Global behavior of a nonlinear higher-order rational difference equation

A. M. Ahmed[1]
Mathematics Department, College of Science, Jouf University, Sakaka (2014), Kingdom of Saudi Arabia
E-mail: amaahmed@ju.edu.sa & ahmedelkb@yahoo.com

Abstract

In this paper, we investigate the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum\limits_{i=1}^{k} x_{n-2m_i}^{p-1} \prod\limits_{j=1}^{k} x_{n-2m_j}}, \quad n = 0, 1, 2, ...$$

with positive parameters and non-negative initial conditions.

Keywords: Recursive sequences; Global asymptotic stability; Oscillation; Period two solutions; Semicycles.

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[1] On leave from: Department of Mathematics, Faculty of Science, Al-Azhar University , Nasr City (11884), Cairo, Egypt.

1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

The study of these equations is quite challenging and rewarding and is still in its infancy.

We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore, that results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations.

El-Owaidy et al [1] investigated the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \qquad n = 0, 1, 2, ...$$

where the parameters $\alpha,\ \beta,\ \gamma$ and p are non-negative real numbers.

Other related results on rational difference equations can be found in refs. [2-15].

In this paper, we investigate the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum\limits_{i=1}^{k} x_{n-2m_i}^{p-1} \prod\limits_{j=1}^{k} x_{n-2m_j}}, \qquad n = 0, 1, 2, ... \tag{1.1}$$

where the parameters $\alpha,\ \beta,\ \gamma$ and p are positive real numbers, $k \in \{1, 2, ...\}$, $\{m_i\}_{i=1}^{k}$ be positive integers such that $m_i > m_{i-1}$; $i = 2, ...k$ and the initial conditions $x_{-2m_k}, x_{-2m_k+1}, ..., x_0$ are non-negative real numbers.

The results in this work are consistent with the results in [1] when $k = 1$ and $m_1 = 1$.

The results in this work are consistent with the results in [3] when $k = 2$, $m_1 = 1$ and $m_2 = 2$.

We need the following definitions.

Definition 1. *Let I be an interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

with $x_{-k}, x_{-k+1}, ..., x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(1.2). The linearized equation of Eq.(1.2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + ... + c_{k+1} y_{n-k} \tag{1.3}$$

where
$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, ..., \overline{x})$, $c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, ..., \overline{x}), ..., c_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, \overline{x}, ..., \overline{x})$.
The characteristic equation of Eq.(1.3) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} c_i \lambda^{k-i+1} = 0. \tag{1.4}$$

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).
(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

Definition 2. *A positive semicycle of* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.2) consists of a 'string' of terms* $\{x_l, x_{l+1}, ..., x_m\}$ *, all greater than or equal to* $\overline{x}$*, with* $l \geq -k$ *and* $m < \infty$ *and such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} < \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} < \overline{x}$.

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.2) consists of a 'string' of terms* $\{x_l, x_{l+1}, ..., x_m\}$ *, all less than* $\overline{x}$*, with* $l \geq -k$ *and* $m < \infty$ *and such that either* $l = -k$ *or* $l > -k$ *and* $x_{l-1} \geq \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} \geq \overline{x}$.

Definition 3. *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.2) is called nonoscillatory if there exists* $N \geq -k$ *such that either*

$$x_n \geq \overline{x} \ \ \forall n \geq N \quad \text{or} \quad x_n < \overline{x} \ \ \forall n \geq N \ ,$$

and it is called oscillatory if it is not nonoscillatory.

(a) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if

$$x_{n+p} = x_n \ \text{ for all } n \geq -k. \tag{1.5}$$

(b) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if it is periodic with period p and p is the least positive integer for which (1.5) holds.

We need the following theorem.

Theorem 1.1. *(i) If all roots of Eq.(1.4) have absolute value less than one, then the equilibrium point* $\overline{x}$ *of Eq.(1.2) is locally asymptotically stable.*

(ii) If at least one of the roots of Eq.(1.4) has absolute value greater than one, then $\overline{x}$ is unstable.

The equilibrium point $\overline{x}$ of Eq.(1.2) is called a saddle point if Eq.(1.4) has roots both inside and outside the unit disk.

2. Main results

In this section, we investigate the dynamics of Eq.(1.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{p+k-1}} y_n$ reduces Eq.(1.1) to the difference equation

$$y_{n+1} = \frac{r y_{n-1}}{1 + \sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}}, \qquad n = 0, 1, 2, ... \tag{2.1}$$

where $r = \frac{\alpha}{\beta} > 0$.

Note that $\overline{y_1} = 0$ is always an equilibrium point of Eq.(2.1). When $r > 1$, Eq.(2.1) also possesses the unique positive equilibrium $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$.

Theorem 2.1. *The following statements are true*

(i) If $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is locally asymptotically stable.

(ii) If $r > 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is a saddle point.

(iii) When $r > 1$, then the positive equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ of Eq.(2.1) is unstable.

Proof: The linearized equation of Eq.(2.1) about the equilibrium point $\overline{y_1} = 0$ is

$$z_{n+1} = r z_{n-1}, \qquad n = 0, 1, 2, ...$$

so, the characteristic equation of Eq.(2.1) about the equilibrium point $\overline{y_1} = 0$ is

$$\lambda^{2m_k+1} - r\lambda^{2m_k-1} = 0,$$

and hence, the proof of (i) and (ii) follows from Theorem A.

For (iii), we assume that $r > 1$, then the linearized equation of Eq.(2.1) about the equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ has the form

$z_{n+1} = z_{n-1} - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_1} - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_2} - ... - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_k}, \quad n = 0, 1, 2, ...$

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so, the characteristic equation of Eq.(2.1) about the equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ is

$$f(\lambda) = \lambda^{2m_k+1} - \lambda^{2m_k-1} + \frac{(r-1)(p+k-1)}{rk}\sum_{i=1}^{k}\lambda^{2m_k-2m_i} = 0,$$

It is clear that $f(\lambda)$ has a root in the interval $(-\infty,-1)$, and so, $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ is an unstable equilibrium point.

This completes the proof.

Theorem 2.2. *Assume that $r<1$, then the equilibrium point $\overline{y_1}=0$ of Eq.(2.1) is globally asymptotically stable.*

Proof: We know by Theorem 2.1 that the equilibrium point $\overline{y_1}=0$ of Eq.(2.1) is locally asymptotically stable. So, let $\{y_n\}_{n=-2m_k}^{\infty}$ be a solution of Eq.(2.1). It suffices to show that $\lim_{n\to\infty} y_n = 0$. Since

$$0 \le y_{n+1} = \frac{ry_{n-1}}{1+\sum_{i=1}^{k} y_{n-2m_i}^{p-1}\prod_{j=1}^{k} y_{n-2m_j}} \le ry_{n-1} < y_{n-1}.$$

So, the even terms of this solution decrease to a limit (say $L_1 \ge 0$), and the odd terms decrease to a limit (say $L_2 \ge 0$), which implies that

$$L_1 = \frac{rL_1}{1+kL_2^{k+p-1}} \qquad \text{and} \qquad L_2 = \frac{rL_2}{1+kL_1^{k+p-1}}.$$

If $L_1 \neq 0 \Rightarrow L_2^{k+p-1} = \frac{r-1}{k} < 0$, which is a contradiction, so $L_1 = 0$, which implies that $L_2 = 0$.

So, $\lim_{n\to\infty} y_n = 0$, which the proof is complete.

Theorem 2.3. *Assume that $r=1$, then Eq.(2.1) possesses the prime period two solution*

$$\ldots,\phi,0,\phi,0,\ldots \tag{2.2}$$

with $\phi>0$. Furthermore, every solution of Eq.(2.1) converges to a period two solution (2.2) with $\phi \ge 0$.

Proof: Let

$$..., \phi, \psi, \phi, \psi, ...$$

be period two solutions of Eq.(2.1). Then

$$\phi = \frac{r\phi}{1 + k\psi^{k+p-1}}, \quad \text{and} \quad \psi = \frac{r\psi}{1 + k\phi^{k+p-1}},$$

so,

$$k\phi\psi = \frac{(r-1)(\phi - \psi)}{\psi^{k+p-2} - \phi^{k+p-2}} \geq 0,$$

If $k + p > 2$, then we have $r - 1 \leq 0$.

If $r < 1$, then this implies that $\phi < 0$ or $\psi < 0$, which is impossible, so $r = 1$.

If $k + p < 2$, then we have $r - 1 \geq 0$.

If $r > 1$, then we have either $\phi = \psi = 0$, which is impossible or $\phi = \psi = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$, which is impossible, so $r = 1$.

If $k + p = 2$, then we have$(r - 1)(\phi - \psi) = 0$, which implies that $r = 1$.

To complete the proof, assume that $r = 1$ and let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of Eq.(2.1), then

$$y_{n+1} - y_{n-1} = \frac{-y_{n-1}\left(\sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}\right)}{1 + \sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}} \leq 0, \qquad n = 0, 1, 2, ...$$

So, the even terms of this solution decrease to a limit (say $\Phi \geq 0$), and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus,

$$\Phi = \frac{\Phi}{1 + k\Psi^{k+p-1}} \text{ and } \Psi = \frac{\Psi}{1 + k\Phi^{k+p-1}},$$

which implies that $k\Phi\Psi^{k+p-1} = 0$ and $k\Phi^{k+p-1}\Psi = 0$. Then the proof is complete.

Theorem 2.4. *Assume that $r > 1$, and let $\{y_n\}_{n=-2m_k}^{\infty}$ be a solution of Eq.(2.1) such that*

$$y_{-2m_k}, y_{-2m_k+2}, ..., y_0 \geq \overline{y_2} \text{ and } y_{-2m_k+1}, y_{-2m_k+3}, ..., y_{-1} < \overline{y_2}, \tag{2.3}$$

or

$$y_{-2m_k}, y_{-2m_k+2}, ..., y_0 < \overline{y_2} \text{ and } y_{-2m_k+1}, y_{-2m_k+3}, ..., y_{-1} \geq \overline{y_2}. \tag{2.4}$$

Then $\{y_n\}_{n=-2m_k}^{\infty}$ oscillates about $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ with a semicycle of length one.

Proof: Assume that (2.3) holds. (The case where (2.4) holds is similar and will be omitted.) Then,

$$y_1 = \frac{ry_{-1}}{1+\sum\limits_{i=1}^{k} y_{-2m_i}^{p-1} \prod\limits_{j=1}^{k} y_{-2m_j}} < \frac{r\overline{y_2}}{1+k\overline{y_2}^{k+p-1}} = \overline{y_2}$$

and

$$y_2 = \frac{ry_0}{1+\sum\limits_{i=1}^{k} y_{-2m_i+1}^{p-1} \prod\limits_{j=1}^{k} y_{-2m_j+1}} > \frac{r\overline{y_2}}{1+k\overline{y_2}^{k+p-1}} = \overline{y_2}$$

and then the proof follows by induction.

Theorem 2.5. *Assume that $r > 1$, then Eq.(2.1) possesses an unbounded solution.*

Proof: From Theorem 2.4, we can assume without loss of generality that the solution $\{y_n\}_{n=-2k}^{\infty}$ of Eq.(2.1) is such that

$$y_{2n-1} < \overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}} \quad \text{and} \quad y_{2n} > \overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}, \quad \text{for } n \geq -m_k+1.$$

Then

$$y_{2n+1} = \frac{ry_{2n-1}}{1+\sum\limits_{i=1}^{k} y_{2n-2m_i}^{p-1} \prod\limits_{j=1}^{k} y_{2n-2m_j}} < \frac{ry_{2n-1}}{1+k\overline{y_2}^{k+p-1}} = y_{2n-1}$$

and

$$y_{2n+2} = \frac{ry_{2n}}{1+\sum\limits_{i=1}^{k} y_{2n-2m_i+1}^{p-1} \prod\limits_{j=1}^{k} y_{2n-2m_j+1}} > \frac{ry_{2n}}{1+k\overline{y_2}^{k+p-1}} = y_{2n}$$

from which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} y_{2n+1} = 0.$$

Then, the proof is complete.

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