New Oscillation Criteria of First Order Neutral Delay Difference Equations of Emden–Fowler Type

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Abstract

In this paper, we will establish some new sufficient condition for oscillation of solutions of a certain class of first-order neutral delay difference equations of the form

$$\Delta\left(x_n - p_n x_{n-1}\right) + q_n x_{n-\tau}^{\gamma} = 0,$$

where γ is a quotient of odd positive integers. We will consider the sublinear and super linear cases. The results will be obtained by using the oscillation theorems of second order delay difference equations.

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1 Introduction

In recent decades there has been much research activity concerning oscillation and nonoscillation of first and second order delay and neutral delay difference equations, we refer the reader to the papers [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and the references cited therein. In the following, we recall some results of first order neutral delay difference equations of sublinear and super linear types that motivate the contents of this paper. Xiaoyan Lin in [12] studied the oscillatory behavior of solutions of the neutral difference equations with nonlinear neutral term of the form

$$\Delta\left(x_n - p_n x_{n-\sigma}^{\alpha}\right) + q_n x_{n-\tau}^{\beta} = 0, \text{ for } n \in \mathbb{N}_{n_0}, \tag{1.1}$$

where α and β are quotient of odd positive integers, τ and σ are nonnegative integers and $\{p_n\}$ and $\{q_n\}$ are two sequences of nonnegative real numbers. The authors obtained necessary and sufficient conditions for existence of oscillatory solutions and studied the two cases when $0 < \alpha < 1$ and when $\alpha > 1$. As usual, a nontrivial solution x_n of (1.1) is called nonoscillatory if it eventually positive or eventually negative, otherwise it is called oscillatory and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\mathbb{N}_i = \{i+1, i+2, ...\}$. Lalli [11] established several sufficient conditions for oscillation of the equation

$$\Delta\left(x_n + p x_{n-\delta k}\right) + q_n f\left(x_{n-\tau}\right) = F_n, \; n \geq n_0, \tag{1.2}$$

where $\delta = \pm 1$, p is a nonnegative real number, $k \in \mathbb{N} = \{1, 2, ...\}$, τ is a sequence of nonnegative integers with $\lim_{n\to\infty} \tau_n = \infty$, and $\{F_n\}$, $\{q_n\}$ are sequences of real numbers and f is a real valued function satisfying $xf(x) > 0$ for $x \neq 0$. El-Morshedy et al. [6] considered the equation

$$\Delta g(x_n + p_n x_{\sigma_n}) + f(n, x_{\tau_n}) = 0, \tag{1.3}$$

where $0 \leq p_n < p < 1$, σ_n and τ_n are sequences of integers such that $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} = \infty$ and $\sigma_{n+1} > \sigma_n$ for all $n \geq n_0$. They established several sufficient conditions for oscillation when the function f satisfies the condition

$$\frac{f(n, x)}{h(x)} \geq q_n, \ x \neq 0 \text{ and } n \geq n_0,$$

where $q_n \geq 0$ for $n \geq n_0$, $h \in C(\mathbb{R},\mathbb{R})$ and $xh(x) > 0$ for all $x \neq 0$. Recently Murugesan and Suganthi [13] discussed the oscillatory behavior of all solutions of the first order nonlinear neutral delay difference equation

$$[\Delta(r_n(a_n x_n - p_n x_{n-\tau}))] + q_n x_{n-\sigma} = 0,$$

where r_n and a_n are sequences of positive real numbers p_n and q_n are sequences of nonnegative real numbers, τ and σ are positive integers. Following this trend in this paper, we will consider the first order neutral delay difference equation

$$\Delta(x_n - p_n x_{n-1}) + q_n x^{\gamma}_{n-\tau} = 0, \ \text{for } n \in \mathbb{N}_{n_0}, \tag{1.4}$$

Our aim in this paper is to establish some new sufficient conditions for oscillation of (1.4) by using a new technique when $0 < p_n \leq p \leq 1$ and we will consider the sublinear and the super linear cases. The new technique depends on the application of an invariant substitution which transforms the first nonlinear neutral difference equation to a second nonlinear difference equation. This allows us to obtain several sufficient conditions for oscillation of (1.4) by employing the oscillation conditions of second order delay difference equations by using the Riccati technique.

2 Main results

In this section, we prove the main results but before we do this, we apply an invariant substitution which transforms the first order neutral equation to a non-neutral second order difference equations. This substitution is given by

$$y_{n+1} = x_n \prod_{i=1}^{n} \frac{1}{p_i}, \quad \text{where} \quad \prod_{i=1}^{n} p_i = O(n), \tag{2.1}$$

This gives us that

$$x_n = y_{n+1} \prod_{i=1}^{n} p_i, \quad x_{n-1} = y_n \prod_{i=1}^{n-1} p_i, \quad \text{and } x_{n-\tau} = y_{n-\tau+1} \prod_{i=1}^{n-\tau} p_i. \tag{2.2}$$

From (2.2), we have

$$x_n - p_n x_{n-1} = \Delta y_n \prod_{i=1}^{n} p_i. \tag{2.3}$$

Substituting (2.3) into (1.4), we obtain

$$\Delta\left(\Delta y_n \prod_{i=1}^{n} p_i\right) + q_n \prod_{i=1}^{n-\tau} p_i y_{n-\tau+1} = 0. \tag{2.4}$$

Setting $d_n = \prod_{i=1}^{n} p_i$, and $Q_n = q_n d_{n-\tau}$ then (2.4) becomes

$$\Delta\left(d_n \Delta y_n\right) + Q_n y^{\gamma}_{n-(\tau-1)} = 0,\ n \in \mathbb{N}_0. \tag{2.5}$$

In this section, we intend to use the Riccati transformation technique for obtaining several new oscillation criteria for (1.4). First we state some fundamental lemmas for second order difference equations that will be used in the proofs of the main results (see [15]).

Lemma 2.1 *Assume that p_n is a real sequence with $0 < p_n \le p < 1$ for all $n \in \mathbb{N}$. Furthermore assume that*

$$\sum_{n=1}^{\infty} \frac{1}{d_n} = \infty. \tag{2.6}$$

Let y be a positive solution of (2.5). Then
(I). $\Delta y(n) \ge 0$, $y(n) \ge n\Delta y(n)$ for $n \ge N$,
(II). y is nondecreasing, while $y(n)/n$ is nonincreasing for $n \ge N$.

Lemma 2.2 *Assume that p_n is a real sequence with $0 < p_n \le p < 1$ for all $n \in \mathbb{N}$. Furthermore assume that (2.6) holds. If y_n be a nonoscillatory solution of (2.5) such that $y_n \ge 0,\ \Delta y_n \le 0$, then $\lim_{n\to\infty} y_n = 0$ and hence*

$$\lim_{n\to\infty} \frac{x_n}{d_n} = 0, \tag{2.7}$$

where x_n is a solution of (1.4).

Throughout this paper, we will assume that the real sequences $p_n,\ q_n$ are nonnegative, γ is a quotient of odd positive integers, τ is a nonnegative integer. Now, we state and prove the sufficient conditions which ensure that each solution of equation (1.4) is oscillatory or satisfies (2.7). We start with the case when $0 < \gamma \le 1$.

Theorem 2.3 *Assume that (H_1) holds and $\Delta d_n \ge 0$. Furthermore, assume that there exists a positive sequence ρ_n such that,*

$$\lim_{n\to\infty} \sup \sum_{i=n_0}^{n} \left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right] = \infty, \tag{2.8}$$

where $d_n = \prod_{i=1}^{n} p_i$ and $Q_n = q_n d_{n-\tau}$. Then every solution of (1.4) oscillates for all $0 < \gamma \le 1$.

Proof. Assume to the contrary that x_n be a nonoscillatory solution of (1.4) such that $x_{n-1},\ x_{n-\tau},\ x_n > 0$ for all large $n \ge n_1 > n_0$ sufficiently large. We shall consider only this case, since the substitution $y_n = -x_n$ transforms equation (1.4) into an equation of the same form. From (2.1) we see that y_n is a positive solution of (2.5) such that $y_n > 0$ and $y_{n-\tau+1} > 0$ for $n > n_1 > n_0$ sufficiently large. From equation (2.5), we have

$$\Delta\left(d_n \Delta y_n\right) = -Q_n y^{\gamma}_{n-\tau+1} \le 0,\ n \ge n_1, \tag{2.9}$$

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and then $d_n\Delta y_n$ is an eventually nonincreasing sequence. We first show that $d_n\Delta y_n \geq 0$ for $n \geq n_0$. In fact, if there exists an integer $n_1 \geq n_0$ such that $d_{n_1}\Delta y_{n_1} = c < 0$ then (2.9) implies that $d_n\Delta y_n \leq c$ for $n \geq n_1$ that is $\Delta y_n \leq c/d_n$, and hence

$$y_n \leq y_{n_1} + c\sum_{i=n_1}^{n-1}\frac{1}{d_i} \to -\infty, \text{ as } n \to \infty, \tag{2.10}$$

which contradicts the fact that $y_n > 0$ for $n \geq n_0$ then $d_n\Delta y_n \geq 0$. Also since $\Delta d_n \geq 0$, we can prove that $\Delta^2 y_n > 0$ for $n \geq n_1$. Therefore we have

$$y_n > 0,\ \Delta y_n \geq 0, \text{ and } \Delta^2 y_n \leq 0, \text{ for } n \geq n_1. \tag{2.11}$$

From (2.9) and (2.11)

$$d_{n-\tau+1}\Delta y_{n-\tau+1} \geq d_{n+1}\Delta\left(y_{n+1}\right) \text{ and } y_{n-\tau+1} \geq y_{n-\tau}. \tag{2.12}$$

Defining the sequence u_n by the Riccati substitution

$$u_n = \rho_n\frac{d_n\Delta y_n}{y_{n-\tau+1}^{\gamma}}, \quad \text{for } n > n_1. \tag{2.13}$$

This implies that $u_n > 0$, and

$$\Delta u_n = d_{n+1}\Delta y_{n+1}\Delta\left[\frac{\rho_n}{y_{n-\tau+1}^{\gamma}}\right] + \rho_n\frac{\Delta\left(d_n\Delta y_n\right)}{y_{n-\tau+1}^{\gamma}}.$$

Hence

$$\Delta u_n = d_{n+1}\Delta y_{n+1}\left[\frac{\Delta\rho_n\left(y_{n-\tau+1}^{\gamma}\right) - \rho_n\left(\Delta y_{n-\tau+1}^{\gamma}\right)}{y_{n-\tau+1}^{\gamma}y_{n-\tau+2}^{\gamma}}\right] + \rho_n\frac{\Delta\left(d_n\Delta y_n\right)}{y_{n-\tau+1}^{\gamma}}. \tag{2.14}$$

From this, (2.5) and (2.14) we see that

$$\Delta u_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \left[\frac{d_{n+1}\Delta y_{n+1}\rho_n\Delta y_{n-\tau+1}^{\gamma}}{y_{n-\tau+2}^{\gamma}y_{n-\tau+1}^{\gamma}}\right] - \rho_n Q_n. \tag{2.15}$$

From (2.5) and (2.14), we have

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \frac{d_{n+1}\Delta y_{n+1}\rho_n\Delta y_{n-\tau+1}^{\gamma}}{y_{n-\tau+2}^{2\gamma}}. \tag{2.16}$$

By using the inequality (see [8]),

$$x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}\left(x-y\right), \text{ for all } x \neq y > 0 \text{ where } 0 < \gamma \leq 1, \tag{2.17}$$

we have

$$\begin{aligned} \Delta y_{n-\tau+1}^{\gamma} &= \left(y_{n+2-\tau}^{\gamma} - y_{n+1-\tau}^{\gamma}\right) \geq \gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(y_{n-\tau+2} - y_{n-\tau+1}\right) \\ &= \gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(\Delta y_{n-\tau+1}\right). \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.16), we obtain that

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \rho_n d_{n+1}\frac{\gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(\Delta y_{n-\tau+1}\right)\Delta y_{n+1}}{y_{n-\tau+2}^{2\gamma}}. \tag{2.19}$$

From (2.12) and (2.19), we have that

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma\rho_n d_{n+1}^2 \left(\Delta y_{n+1}\right)^2}{d_{n-\tau+1}\left(y_{n+2-\tau}\right)^{1-\gamma}\left(y_{n-\tau+2}^{\gamma}\right)^2}.$$

Hence,

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma\rho_n}{\left(\rho_{n+1}\right)^2 d_{n-\tau+1}\left(y_{n+2-\tau}\right)^{1-\gamma}}\left(u_{n+1}\right)^2. \tag{2.20}$$

From (2.11), we conclude that

$$y_n \leq y_{n_0} + \Delta y_{n_0}\left(n - n_0\right), \quad n \geq n_1,$$

and consequently there exists a $n_2 \geq n_2$ and appropriate constant $\beta \geq 1$ such that

$$y_n \leq \beta n, \quad \text{for} \quad n \geq n_2,$$

and this implies that

$$y_{n+2-\tau} \leq \beta\left(n+2-\tau\right), \text{ for } \ n \geq n_3 = n_2 + \tau + 2,$$

and then

$$\frac{1}{\left(y_{n+2-\tau}\right)^{1-\gamma}} \geq \frac{1}{\left(\beta\left(n+2-\tau\right)\right)^{1-\gamma}}. \tag{2.21}$$

Substituting (2.21) into (2.20) we obtain

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma\rho_n}{\left(\rho_{n+1}\right)^2 d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}}\left(u_{n+1}\right)^2 \tag{2.22}$$

Hence

$$\begin{aligned}\Delta u_n \quad &\leq \quad -\rho_n Q_n + \frac{d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}\left(\Delta\rho_n\right)^2}{\rho_n} \\ &\quad - \left[\frac{\sqrt{\rho_n}}{\rho_{n+1}\sqrt{\left(\beta\left(n+2-\tau\right)\right)^{1-\gamma}} d_{n-\tau+1}} u_{n+1} - \frac{\Delta\rho_n\sqrt{d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}}}{2\rho_n}\right]^2\end{aligned}$$

Then, we have

$$\Delta u_n \leq -\left[\rho_n Q_n - \frac{d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}\left(\Delta\rho_n\right)^2}{\rho_n}\right] \tag{2.23}$$

Summing (2.23) from n_3 to n we obtain

$$-u_{n_3} < u_{n+1} - u_{n_3} \leq -\sum_{i=n_3}^{n}\left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right]$$

which yields

$$\sum_{i=n_3}^{n}\left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right] < c_1,$$

for all large n, and this contrary to (2.8). The proof is complete. ■

From the Theorem 2.3, we can obtain different condition for oscillation of all solutions of (1.4) by different choices of ρ_n. For example if we take $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda > 1$ is a constant we have the following result.

Corollary 2.4 *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.8) is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[s^{\lambda}Q_s-\frac{d_{s-\tau+1}\beta^{1-\gamma}\left(s+2-\tau\right)^{1-\gamma}\left(\Delta s^{\lambda}\right)^2}{s^{\lambda}}\right]=\infty.$$

Then every solution of (1.4) oscillates for all $0<\gamma\leq 1$.

Remark 2.5 *When* $\gamma=1$ *the equation (1.4) reduced to linear delay difference equation*

$$\Delta\left(x_n-p_nx_{n-1}\right)+q_nx_{n-\tau}=0,\ \textit{for}\ n\in\mathbb{N}_{n_0},$$

and the condition (2.8) in Theorem 2.3 reduced to

$$\lim_{n\to\infty}\sup\sum_{i=n_0}^{n}\left[\rho_iQ_i-\frac{d_{i-\tau+1}\left(\Delta\rho_i\right)^2}{\rho_i}\right]=\infty, \tag{2.24}$$

where $d_n=\prod_{i=1}^{n}p_i$ *and* $Q_n=q_nd_{n-\tau}$ *for all* $0<\gamma\leq 1$.

Now, we consider the case when $\gamma\geq 1$.

Theorem 2.6 *Assume that (2.6) holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=1}^{\infty}$ *such that for every positive constant* M,

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[\rho_lq_l-\frac{\left(d_{l-\sigma}\right)^{\gamma}\left(\Delta\rho_l\right)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{l+1}\right)^{2\gamma-2}\rho_l}\right]=\infty, \tag{2.25}$$

where $\sigma=\tau-1$. *Then every solution of (1.4) oscillates for all* $\gamma\geq 1$.

Proof. Suppose to the contrary that x_n is a nonoscillatory solution of (1.4). Without loss of generality, we may assume that x_n is an eventually positive solution of (1.4) such that $x_{n-1},\ x_{n-\tau},\ x_n>0$ for all large $n\geq n_1>n_0$ sufficiently large. We shall consider only this case, since the substitution $y_n=-x_n$ transforms equation (1.4) into an equation of the same form. As in the proof of Theorem 2.3, we have by (2.6) that

$$y_n>0,\ \Delta y_n\geq 0,\ \Delta(d_n\left(\Delta y_n\right))\leq 0,\ n\geq n_1. \tag{2.26}$$

Define the sequence u_n by

$$u_n:=\rho_n\frac{d_n\Delta y_n}{y_{n-\sigma}^{\gamma}}. \tag{2.27}$$

Then $u_n>0$, and

$$\Delta u_n=d_{n+1}\Delta y_{n+1}\Delta\left[\frac{\rho_n}{y_{n-\sigma}^{\gamma}}\right]+\frac{\rho_n\Delta(d_n\Delta y_n)}{y_{n-\sigma}^{\gamma}}. \tag{2.28}$$

In view of (2.5), (2.28), we have

$$\Delta u_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1}-\frac{\rho_nd_{n+1}\Delta y_{n+1}\Delta y_{n-\sigma}^{\gamma}}{y_{n+1-\sigma}^{\gamma}y_{n-\sigma}^{\gamma}}. \tag{2.29}$$

From (2.26), we see that

$$d_{n-\sigma}\Delta y_{n-\sigma}\geq d_{n+1}\Delta y_{n+1},\quad\text{and}\quad y_{n+1-\sigma}\geq y_{n-\sigma}. \tag{2.30}$$

Substituting (2.30) into (2.29), we have

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\rho_n d_{n+1} \Delta y_{n+1} \Delta y^{\gamma}_{n-\sigma}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2}. \tag{2.31}$$

Now, by using the inequality

$$x^{\gamma} - y^{\gamma} \geqslant 2^{1-\gamma}(x-y)^{\gamma}, \text{ for all } x \geqslant y > 0 \text{ and } \gamma \geqslant 1,$$

we find that

$$\Delta y^{\gamma}_{n-\sigma} = y^{\gamma}_{n+1-\sigma} - y^{\gamma}_{n-\sigma} \geqslant 2^{1-\gamma}(y_{n+1-\sigma} - y_{n-\sigma})^{\gamma} = 2^{1-\gamma}\left(\Delta y_{n-\sigma}\right)^{\gamma}. \tag{2.32}$$

Substituting (2.32) into (2.31), we have

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - 2^{1-\gamma}\rho_n d_{n+1} \frac{\Delta y_{n+1}\left(\Delta y_{n-\sigma}\right)^{\gamma}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2}. \tag{2.33}$$

From (2.30) and (2.33), we obtain

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - 2^{1-\gamma}\rho_n \frac{\left(d_{n+1}\right)^{\gamma+1}}{\left(d_{n-\sigma}\right)^{\gamma}} \frac{\left(\Delta y_{n+1}\right)^{\gamma+1}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2}. \tag{2.34}$$

Hence,

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\left(d_{n+1}\right)^{\gamma+1}}{\left(d_{n-\sigma}\right)^{\gamma}} \frac{2^{1-\gamma}\rho_n\left(\Delta y_{n+1}\right)^{2\gamma}}{\left(y^{\gamma}_{n+1-\sigma}\right)^2\left(\Delta y_{n+1}\right)^{\gamma-1}}. \tag{2.35}$$

From the definition of u_n, we get that

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{2^{1-\gamma}\rho_n}{\left(\rho_{n+1}\right)^2} \frac{\left(d_{n+1}\right)^{\gamma-1}}{\left(d_{n-\sigma}\right)^{\gamma}} \frac{u^2_{n+1}}{\left(\Delta y_{n+1}\right)^{\gamma-1}}. \tag{2.36}$$

Since $\{d_n\left(\Delta y_n\right)\}$ is a positive and nonincreasing sequence, there exists a $n_2 \geq n_1$ sufficiently large such that $d_n\left(\Delta y_n\right) \leq 1/M$ for some positive constant M and $n \geq n_1$, and hence by (2.26), we have

$$\frac{1}{\left(\Delta y_{n+1}\right)^{\gamma-1}} \geqslant \left(M d_{n+1}\right)^{\gamma-1}.$$

Substituting the last inequality into (2.36), we obtain

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \left(\frac{M}{2}\right)^{\gamma-1} \frac{\rho_n\left(d_{n+1}\right)^{2\gamma-2}}{\left(\rho_{n+1}\right)^2} \frac{1}{\left(d_{n-\sigma}\right)^{\gamma}} u^2_{n+1}, \tag{2.37}$$

so that

$$\begin{aligned}
\Delta u_n \quad &\leq \quad -\rho_n q_n + \frac{\left(d_{n-\sigma}\right)^{\gamma}\left(\Delta\rho_n\right)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2}\rho_n} - \\
&\quad \left[\frac{\sqrt{\left(\frac{M}{2}\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2}\rho_n}}{\rho_{n+1}\sqrt{\left(d_{n-\sigma}\right)^{\gamma}}} u_{n+1} - \frac{\sqrt{\left(d_{n-\sigma}\right)^{\gamma}}\Delta\rho_n}{2\sqrt{\left(\frac{M}{2}\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2}\rho_n}}\right]^2 \\
&< \quad -\left[\rho_n q_n - \frac{\left(d_{n-\sigma}\right)^{\gamma}\left(\Delta\rho_n\right)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2}\rho_n}\right]
\end{aligned}$$

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Then, we have

$$\Delta u_n < -\left[\rho_n q_n - \frac{(d_{n-\sigma})^\gamma (\Delta\rho_n)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}\right]. \tag{2.38}$$

Summing (2.38) from n_2 to n, we obtain

$$-u_{n_2} < u_{n+1} - u_{n_2} < -\sum_{l=n_2}^{n}\left[\rho_l q_l - \frac{(d_{l-\sigma})^\gamma (\Delta\rho_l)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{l+1})^{2\gamma-2}\rho_l}\right],$$

which yields

$$\sum_{l=n_2}^{n}\left[\rho_l q_l - \frac{(d_{l-\sigma})^\gamma (\Delta\rho_l)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{l+1})^{2\gamma-2}\rho_l}\right] < c_1,$$

for all large n. This contradicts (2.25). The proof is complete. ■

From Theorem 2.6, we can obtain different conditions for oscillation of all solutions of (1.4) when (2.6) holds by different choices of $\{\rho_n\}$. For example, let $\rho_n = n^\lambda$, $n \geq n_0$ and $\lambda > 1$ is a constant. From Theorem 2.6 we have the following result.

Corollary 2.7 *Assume that all the assumptions of Theorem 2.6 hold, except the condition (2.25) is replaced by*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[s^\lambda q_s - \frac{(d_{s-\sigma})^\gamma ((s+1)^\lambda - s^\lambda)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{s+1})^{2\gamma-2}s^\lambda}\right] = \infty. \tag{2.39}$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

As a variant of the Riccati transformation technique used above, we will derive some oscillation criterion which can be considered as a discrete analogy of the Philos condition for oscillation of second order differential equation by introducing the following class of sequences that will be used in this chapter and later. Let

$$\pounds_0 = \{(m,n):\ m > n \geq n_0\}, \quad \pounds = \{(m,n):\ m \geq n \geq n_0\}.$$

The double sequence $H_{m,n} \in \Sigma$ if:

(I). $H(m,m) = 0$ on $\pounds$,

(II). $H(m,n) > 0$ on $\pounds_0$;

(III). $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$, and there exists a double sequence $h_{m,n}$ such that

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}, \text{ for } \ m > n \geq 0.$$

Theorem 2.8 *Assume that (2.6) hold. Let $\{\rho_n\}_{n=1}^\infty$ be a positive sequence and $H_{m,n} \in \Sigma$. If*

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - B_n\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] = \infty, \tag{2.40}$$

where

$$B_n := \frac{(d_{n-\sigma})^\gamma \rho_{n+1}^2}{2^{3-\gamma}M^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}.$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

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Proof. We proceed as in the proof of Theorem 2.6, we may assume that (1.4) has a nonoscillatory solution x_n such that $x_n > 0$. As in the proof of Theorem 2.6 we get that (2.26) holds. Define $\{u_n\}$ by (2.27) as before, then we have $u_n > 0$ and there is some $M > 0$ such that (2.37) holds. For the sake of convenience, let us set

$$\bar{\rho}_n = \frac{2^{1-\gamma}(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}{(d_{n-\sigma})^{\gamma}}.$$

Then, we have from (2.37) that

$$\rho_n q_n \leq -\Delta u_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\bar{\rho}_n}{(\rho_{n+1})^2} u_{n+1}^2. \tag{2.41}$$

Therefore, we get

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta u_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\bar{\rho}_n u_{n+1}^2}{(\rho_{n+1})^2}. \tag{2.42}$$

The rest of the proof is similar to the proof of [15, Theorem 2.3.6]. ■

As an immediate consequence of Theorem 2.8, we get the following:

Corollary 2.9 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=n_0}^{m-1} H_{m,n}\rho_n q_n = \infty,$$

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=n_0}^{m-1} \frac{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}{(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2 < \infty.$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

By choosing the sequence $H_{m,n}$ in appropriate manners, we can derive several oscillation criteria for (1.4). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\left.\begin{array}{l} H_{m,n} = (m-n)^{\lambda}, \quad \lambda \geq 1, m \geq n \geq 0, \\ H_{m,n} = \left(\log \frac{m+1}{n+1}\right)^{\lambda}, \lambda \geq 1, m \geq n \geq 0, \\ H_{m,n} = (m-n)^{(\lambda)} \quad \lambda > 2,\ m \geq n \geq 0, \end{array}\right\} \tag{2.43}$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)...(m-n+\lambda-1)$, and

$$\Delta_2(m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Hence we have the following result which gives new sufficient conditions for the oscillation of (1.4) of Kamenev type.

Corollary 2.10 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=0}^{m-1} \left[(m-n)^{\lambda}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} V_{m,n}^2\right] = \infty, \tag{2.44}$$

where

$$V_{m,n} := \left(\lambda(m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{(m-n)^{\lambda}} \right).$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

Corollary 2.11 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{(\log(m+1))^{\lambda}} \sum_{n=0}^{m-1} \left[\left(\log \frac{m+1}{n+1} \right)^{\lambda} \rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}^2 R_{m,n}^2 \right] = \infty, \tag{2.45}$$

where

$$R_{m,n} = \left(\frac{\lambda}{n+1} \left(\log \frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{\left(\log \frac{m+1}{n+1} \right)^{\lambda}} \right).$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

Corollary 2.12 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^{(\lambda)}} \sum_{n=0}^{m-1} (m-n)^{(\lambda)} \left[\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} U_n^2 \right] = \infty, \tag{2.46}$$

where

$$U_n := \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta\rho_n}{\rho_{n+1}} \right)^2.$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

In the following theorem, we consider the case when $0 < \gamma < 1$.

Theorem 2.13 *Assume that (2.6) holds and* $\Delta d_n \geq 0$. *If*

$$\sum_{n=n_0}^{\infty} \left(\frac{n-\sigma}{d_n} \right)^{\gamma} q_n = \infty. \tag{2.47}$$

Then every solution of (1.4) oscillates for all $0 < \gamma < 1$.

Proof. Proceeding as in Theorem *2.6*, we assume that (1.4) has a nonoscillatory solution, say $x_n > 0$ and $x_{n-\tau} > 0$ for all $n \geq n_0$. From the proof of Theorem 2.6 we know that $\Delta y_n > 0$, then y_n is nondecreasing sequence. Since $\Delta d_n \geq 0$ we obtain that $\Delta^2 y_n \leq 0$ and then Δy_n is a nonincreasing for all $n \geq n_1 \geq n_0$. Then, we have $y_n \geq (n-n_1)\Delta y_n$ which implies that $y_n \geq \frac{n}{2}\Delta y_n$ for $n \geq n_2 \geq 2n_1 + 1$. Then

$$y_{n-\sigma} \geq \frac{n-\sigma}{2} \Delta y_{n-\sigma} \geq \frac{n-\sigma}{2} \Delta y_{n+1}, \quad for\ n \geq N = n_2 + \sigma. \tag{2.48}$$

From (2.5) and (2.48) by dividing by $z_{n+1} = (d_n \Delta y_{n+1})^{\gamma} > 0$ and summing from $2N$ to k, we obtain

$$\sum_{n=2N}^{k} \left(\frac{n-\sigma}{2d_n} \right)^{\gamma} q_n \leq - \sum_{n=2N}^{k} \frac{\Delta(z_n)}{(z_{n+1})^{\gamma}}, \quad k \geq 2N. \tag{2.49}$$

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Since

$$y^{\gamma}-z^{\gamma}\leq \gamma y^{\gamma-1}(y-z) \text{ for } \gamma<1 \text{ and } y>z>0,$$

we see that

$$\Delta\left(z_n^{1-\gamma}\right)=\left(z_{n+1}^{1-\gamma}\right)-\left(z_n^{1-\gamma}\right)\leq(1-\gamma)(z(n+1))^{-\gamma}\Delta z(n).$$

Substituting in (2.49), we see that

$$\begin{aligned}\sum_{n=2N}^{k}\left(\frac{n-\sigma}{2d_n}\right)^{\gamma}q_n &\leq -\sum_{n=2N}^{k}\frac{\Delta\left(z_n^{1-\gamma}\right)}{(1-\gamma)}=-\frac{\left(z_{k-1}^{1-\gamma}\right)}{(1-\gamma)}+\frac{\left(z_{2N}^{1-\gamma}\right)}{(1-\gamma)}\\ &< \frac{\left(z_{2N}^{1-\gamma}\right)}{(1-\gamma)}<\infty, \text{ as } n\to\infty\end{aligned}$$

which contradicts (2.47). The proof is complete. ■

Now, we consider the case when

$$\sum_{n=0}^{\infty}\left(\frac{1}{d_n}\right)<\infty, \tag{2.50}$$

holds and establish some oscillation criteria for (1.4) in the sublinear and superlinear cases.

Theorem 2.14 *Assume that (2.50) holds and there exist positive sequences $\{\rho_n\}_{n=1}^{\infty}$ such that (2.25) holds for every positive constant M, and*

$$\sum_{n=0}^{\infty}\left(\frac{1}{d_n}\sum_{i=n_0}^{n-1}q_i\right)=\infty. \tag{2.51}$$

Then every solution of (1.4) oscillates or $\lim_{n\to\infty}x_n/d_n=0$ for all $\gamma\geq 1$.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1.4). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. From (2.5), we have

$$\Delta(d_n\Delta y_n)\leq -q_n y_{n-\sigma}^{\gamma}\leq 0, \qquad n\geq n_0, \tag{2.52}$$

and so $\{d_n(\Delta y_n)\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, either $\{\Delta y_n\}$ is eventually negative or eventually positive. If $\{\Delta y_n\}$ is eventually positive, we are then back to the case where (2.26) holds. Thus the proof of Theorem 2.6 goes through, and we may conclude that $\{y_n\}$ cannot be eventually positive, which is not possible. If $\{\Delta y_n\}$ is eventually negative, then $\lim_{n\to\infty}y_n=b\geq 0$. We assert that $b=0$. If not then $y_{n-\sigma}^{\gamma}\to b^{\gamma}>0$ as $n\to\infty$, and hence there exists $n_1\geq n_0>0$ such that $y_{n-\sigma}^{\gamma}\geq b^{\gamma}$. Therefore from (2.52) we have

$$\Delta(d_n\Delta y_n)\leq -q_n b^{\gamma}.$$

The rest of the proof is similar to the proof of [15, Theorem 2.3.7] and hence is omitted. ■

By choosing $\{\rho_n\}_{n=1}^{\infty}$ in appropriate manners, we may obtain different oscillation criteria. For instance, let $\rho_n=n^{\lambda}$ for $n\geq 0$ and $\lambda>1$. Then we have the following oscillation conditions of all solutions of (1.4) when (2.50) holds.

Corollary 2.15 *Assume that all assumptions of Theorem 2.14 hold, except that the condition (2.25) is replaced by (2.39). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Theorem 2.16 *Assume that (2.50) and (2.51) hold. Furthermore, assume that there exists a double sequence* $H_{m,n} \in \Sigma$ *such that (2.40) holds. Then every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.4). Then as seen in the proof of Theorem *2.3*, either $\{\Delta x_n\}$ is eventually positive or is eventually negative. In the case when $\{\Delta y_n\}$ is eventually positive, we may follow the proof of Theorem *2.8* and obtain a contradiction. If $\{\Delta y_n\}$ is eventually negative, then we may follow the proof of Theorem *2.14* to show that $\{y_n\}$ converges to zero.

By choosing $H_{m,n}$ in appropriate manners, we can derive several oscillation criteria for (2.5) when (2.50) holds. For instance, let us consider the double sequence $H_{m,n}$ defined again by (2.43). Hence we have the following results.

Corollary 2.17 *Assume that all the assumptions of Theorem 2.16 hold, except that the condition (2.40) is replaced by (2.44). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Corollary 2.18 *Assume that all the assumptions of Theorem 2.16 hold, except that the condition (2.40) is replaced by(2.45) or (2.46). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Theorem 2.19 *Assume that (2.50) and (2.47) hold. Let* $\{\rho_n\}_{n=1}^{\infty}$ *such that (2.51) holds. Then every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $0 < \gamma < 1$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.4). Then as seen in the proof of Theorem 2.6, either $\{\Delta y_n\}$ is eventually positive or is eventually negative. In the case when $\{\Delta y_n\}$ is eventually positive , we may follow the proof of Theorem 2.13 and obtain a contradiction. If $\{\Delta y_n\}$ is eventually negative, then we may follow the proof of Theorem 2.14 to show that $\{x_n/d_n\}$ converges to zero.

From Theorem 2.14 if $\rho_n = 1$, we see that the Riccati inequality associated with the equation (1.4) is given by

$$\Delta u_n + \rho_n q_n + \frac{1}{a_n} u_{n+1}^2 \leq 0, \tag{2.53}$$

where

$$A_n = \frac{2^{\gamma-1}\left(d_{n-\sigma}\right)^{\gamma}}{\left(M\right)^{\gamma-1}\left(d_{n+1}\right)^{2\gamma-2}} > 0, \tag{2.54}$$

for every positive constant $M > 0$. Using the inequality (2.53) and proceeding as in the proof [15, Theorem 2.3.8], we can prove the following Hille and Nehari type results.

Theorem 2.20 *Assume that* (H_1) *holds and* $\Delta d_n \geq 0$. *Furthermore, assume that*

$$\liminf_{n\to\infty} \frac{n}{A_n} \sum_{n+1}^{\infty} q(s) > \frac{1}{4},$$

or

$$\liminf_{n\to\infty} \frac{n}{A_n} \sum_{n+1}^{\infty} q_s + \liminf_{n\to\infty} \frac{1}{n} \sum_{N}^{n-1} \frac{s^2}{A_n} q_s > 4.$$

Then every solution of (1.4) is oscillatory.

References

[1] R. P. Agarwal, M. M. S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations, Math. Comput. Modelling 24 (1996), 5–11.

[2] R. P. Agarwal, M. M. S. Manuel, and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations II, Appl. Math. Lett. 10 (1997), 103–109.

[3] A. Bezubik, M. Migda, M. Nockowska and E. Schmeidel, Trichotomy of nonoscillatory solutions to second-order neutral difference equation with quasi-difference, Adv. Diff. Eqns. (2015).

[4] M. Budericenic, Oscillation of a second order neutral difference equation, Bull Cl. Sci. Math. Nat. Sci. Math. 22 (1994), 1–8.

[5] M. P. Chen, B. S. Lalli and J. S. Yu, Oscillation in neutral delay difference equations with variable coefficients, Comp. Math. Applic. 29 (3), 5-12, (1995).

[6] H. A. El-Morshedy and S. R. Grace, Oscillation of some nonlinear difference equations, J. Math. Anal. Appl. 281 (2003) 10–21.

[7] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991).

[8] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2^{nd} Ed. Cambridge Univ. Press 1952.

[9] H. J. Li and C. C. Yeh, Oscillation criteria for second order neutral delay difference equations, Comp. Math. Appl. 36 (1998), 123–132.

[10] B. S. Lalli, B. G. Zhang and J. Z. Li, On the oscillation of solutions and existence of positive solutions of neutral difference equations, J. Math. Anal. Appl. 158 (1991), 213-233.

[11] B. S. Lalli, Oscillation theorems for neutral difference equations, Comp. Math. Appl. 28, (1994) 191-202.

[12] X. Lin, Oscillation of solutions of neutral difference equations with a nonlinear neutral term, Comp. Math. Appl. 52 (2006) 439-448.

[13] A. Murugesan, and R. Suganthi, Oscillation criteria for first order nonlinear neutral delay difference equations with variable coefficients, Inter. J. Math. Stat. Inven. 4 (2016) 35-40.

[14] S. H. Saker, New oscillation criteria for second order nonlinear neutral delay difference equations, Appl. Math. Comput. 142 (2003), 99–111.

[15] S. H. Saker, *Oscillation Theory of Delay Differential and Difference Equations Second and Third Orders,* Verlag Dr. Muller (2010).

[16] S. H. Saker, and S. S. Cheng, Kamenev type oscillation criteria for nonlinear difference equations, Czechoslovak Math. J. 54 (2004), 955-967.

[17] Y. G. Sun and S. H. Saker, Oscillation for second order nonlinear neutral delay difference equations, Appl. Math. Comput. 163 (2005), 909–918.

[18] A. Sternal and B. Szmanda, Asymptotic and oscillatory behavior of certain difference equations, Le Mat. 51 (1996), 77–86.

[19] Y. Shoukaku, On the oscillation of solutions of first-order difference equations with delay, Comm. Math. Analysis 20 (2017), 62-67.

[20] X. H. Tang and Y. J. Liu, Oscillations for nonlinear delay difference equations, Tamkang J. Math. 32 (2001), 275–280.

[21] X. H. Tang, Necessary and sufficient conditions of oscillation for a class of neutral difference equations with variable coefficients (in Chinese), J. Hunan University 26 (6) (1996), 20-26.

[22] E. Thandapani and R. Arul, Oscillation properties of second order nonlinear neutral delay difference equations, Indian J. Pure Appl. Math. 28 (1997), 1567–1571.

[23] G. Zhang and Y. Gao, *Oscillation Theory for Difference Equations*, Publishing House of Higher Education, Beijing, (2001).