RICCATI TECHNIQUE AND OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS

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Abstract. In this paper, by using the Riccati technique which reduces the higher order dynamic equations to a Riccati dynamic inequality, we will establish some new sufficient conditions for oscillation of the second order nonlinear neutral dynamic equation

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))+v(t)x^{\beta}(\eta(t))=0,$$

on time scales where γ, α β are quotient of odd positive integers.

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1. Introduction

The theory of time scales has been introduced by Stefan Hilger in [14] in 1988 in his Ph.D thesis in order to unify continuous and discrete analysis. In the last decades the subject is going fast and simultaneously extending to the other areas of research and many researchers have contributed on different aspects of this new theory, see the survey paper by Agarwal et al. [1] and the references cited therein. In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of different classes of dynamic equations on a time scale $\mathbb{T}$ which may be an arbitrary closed subset of real numbers $\mathbb{R}$, and as special cases contains the continuous and the discrete results as well, we refer the reader to papers ([3],[6], [7], [21]) and the references cited therein.

Following this trend, in this paper, we are concerned with oscillation of a certain class of nonlinear neutral delay dynamic equations of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))+v(t)x^{\beta}(\eta(t))=0, \text{ for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where γ, α, β are quotient of odd positive integers, $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and p, $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ with $0\le p(t)<1$, $q(t)$, $v(t)\ge 0$ and τ, δ, $\eta\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $\tau(t)\le t$, $\delta(t)\le t$, $\eta(t)\le t$ with $\lim_{t\to\infty}\tau(t)=\infty=\lim_{t\to\infty}\delta(t)=\infty=\lim_{t\to\infty}\eta(t)$. By a solution of (1.1), we mean a nontrivial real-valued function $x(t)\in C^1_{rd}([T_x,\infty),\mathbb{R})$, $T_x\ge t_0$ which has the properties that $r(z^{\Delta})^{\gamma})^{\Delta}\in C^1_{rd}([T_x,\infty),\mathbb{R})$ such that $x(t)$ satisfies (1.1) for all $[T_x,\infty)_{\mathbb{T}}$.

We mention here that the neutral delay differential equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role, we refer the reader to the papers by Boe and Chang [4], Brayton and Willoughby [8] and to the books by Driver [9], Hale [13] and Popov [16] and reference cited therein.

For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [5], [6] which summarize and organize much of the time scale calculus. Throughout the paper, we will denote the time scale by the symbol $\mathbb{T}$. For example, the real numbers $\mathbb{R}$,

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the integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$ are time scales. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. A time-scale $\mathbb{T}$ equipped with the order topology is metrizable and is a K_σ −space; i.e. it is a union of at most countably many compact sets. The metric on $\mathbb{T}$ which generates the order topology is given by $d(r; s) := |\mu(r; s)|$, where $\mu(.) = \mu(.; \tau)$ for a fixed $\tau \in \mathbb{T}$ is defined as follows: The mapping $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called graininess.

When $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) \equiv 0$ for all $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t+1$ and $\mu(t) \equiv 1$ for all $t \in \mathbb{T}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The mapping : $\nu : \mathbb{T} \to \mathbb{R}_0^+$ such that $\nu(t) = t - \rho(t)$ is called the backward graininess. If $\sigma(t) > t$, we say that t is right-scattered , while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous ($rd-$continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we define the derivative f^Δ as follows: Let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be differentiable at t, and we call α the delta derivative of f at t and denote it by $f^\Delta(t)$. For example, if $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^{'}(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \text{ for all } \ t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{N}$, then $f^\Delta(t) = f(t + 1) - f(t)$ for all $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right–scattered. If t is not right–scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{t \to \infty} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right–dense continuous ($rd-$continuous) if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. The space of $rd-$continuous functions is denoted by $C_r(\mathbb{T}, \mathbb{R})$. A useful formula is

$$f^\sigma = f + \mu f^\Delta, \quad \text{where} f^\sigma := f \circ \sigma.$$

A time scale $\mathbb{T}$ is said to be regular if the following two conditions are satisfied simultaneously:
(a). For all $t \in \mathbb{T}$, $\sigma(\rho(t)) = t$,
(b). For all $t \in \mathbb{T}$, $\rho(\sigma(t)) = t$.

Remark 1.1. *If $\mathbb{T}$ is a regular time scale, then both operators and are invertible with $\sigma^{-1} = \rho$ and $\rho^{-1} = \sigma$.*

The following formulae give the product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable function f and g. Assume f; $g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}$, then

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \tag{1.2}$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{1.3}$$

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The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_{0}^{1}\left[hx^{\sigma}+(1-h)x\right]^{\gamma-1}dhx^{\Delta}(t), \tag{1.4}$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90]. Note that when $\mathbb{T}=\mathbb{R}$, we have

$$\sigma(t)=t,\ \mu(t)=0,\quad f^{\Delta}(t)=f^{'}(t),\quad \int_{a}^{b}f(t)\Delta t=\int_{a}^{b}f(t)dt.$$

When $\mathbb{T}=\mathbb{Z}$, we have

$$\sigma(t)=t+1,\ \mu(t)=1,\ f^{\Delta}(t)=\Delta f(t),\quad \int_{a}^{b}f(t)\Delta t=\sum_{t=a}^{b-1}f(t).$$

When $\mathbb{T}=h\mathbb{Z}$, $h>0$, we have $\sigma(t)=t+h$, $\mu(t)=h$,

$$f^{\Delta}(t)=\Delta_{h}f(t)=\frac{(f(t+h)-f(t))}{h},\quad \int_{a}^{b}f(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}}f(a+kh)h.$$

When $\mathbb{T}=\{t:t=q^{k},\ k\in\mathbb{N}_{0},\ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$$f^{\Delta}(t)=\Delta_{q}f(t)=\frac{(f(q\,t)-f(t))}{(q-1)\,t},\quad \int_{t_{0}}^{\infty}f(t)\Delta t=\sum_{k=0}^{\infty}f(q^{k})\mu(q^{k}).$$

When $\mathbb{T}=\mathbb{N}_{0}^{2}=\{t^{2}:t\in\mathbb{N}\}$, we have $\sigma(t)=(\sqrt{t}+1)^{2}$ and

$$\mu(t)=1+2\sqrt{t},\ f^{\Delta}(t)=\Delta_{0}f(t)=(f((\sqrt{t}+1)^{2})-f(t))/1+2\sqrt{t}.$$

When $\mathbb{T}=\mathbb{T}_{n}=\{t_{n}:n\in\mathbb{N}\}$ where $(t_{n}\}$ is the harmonic numbers that are defined by $t_{0}=0$ and $t_{n}=\sum_{k=1}^{n}\frac{1}{k},n\in\mathbb{N}_{0}$, we have

$$\sigma(t_{n})=t_{n+1},\ \mu(t_{n})=\frac{1}{n+1},\ f^{\Delta}(t)=\Delta_{1}f(t_{n})=(n+1)f(t_{n}).$$

When $\mathbb{T}_{2}$=$\{\sqrt{n}:n\in\mathbb{N}\}$, we have $\sigma(t)=\sqrt{t^{2}+1}$,

$$\mu(t)=\sqrt{t^{2}+1}-t,\ f^{\Delta}(t)=\Delta_{2}f(t)=\frac{(f(\sqrt{t^{2}+1})-f(t))}{\sqrt{t^{2}+1}-t}.$$

When $\mathbb{T}_{3}$=$\{\sqrt[3]{n}:n\in\mathbb{N}\}$, we have $\sigma(t)=\sqrt[3]{t^{3}+1}$ and

$$\mu(t)=\sqrt[3]{t^{3}+1}-t,\ f^{\Delta}(t)=\Delta_{3}f(t)=\frac{(f(\sqrt[3]{t^{3}+1})-f(t))}{\sqrt[3]{t^{3}+1}-t}.$$

Now, we pass to the antiderivative and the integration on time scales for detla differentiable functions. For $a,b\in\mathbb{T}$, and a delta differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_{a}^{b}f^{\Delta}(t)\Delta t=f(b)-f(a).$$

An integration by parts formula reads

$$\int_{a}^{b}f(t)g^{\Delta}(t)\Delta t=\ f(t)g(t)|_{a}^{b}-\int_{a}^{b}f^{\Delta}(t)g^{\sigma}(t)\Delta t, \tag{1.5}$$

and infinite integrals are defined as

$$\int_{a}^{\infty}f(t)\Delta t=\lim_{b\to\infty}\int_{a}^{b}f(t)\Delta t.$$

It is well known that $rd-$continuous functions possess antiderivative. If f is $rd-$continuous and $F^{\Delta}=f$, then

$$\int_{t}^{\sigma(t)} f(s)\Delta s = F(\sigma(t)) - F(t) = \mu(t)F^{\Delta}(t) = \mu(t)f(t).$$

Note that the integration formula on a discrete time scale is defined by

$$\int_{a}^{b} f(t)\Delta t = \sum_{t\in(a,b)} f(t)\mu(t).$$

We say that a solution x of (1.1) has a generalized zero at t if $x\,(t)=0$ and has a generalized zero in $(t,\sigma(t))$ in case $x\,(t)\,x^{\sigma}\,(t)<0$ and $\mu(t)>0$. To investigate the oscillation properties of (1.1) it is proper to use the notions such as conjugacy and disconjugacy of the equation (1.1). Equation (1.1) is disconjugate on the interval $[t_0,b]_{\mathbb{T}}$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[t_0,b]_{\mathbb{T}}$.

Equation (1.1) is said to be nonoscillatory on $[t_0,\infty]_{\mathbb{T}}$ if there exists $c\in[t_0,\infty]_{\mathbb{T}}$ such that this equation is disconjugate on $[c,d]_{\mathbb{T}}$ for every $d>c$. In the opposite case (1.1) is said to be oscillatory on $[t_0,\infty]_{\mathbb{T}}$. A solution $x\,(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is oscillatory. We say that (1.1) is right disfocal (left disfocal) on $[a,b]_{\mathbb{T}}$ if the solutions of (1.1) such that $x^{\Delta}\,(a)=0$ ($x^{\Delta}\,(b)=0$) have no generalized zeros in $[a,b]_{\mathbb{T}}$.

In recent two decades some authors have been studied the oscillation of the second order nonlinear neutral delay dynamic equations on time scales and established several sufficient conditions for oscillation of some different types of equations by employing the Riccati transformation technique. For example, Saker [18] has studied the oscillation of second order neutral delay dynamic equations of Emden-fowler type of the form

$$[a(t)(y(t)+r(t)y(\tau(t))]^{\Delta}+p(t)|y(\delta(t))|^{\gamma}signy(\delta(t)))=0,$$

on time scale $\mathbb{T}$, where, $\gamma>1$, $a(t)$, $p(t)$, $r(t)$ and $\delta(t)$ are real-valued function defined on $\mathbb{T}$. Also Saker [19] studied the oscillation of the superlinear and sublinear neutral delay dynamic equations of the form

$$[a(t)([y(t)+p(t)y(\tau(t)))]^{\Delta})^{\gamma}]^{\Delta}+q(t)y^{\gamma}(\delta(t)))=0,$$

on time scale, where $\gamma>0$ is a quotient of odd positive integers. The main results has been obtained under the conditions $\tau(t):\mathbb{T}\to\mathbb{T}$, $\delta(t):\mathbb{T}\to\mathbb{T}$, $\tau(t)\leq t$, $\delta(t)\leq t$ for all $t\in\mathbb{T}$ and $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\delta(t)=\infty$, $\int_{t_0}^{\infty}\frac{1}{a(t)}^{\frac{1}{\gamma}}\Delta t=\infty$, $a^{\Delta}(t)\geq t$ and $0\leq p(t)<1$.

Thandapani et. al [24] studied the oscillation of second order nonlinear neutral dynamic equations on time scale of the form

$$(r(t)((y(t)+p(t)y(t-\tau))^{\Delta})^{\gamma})^{\Delta}+q(t)y^{\beta}(t-\delta)=0,\ t\in\mathbb{T},$$

where $\mathbb{T}$ is a time scales. They obtained their results under the conditions $\gamma\geq 1$ and $\beta>0$ are quotients of odd positive integers, τ,δ are fixed nonnegative constants such that the delay function $\tau(t)=t-\tau<t$ and $\delta(t)=t-\delta<t$ satisfying $\tau:\mathbb{T}\to\mathbb{T}$ and $\delta:\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$, $q(t)$ and $\tau(t)$ real valued rd-continuous functions defined on $\mathbb{T}$, $p(t)$ is a positive and rd-continuous function $\mathbb{T}$ such that $0\leq p(t)<1$.

Sun et al. [22] studied the oscillation of a second order quasiliniear neutral delay dynamic equation on time scales of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\beta}(\tau_2(t))=0,$$

on time scale $\mathbb{T}$, where α,β,γ are quotients of odd positive integers, r, p, q_1, q_2 are rd-continuous function on $\mathbb{T}$ and r,q_1,q_2 are positive, $-1<-p_0\leq p(t)<1$, $p_0>0$, the delay functions $\tau_i:\mathbb{T}\to\mathbb{T}$ satisfying $\tau_i(t)\leq t$ for $t\in\mathbb{T}$ and $\tau_i(t)\to\infty$ as $t\to\infty$, for $i=0,1,2$ and there exists a function $\tau:\mathbb{T}\to\mathbb{T}$ which satisfying $\tau(t)\leq\tau_1(t)$, $\tau(t)\leq\tau_2(t)$, $\tau(t)\to\infty$ as $t\to\infty$.

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Gao et al. [12] established some oscillation theorems for second order neutral functional dynamic equations on time scale of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q_1(t)x^{\alpha}(\delta(t))+q_2(t)x^{\beta}(\eta(t))=0,$$

where γ,α,β are ratios of odd positive integers by using the comparison theorems for oscillation. Sethi [26] considered the second order sublinear neutral delay dynamic equations of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\alpha(t))+v(t)x^{\gamma}(\eta(t))=0,$$

under the assumptions:

(H_0) $\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=+\infty$,

(H_1). $\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t<\infty$,

where $0<\gamma\leq 1$ is a quotient of odd positive integers, q, $v\to[0,\infty)$ and p, q, $v:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions and τ, σ, $\eta:\mathbb{T}\to\mathbb{T}$ are positive rd-continuous functions such that $\lim_{t\to\infty}\tau(t)=\infty=\lim\limits_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\eta(t)$ and obtained some sufficient conditions for oscillation. Our aim in this paper is to establish some new sufficient conditions for oscillation of the equation (1.1) by employing the Riccati technique and some basic lemmas studied the behavior of nonoscillatory solutions. Our motivation of the present work has come under two ways. First is due to the work in [17] and [22] and second is due to the work in [10].

2. Main Results

In this section, we establish some sufficient conditions for oscillation of all solutions of (1.1) under the hypothesis (H_0). Throughout the paper, we use the notation

$$z(t)=x(t)+p(t)x(\tau(t)). \tag{2.1}$$

Lemma 2.1. [2] *Assume that (H_0) holds and $r(t)\in C^1_{rd}([(a,\infty),\mathbb{R}^+)$ such that $r^{\Delta}(t)\geq 0$. Let $x(t)$ be an eventually positive real valued function such that $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}\leq 0$, for $t\geq t_1>t_0$. Then $x^{\Delta}(t)>0$ and $x^{\Delta\Delta}(t)<0$ for $t\geq t_1>t_0$.*

Lemma 2.2. [2] *Assume that the assumptions of Lemma 2.1 holds and let $\tau(t)$ be a positive continuous function such that $\tau(t)\leq t$ and $\lim\limits_{t\to\infty}\tau(t)=\infty$. Then there exists $t_l>t_1$ such that for each $l\in(0,1)$*

$$\frac{x(\tau(t))}{x(\delta(t))}\geq l\frac{\tau(t)}{\delta(t)}.$$

Proof. Indeed, for $t\geq t_1$

$$u(\delta(t))-u(\tau(t))=\int_{\tau(t)}^{\delta(t)}u^{\Delta}(s)\Delta s\leq(\delta(t)-\tau(t)))u^{\Delta}(\tau(t),$$

which implies that

$$\frac{u(\delta(t))}{u(\tau(t))}\leq 1+(\delta(t)-\tau(t)))\frac{u^{\Delta}(\tau(t)}{u(\tau(t))}.$$

On the other hand, it follows that

$$u(\tau(t))-u(t_1))=\int_{t_1}^{\tau(t)}u^{\Delta}(s)\Delta s\geq(u(t)-t_1)u^{\Delta}(\tau(t)).$$

That is for each $l\in(0,1)$, there exists a $t_l>t_1$ such that

$$l(\tau(t))\leq\frac{u(\tau(t))}{u^{\Delta}(\tau(t))},\ t\geq t_l.$$

Consequently,

$$\frac{u(\delta(t))}{u(\tau(t))}\leq 1+(\delta(t)-\tau(t)))\frac{u^{\Delta}(\tau(t))}{u(\tau(t))}\leq\frac{\delta(t)}{l\tau(t)}.$$

The proof is complete. $\square$

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In the following, for simplicity, we denote

$$a_1(t) := \int_t^\infty [q(s)(1-p(\delta(s))]\left(\frac{l\delta(s)}{\sigma(s)}\right)^\alpha \Delta s + \int_t^\infty [v(s)(1-p(\delta(s)))]\left(\frac{l\delta(s)}{\delta(s)}\right)^\alpha \Delta s,$$

and

$$A_1(t,K_1) := \left[a_1(t) + K_1\int_t^\infty \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}(a_1^\delta(s))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}}, \text{ for } t \in [t_0,\infty)_{\mathbb{T}},$$

where $K_1 > 0$ is an arbitrary constant.

Theorem 2.1. *Assume that* (H_0) *holds and let* $0 \le p(t) \le a < 1$, $r^\Delta(t) > 0$ *and* $\gamma < \alpha < \beta$, $\eta(t) \ge \delta(t)$ *and* $\delta^\Delta(t) \ge 1$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$. *If*

(H_1). $\limsup_{t\to\infty} a_1(t) < \infty$,

(H_2). $\int_{t_0}^\infty (\frac{1}{r(s)})^{\frac{1}{\gamma}} A_1^\sigma(s,K_1)\Delta s = \infty$.

Then every solution of (1.1) oscillates on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \ge t_0$. Hence there exists $t \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ and $x(\eta(t)) > 0$ for $t \ge t_1$. Using (2.1), we see that (1.1) becomes

$$(r(t)(z^\Delta(t))^\gamma)^\Delta = -q(t)x^\alpha(\delta(t)) - v(t)x^\beta(\eta(t)) \le 0, \text{ for } t \ge t_2. \tag{2.2}$$

So $r(t)(z^\Delta(t))^\gamma$ is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$, that is, either $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$. By Lemma 2.1, it follows that $z^\Delta(t) > 0$ for $t \ge t_2$. Hence there exists $t_3 > t_2$ such that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\ &\quad - p(t)p(\tau(t))p(\tau(\tau(t))) \\ &= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \le x(t), \end{aligned}$$

which implies that

$$x(t) \ge (1-p(t))z(t), \text{ for } t \in [t_3,\infty)_{\mathbb{T}}.$$

Therefore (1.1) can be written as

$$(r(t)(z^\Delta(t))^\gamma)^\Delta + q(t)(1-p(\delta(t)))^\alpha z^\alpha(\delta(t)) + v(t)(1-p(\eta(t)))^\alpha z^\alpha(\eta(t)) \le 0,$$

where $\gamma < \alpha < \beta$. Define $w(t)$ by the Riccati transformation

$$w(t) = r(t)\frac{(z^\Delta(t))^\gamma}{z^\alpha(t)}, \quad \text{for} \quad t \in [t_3,\infty)_{\mathbb{T}}. \tag{2.3}$$

By using the product and quotient rules, we see that

$$w^\Delta(t) = \frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\sigma)^\alpha} - \frac{(r(z^\Delta)^\gamma)^\sigma (z^\alpha)^\Delta}{z^\alpha(z^\sigma)^\alpha}, \quad \text{for } t \in [t_3,\infty)_{\mathbb{T}}. \tag{2.4}$$

Now, since $\eta(t) > \delta(t)$ and due to (2.3) and (2.4), we have

$$w^\Delta(t) \le -q(1-p^\delta)^\alpha - v(1-p^\delta)^\alpha\frac{(z^\delta)^\alpha}{(z^\sigma)^\alpha} - \frac{w^\sigma(z^\alpha)^\Delta}{z^\alpha}, \text{ for } t \in [t_3,\infty)_{\mathbb{T}},$$

Now, by using the chain rule [6], we get that

$$\begin{aligned} (z^\alpha(t))^\Delta &= \alpha\int_0^1 [(1-h)z(t) + hz(\sigma(t))]^{\alpha-1} dh z^\Delta(t) \\ &\ge \begin{cases} \alpha(z(t))]^{\alpha-1}z^\Delta(t), \ \alpha > 1, \\ \alpha(z(\sigma(t)))]^{\alpha-1}z^\Delta(t), \ 0 < \alpha \le 1. \end{cases} \end{aligned}$$

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Since $z(t)$ is nondecreasing function on $[t_3,\infty)_{\mathbb{T}}$, then for $t \geq t_3$,

$$\frac{(z^{\alpha}(t))^{\Delta}}{z^{\alpha}(t)} \geq \begin{cases} \alpha\frac{z^{\Delta}(t)}{z(t)}, & \text{for} \quad \alpha > 1 \\ \alpha\frac{(z(\sigma(t)))^{\alpha-1}}{z^{\alpha}(t)}z^{\Delta}(t), & for\ 0 < \alpha \leq 1. \end{cases}$$

Using the fact that $t \leq \sigma(t)$, we have

$$\frac{(z^{\alpha})^{\Delta}}{z^{\alpha}} \geq \alpha\frac{z^{\Delta}}{z^{\sigma}},\ \alpha > 0 \quad \text{on} \quad [t_3,\infty)_{\mathbb{T}}.$$

Therefore (2.4) yields that

$$w^{\Delta} \leq -q(1-p^{\delta})^{\alpha} - v(1-p^{\delta})^{\alpha}\frac{(z^{\sigma})^{\alpha}}{(z^{\delta})^{\alpha}} - \alpha w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}},\ t \geq t_3. \tag{2.5}$$

Now, since $\left(r^{\frac{1}{\gamma}}z^{\Delta}\right)$ is nonincreasing on $[t_3,\infty)_{\mathbb{T}}$, then for $t \leq \sigma(t)$, we have that

$$z^{\Delta} \geq r^{-\frac{1}{\gamma}}(w^{\sigma})^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\alpha}{\gamma}},\ t \geq t_3. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$w^{\Delta} \leq -q(1-p^{\delta})^{\alpha}\frac{(z^{\delta})^{\alpha}}{(z^{\sigma})^{\alpha}} - v(1-p^{\delta})^{\alpha}\frac{(z^{\delta})^{\alpha}}{(z^{\sigma})^{\alpha}} - \alpha r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}(z^{\sigma})\frac{\alpha}{\gamma} - 1,\ t \geq t_3.$$

Since $z(t)$ is nondecreasing on $[t_3,\infty)_{\mathbb{T}}$, then there exists $t_4 > t_3$ and $C > 0$ such that

$$(z(\sigma(t)))^{\frac{\alpha}{\gamma}-1} \geq (z(t))^{\frac{\alpha}{\gamma}-1} \geq C, \quad \text{for} \quad t \geq t_4.$$

By using Lemma 2.2, it follows from the last inequality that

$$\begin{aligned} w^{\Delta}(t) \quad &\leq \quad -q(1-p(\delta(t)))^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} - v(1-p(\delta(t)))^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} \\ &-\alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}},\ t \quad \geq \quad t_l > t_4. \end{aligned}$$

Integrating the above inequality from t to u $(t < u)$ for $t,\ u \in [t_4,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} -w(t) \quad &\leq \quad w(u) - w(t) \\ &\leq \quad -\int_t^u \left[q(1-p^{\delta})^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} + v(1-p^{\delta})^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} + \alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}}\right]\Delta s, \end{aligned}$$

that is,

$$w(t) \geq a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s,\ t \geq t_1,$$

where $K_{1\alpha} = C\alpha$. Indeed, $w(t) > a_1(t)$ implies that

$$w(t) \geq a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(\sigma(s)))^{1+\frac{1}{\gamma}}\Delta s = A_1^{\gamma}(t,K_1).$$

Since $t \leq \sigma(t)$ we see

$$r(z^{\Delta})^{\gamma} \geq (r(z^{\Delta})^{\gamma})^{\sigma},$$

which implies that

$$\frac{r(z^{\Delta})^{\gamma}}{(z^{\sigma})^{\alpha}} \geq \frac{(r(z^{\Delta})^{\gamma})^{\sigma}}{(z^{\sigma})^{\alpha}} = w^{\sigma} \geq (A_1^{\gamma}(t,k_1))^{\sigma},$$

that is,

$$(z^{\sigma})^{\delta}z^{\Delta} \geq r^{-\frac{1}{\gamma}}(A_1^{\sigma}(t,k_1)),\ t \in [t_5,\infty]_{\mathbb{T}},$$

where $\delta = (\frac{\alpha}{\gamma}) > 1$. Using the chain rule, we have

$$\begin{aligned} (z^{1-\delta}(t))^{\Delta} &= (1-\delta)\int_0^1 [(1-h)z(t) + hz(\sigma(t))]^{\delta}dhz^{\Delta}(t) \\ &\leq (1-\delta)(z(\sigma(t)))^{-\delta}z^{\Delta}(t), \end{aligned}$$

that is,

$$\frac{(z^{1-\delta}(\sigma(t)))^{\Delta}}{1-\delta} \geq z(\sigma(t))^{-\delta} z^{\Delta}(\sigma(t)).$$

Hence

$$\frac{(z^{1-\delta}(t))^{\Delta}}{1-\delta} \geq (z(\sigma(t)))^{-\delta} z^{\Delta}(t),$$

and then due to (2.6), we see that

$$\frac{(z^{1-\delta}(t))^{\Delta}}{1-\delta} \geq r^{-\frac{1}{\gamma}}(t)(A_1^{\sigma}(t,k_1)),\ t \in [t_5,\infty)_{\mathbb{T}}.$$

Integrating above inequality from t_5 to t, we get

$$\int_{t_5}^{t} r(s)^{-\frac{1}{\gamma}} \Big(A_1^{\sigma}(s,K_1)\Big)^{\frac{1}{\gamma}} \Delta s < \infty,$$

which is a contradiction to (H_2). The proof is complete. □

Theorem 2.2. *Let $0 \leq p(t) \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\gamma = \alpha = \beta$, $\eta(t) \geq \sigma(t)$ and assume that (H_0), and (H_1) hold. Furthermore assume that*

(H_3). $\limsup_{t\to\infty} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s) A_1(s,K_1)\Delta s\right) > 1.$

Then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we have

$$w(t) \geq A_1^{\gamma}(t,K_1)\ for\ t \in [t_4,\infty)_{\mathbb{T}}.$$

Using the fact that $r^{\frac{1}{\gamma}} z^{\Delta}$ is nonincreasing on $[t_4,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} z(t) = z(t_4) + \int_{t_4}^{t} z^{\Delta}(s)\Delta s &= z(t_4) + \int_{t_4}^{t} r^{-\frac{1}{\gamma}}(s)\Big(r(s)^{-\frac{1}{\gamma}} z^{\Delta}(s)\Big)\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t) z^{\Delta}(t) r^{-\frac{1}{\gamma}}(s)\Delta s, \end{aligned}$$

that is,

$$\frac{r(t)^{\frac{1}{\gamma}} z^{\Delta}(t)}{z(t)} \leq \Big(\int_{t_4}^{t} r(s)^{-\frac{1}{\gamma}}\Delta s\Big)^{-1},\ t \geq t_4, \tag{2.7}$$

Consequently,

$$A_1(t,K_1) \leq w^{\frac{1}{\gamma}}(t) = \frac{r(t)^{\frac{1}{\gamma}} z'(t)}{z(t} \leq \Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1},$$

implies that

$$\Big(\int_{t_4}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big) A_1(t,K_1) \leq 1,$$

which contradicts (H_3). Hence the theorem is proved. □

Theorem 2.3. *Let $0 \leq p(t) \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\gamma > \alpha > \beta$, $\eta(t) \geq \sigma(t)$ and assume that (H_0) and (H_2) hold. Furthermore assume that*

(H_4). $\limsup_{t\to\infty} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\right) \left[a_1(t) + K_1 \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^{\sigma}(s))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}} = \infty.$

Then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we obtain (2.2) and (2.3) and hence $w(t) > a_1(t)$, for $t \in [t_4,\infty)$. Consequently, it follows from (2.3) that

$$r^{\frac{1}{\gamma}} z^{\Delta} > z^{\frac{\alpha}{\gamma}} a_1^{\frac{1}{\gamma}},\quad \text{for}\quad t \geq t_4.$$

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We have $(rz^{\Delta})^{\gamma})^{\Delta} \leq 0$ implies that there exists a constant $C > 0$ and $t_5 > t_4$ such that $r^{\frac{1}{\gamma}} z^{\Delta} \leq C$, for $t \geq t_5$, that is $C \geq r^{\frac{1}{\gamma}} z^{\Delta} > z^{\frac{\alpha}{\gamma}} a_1^{\frac{1}{\gamma}}$ and hence

$$z(t) \leq C^{\frac{\gamma}{\alpha}} a_1(t)^{-\frac{1}{\alpha}}, \text{ for } t \in [t_5, \infty)_{\mathbb{T}}, \tag{2.8}$$

which implies that

$$(z^{\sigma})^{\frac{(\alpha-\gamma)}{\gamma}} \geq C^{\frac{(\alpha-\gamma)}{\alpha}} (a_1^{\sigma})^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \text{ for } t \in [t_5, \infty)_{\mathbb{T}}. \tag{2.9}$$

Due to (2.5), (2.6) and using Lemma 2.2, we have that

$$\begin{aligned} w^{\Delta}(t) \quad \leq \quad & -q(1-p(\delta(t)))^{\alpha}\Big(\frac{l\delta(t)}{\sigma(t)}\Big)^{\alpha} - v(1-p(\delta(t)))^{\alpha}\Big(\frac{l\delta(t)}{\sigma(t)}\Big)^{\alpha} \\ & -\alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}}(z^{\sigma}(t))^{\frac{(\alpha-\gamma)}{\alpha}}. \end{aligned}$$

Integrating the last inequality as in the proof of Theorem 2.1 and using (2.8), we obtain for $t \geq t_1 \geq t_5$ that

$$w(t) \geq a_1(t) + K_3 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(s))^{1+\frac{1}{\gamma}} \Delta s, \ for \ t \in [t_l, \infty)_{\mathbb{T}}, \tag{2.10}$$

where $K_1 = \alpha C^{\frac{(\alpha-\gamma)}{\gamma}}$. Substitute (2.9) into (2.3), it is easy to verify that

$$(z(t)^{\frac{(\alpha-\gamma)}{\gamma}} \frac{r^{\frac{1}{\gamma}}(t) z^{\Delta}(t)}{z(t)} \geq \Big[a_1(t) + K_1 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\Big]^{\frac{1}{\gamma}}. \tag{2.11}$$

Using (2.7) and (2.9) in (2.11), we can find

$$\begin{aligned} C^{\frac{\alpha-\gamma}{\alpha}} a_1(t)^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1} \quad & \geq \quad \Big[a_1(t) \\ +K_1 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\Big]^{\frac{1}{\gamma}}, \text{ for } t \quad & \in \quad [t_1, \infty)_{\mathbb{T}}. \end{aligned}$$

Therefore, for $t \geq t_1$ we have

$$(a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)\Big[a_1(t) + K_1 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\Big]^{\frac{1}{\gamma}} \leq C^{\frac{\alpha-\gamma}{\alpha}},$$

which contradicts (H_4). This completes the proof of theorem. □

Theorem 2.4. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\gamma < \beta < \alpha$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_2) and (H_3) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1. Hence the details are omitted. □

Theorem 2.5. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\alpha > \gamma > \beta$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_2) and (H_3) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1. □

Theorem 2.6. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\alpha < \beta < \gamma$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_1) and (H_4) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1 and Theorem 2.3. □

Theorem 2.7. *Let $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\alpha < \gamma < \beta$, $\eta(t) \geq \sigma(t)$. If (H_0), (H_1) and (H_4) hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1 and Theorem 2.3. □

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In the following theorems we will denote

$$a_2(t)=\int_t^{\infty}\Big[\lambda Q(s)\left(\frac{l\delta(t)}{\delta(t)}\right)^{\alpha}+\mu V(s)\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha}\Big]\Delta s,\ t\in[t_0,\infty)_{\mathbb{T}},$$

and

$$A_2(t,K_2)=\left[\frac{\lambda}{1+a^{\alpha}}a_2(\tau^{-1}(t))+\frac{\mu a_2(\tau^{-1}(t))}{1+a^{\alpha}}+K_2\int_{\tau^{-1}(t)}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}((a_2(\tau^{-1\delta}(s)))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}},$$

where K_2 is an arbitrary positive constant and $a>0$ $\lambda,\mu>0$ are positive constants, $Q(t)=min\{q(t),q(\tau(t))\}$, $V(t)=min\{v(t),v(\tau(t))\}$. From the definitions of τ, δ, η, we see that τ^{-1}, δ^{-1}, $\eta^{-1}:\mathbb{T}\to\mathbb{T}$ and τ^{-1}, δ^{-1}, η^{-1} are rd-continuous functions and $\tau^{-1}(t)\ge t,\delta^{-1}(t)\ge t$ and $\eta^{-1}(t)\ge t$.

Theorem 2.8. *Let $1\le p(t)\le p<\infty$, $r^{\Delta}(t)\ge 0$ $\tau(\delta(t))=\delta(\tau(t))$, $\tau(\eta(t))=\eta(\tau(t))$ and $\gamma<\alpha<\beta$, $\eta(t)\ge\delta(t)$ and If (H_0) holds and the following conditions hold:*
(H_5). $\limsup\limits_{t\to\infty}a_2(t)<\infty$,
(H_6). $\int_{t_0}^{\infty}(\frac{1}{r(s)})^{\frac{1}{\gamma}}A_2^{\sigma}(s,K_2)\Delta s=\infty$,
Then every solution of (1.1) oscillates.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t)>0$ for $t\ge t_0$. Proceeding as in the proof of Theorem 2.1, we get (2.2) for $t\in[t_2,\infty)$, that is either $z^{\Delta}(t)>0$ or $z^{\Delta}(t)<0$. By lemma 2.1, it follows that $z^{\Delta}(t)>0$. From (1.1), it is easy to see for $t\ge t_1$, that

$$\begin{aligned}&(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p^{\beta}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))\\&+p^{\beta}q(\tau(t))x^{\alpha}(\delta(\tau(t))+v(t)x^{\beta}(\eta(t))+p^{\beta}v(\tau(t))x^{\beta}(\eta(\tau(t))=0.\end{aligned}\tag{2.12}$$

By assuming that there exists $\lambda>0$ such that $u^{\gamma}(x)+u^{\gamma}(y)\ge\lambda u^{\gamma}(x+y)$, x, $y\in\mathbb{R}^{+}$, and there exists $\mu>0$ such that $u^{\gamma}(x)+u^{\gamma}(y)\ge\mu u^{\gamma}(x+y)$, x, $y\in\mathbb{R}^{+}$, we obtain (note that $\gamma<\alpha<\beta$) that

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p^{\alpha}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta}+\lambda Q(t)z^{\alpha}(\delta(t))+\mu V(t)z^{\alpha}(\eta(t))\le 0.$$

for $t\in[t_2,\infty)_{\mathbb{T}}$, where $z(t)\le x(t)+px(\tau(t))$. Define $w(t)$ as in (2.3). Upon using the fact that

$$w^{\Delta}(t)=\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\delta})^{\alpha}}-\frac{r(z^{\Delta})^{\gamma})^{\delta}(z^{\alpha})^{\Delta}}{z^{\alpha}(z^{\delta})^{\alpha}}$$

and

$$\frac{(z^{\alpha})^{\Delta}}{(z^{\sigma})^{\alpha}}\ge\alpha\frac{(z^{\Delta})}{z^{\sigma}},\ \alpha>0\ for\ t\in[t_3,\infty)_{\mathbb{T}}.$$

By using the fact that $z(t)$ is nondecreasing and using (2.12) into (2.11) we obtain

$$w^{\Delta}\le\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}-\alpha w^{\sigma}\frac{z^{\Delta}}{z^{\rho}},\ t\ge t_3.$$

Due to (2.6) and $(z(\sigma(t)))^{\frac{\alpha}{\gamma}}\ge C$, there exists $t_4>t_3$ such that, for $t\in[t_4,\infty)_{\mathbb{T}}$,

$$w^{\Delta}\le\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}-\alpha Cr^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}.\tag{2.13}$$

From (2.13), we find

$$w^{\Delta}+a^{\alpha}w^{\tau\Delta}\le\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}-\alpha Cr^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}\frac{(r(z^{\Delta})^{\gamma})^{\tau\Delta}}{(z^{\sigma\Delta})^{\alpha}}-\alpha C(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}},$$

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that is,

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau\Delta} \le & \frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}+a^{\alpha}\frac{(r(z^{\Delta})^{\gamma})^{\tau\Delta}}{(z^{\sigma\Delta})^{\alpha}} \\ & -\alpha C\left[r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}}\right]. \end{aligned}$$

Applying Lemma 2.2 on the above inequality, we get

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau\Delta} \quad \le & \quad -\lambda Q\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\mu V\left(\frac{l\delta}{\sigma}\right)^{\alpha} \\ & -\alpha C\left[r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}}\right] \end{aligned}$$

for $t\in[t_1,\infty)_{\mathbb{T}}$, that is

$$w^{\Delta}+a^{\alpha}w^{\tau\Delta}\le-\lambda Q(t)\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\mu V(t)\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\alpha Cr^{-\frac{1}{\gamma}}(1+a^{\alpha})(w^{\sigma})^{1+\frac{1}{\gamma}}, \tag{2.14}$$

where we used the fact that $r^{\Delta}(t)\ge 0$ and $w(t)$ is a decreasing function due to (2.6) and (2.14) on $[t_1,\infty)_{\mathbb{T}}$. Integrating (2.14) from t to v for $t,\ v\in[t_1,\infty)_{\mathbb{T}}$, it is easy to verify that

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau(t)} \ge & \int_t^{\infty}\lambda Q(s)\left(\frac{l\delta}{\sigma}\right)^{\alpha}\Delta s+\int_t^{\infty}\mu V(s)\left(\frac{l\delta}{\sigma}\right)^{\alpha}\Delta s \\ & +\alpha C(1+a^{\alpha})\int_t^{\infty}\left[r(s)^{-\frac{1}{\gamma}}w(\sigma(s))^{1+\frac{1}{\gamma}}\right]\Delta s, \end{aligned}$$

that is,

$$w^{\Delta}+a^{\alpha}w^{\tau(t)}=a_2(t)+\alpha C(1+a^{\alpha})\int_t^{\infty}\left[r(s)^{-\frac{1}{\gamma}}w(\sigma(s))^{1+\frac{1}{\gamma}}\right]\Delta s,$$

which implies that

$$(1+a^{\alpha})w(\tau(t))\ge a_2(t)+\alpha C(1+a^{\alpha})\int_t^{\infty}r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s. \tag{2.15}$$

Then (H_2) and (2.15) yield that

$$w(t)\ge\frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+\alpha C\int_{\tau^{-1}(t)}^{\infty}r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s.$$

Indeed

$$w(t)\ge\frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}.$$

Hence the last inequality becomes

$$\begin{aligned} w(t) \quad \ge & \quad \frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+\alpha C\int_{\tau^{-1}(t)}^{\infty}\left[r^{-\frac{1}{\gamma}}(s)\left(\frac{1}{1+a^{\alpha}}\right)^{1+\frac{1}{\gamma}}(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right]\Delta s \\ = & \quad \frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+K_2\int_{\tau^{-1}(t)}^{\infty}\left[r^{-\frac{1}{\gamma}}(s)(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right]\Delta s \\ = & \quad A_2^{\gamma}(t,K_2), K_2=\alpha C\left(\frac{1}{1+a^{\alpha}}\right)^{1+\frac{1}{\gamma}}. \end{aligned}$$

Proceeding as in the proof of theorem 2.1, we obtain

$$\int_{t_4}^{t}r^{-\frac{1}{\gamma}}(s)A_2^{\sigma}(s,K_2)\Delta s<\infty,$$

a contradiction due to (H_6). The proof is complete. □

Theorem 2.9. *Let $1 \le p(t) \le p < \infty$, $r^{\Delta}(t) \ge 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, $\tau(\delta(t)) = \delta(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$ and $\gamma = \alpha = \beta$, $\eta(t) \ge \delta(t)$. If (H_0), $(H_5) - (H_7)$ and*

(H_7). $\limsup\limits_{t\to\infty} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s) A_2(s, K_2)\Delta s \right) > 1.$

Then every solution of (1.1) oscillates.

Proof. The proof of the theorem follows from Theorem 2.2 and Theorem 2.8. Hence the details are omitted. □

Theorem 2.10. *Let $1 \le p(t) \le a < \infty$, $r^{\Delta}(t) \ge 0$ $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\gamma > \alpha > \beta$, $\eta(t) \ge \delta(t)$. If (H_0), (H_2), $(H_5) - (H_7)$ and*

(H_8). $\limsup\limits_{t\to\infty} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s \right) \left[a_1(t) + K_3 \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s \right]^{\frac{1}{\gamma}} = \infty.$

Then every solution of (1.1) oscillates.

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