spaces, H^p ,and BMOA, J. Lond. Math. Soc. 65(2)(2002), 411-417.

[11] M. Essén, H. Wulan, and J. Xiao, Several Function- theoretic aspects characterizations of Möbius invariant $\mathcal{Q}_K$-spaces, J. Funct. Anal. 230 (2006), 78-115.

[12] O. Hyvärinen and M. Lindström, Estimates of essential norms of weighted composition operators between Bloch-type spaces, J. Math. Anal. Appl. 393(1)(2012), 38-44.

[13] A. S. Kucik, Weighted composition operators on spaces of analytic functions on the complex half-plane, Complex Anal. Oper. Theory 12(8)(2018), 1817-1833.

[14] Y. Liang, New characterizations for differences of weighted differentiation composition operators from a Bloch-type space to a weighted-type space, Period. Math. Hung. 77(1)(2018), 119-138.

[15] H. Li and T. Ma, Generalized composition operators from $\mathcal{B}_\mu$ spaces to $Q_{K,\omega}(p,q)$ spaces, Abstr. Appl. Anal. Volume 2014, Special Issue (2014), Article ID 897389, 6 pages.

[16] J. S. Manhas and R. Zhao, New estimates of essential norms of weighted composition operators between Bloch type spaces, J Math. Anal. Appl. 389(1)(2012), 32-47.

[17] V. G. Miller and T. L. Miller, The Cesáro operator on the Bergman space, Arch. Math. 78 (2002), 409-416.

[18] A. Montes-Rodriguez, Weighted composition operators on weighted Banach spaces of analytic functions, J. Lond. Math. Soc. 61(3)(2000), 872-884.

[19] N. Palmberg, Composition operators acting on $\mathcal{N}_p$- spaces, Bull. Belg. Math. Soc. Simon Stevin, 14 (2007), 545 - 554.

[20] J .B. Garnett, Bounded analytic functions, Academic Press, New York, (1981).

[21] R. A. Rashwan, A. El-Sayed Ahmed and A. Kamal, Some characterizations of weighted holomorphic Bloch space, Eur. J. Pure Appl. Math, 2 (2009), 250-267.

[22] X. Song and Z. Zhou, Differences of weighted composition operators from Bloch space to H^1 on the unit ball, J. Math. Anal. Appl. 401(1)(2013), 447-457.

[23] M. H. Shaabani, Fredholmness of multiplication of a weighted composition operator with its adjoint on H^2 and A^2_α, J. Inequal. Appl. (2018): 23.

[24] D. Thompson, Normaloid weighted composition operators on H^2, J. Math. Anal. Appl. 467(2)(2018), 1153-1162.

[25] P. T. Tien and L. H. Khoi, Weighted composition operators between different Fock spaces, Potential Anal. 50(2)(2019), 171-195.

[26] P. T. Tien and L. H. Khoi, Differences of weighted composition operators between the Fock spaces, Monatsh. Math. 188(1)(2019), 183-193.

[27] S. Ueki, Weighted composition operators on the Fock space, Proc. Amer. Math. Soc. 135 (2007), 1405-1410.

[28] X. Zhang and J. Xiao, Weighted composition operator between two analytic function spaces, Adv. Math. 35(4) (2006), 477-486.

[29] L. Zhang and H. Zeng, Weighted differentiation composition operators from weighted Bergman space to nth weighted space on the unit disk, J. Ineq. Appl. (2011) :65.