Explicit identities involving truncated exponential polynomials and phenomenon of scattering of their zeros

C. S. RYOO

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In this paper, we study differential equations arising from the generating functions of truncated exponential polynomials. We give explicit identities for the truncated polynomials. Using numerical investigation, we observe the behavior of complex roots of the truncated polynomials $e_n(x)$. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the truncated polynomials $e_n(x)$.

Key words : Differential equations, complex roots, truncated polynomials.

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1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, tangent numbers and polynomials, Genocchi numbers and polynomials, Laguerre polynomials, and Hermite polynomials. These numbers and polynomials possess many interesting properties and arising in many areas of mathematics, physics, and applied engineering(see [1-14]). By using software, many mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Numerical experiments of Euler polynomials, Bernoulli polynomials, tangent polynomials, Genocchi polynomials, Laguerre polynomials, and Hermite polynomials have been the subject of extensive study in recent year and much progress have been made both mathematically and computationally. Using computer, a realistic study for the zeros of truncated polynomials $e_n(x)$ is very interesting. The main purpose of this paper is to observe an interesting phenomenon of 'scattering' of the zeros of the truncated polynomials $e_n(x)$ in complex plane. Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. We first give the definitions of the truncated exponential polynomials. It should be mentioned that the definition of truncated exponential polynomials $e_n(x)$ can be found in [1, 3]. The truncated exponential polynomials $e_n(x)$ are defined by means of the generating function:

$$\left(\frac{1}{1-t}\right) e^{xt} = \sum_{n=0}^{\infty} e_n(x)t^n, \quad |t| < 1. \tag{1.1}$$

We recall that G. Dattoli and M. Migliorati(see [3]) studied some properties of truncated exponential polynomials $e_n(x)$. The truncated exponential polynomials $e_n(x)$ satisfy the following relations

$$\frac{d}{dx} e_n(x) = e_{n-1}(x),$$

$$e_{n+1}(x) = \left(1 + \frac{x}{n+1}\left(1 - \frac{d}{dx}\right)\right) e_n(x).$$

The Miller-Lee polynomials $G_n^{(k)}(x)$(see [1]), are defined by means of the following generating function

$$\left(\frac{1}{1-t}\right)^{k+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)t^n. \tag{1.2}$$

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials. In this paper, we study linear differential equations arising from the generating functions of truncated exponential polynomials $e_n(x)$. We give explicit identities for truncated exponential polynomials $e_n(x)$.

2. Differential equations associated with truncated exponential polynomials

In this section, we study linear differential equations arising from the generating functions of truncated exponential polynomials. Let

$$F = F(t,x) = \left(\frac{1}{1-t}\right) e^{xt}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x) = \frac{d}{dt}\left(\frac{1}{1-t}\right) e^{xt} \\ &= \left(\frac{1}{1-t}\right)^2 e^{xt} + x\left(\frac{1}{1-t}\right) e^{xt} \\ &= \left(\frac{1}{1-t} + x\right) F(t,x), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(2)} &= \left(\frac{d}{dt}\right)^2 F(t,x) \\ &= \left(\frac{1}{1-t}\right)^2 e^{xt} F(t,x) + \left(\frac{1}{1-t} + x\right) F(t,x) F^{(1)} \\ &= \left(\left(\frac{2}{1-t}\right)^2 + \left(\frac{2x}{1-t} + x^2\right)\right) F(t,x). \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right) F(t,x), \quad (N = 0, 1, 2, \ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we obtain

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= \left(\sum_{i=0}^{N} i a_i(N,x)(1-t)^{-i-1}\right) F(t,x) + \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right) F^{(1)}(t,x) \\ &= \left(\sum_{i=0}^{N} i a_i(N,x)(1-t)^{-i-1}\right) F(t,x) \\ &\quad + \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right) \left((1-t)^{-1}+x\right) F(t,x) \\ &= \left(\sum_{i=0}^{N} (i+1) a_i(N,x)(1-t)^{-i-1}\right) F(t,x) + \left(\sum_{i=0}^{N} x a_i(N,x)(1-t)^{-i}\right) F(t,x) \\ &= \left(\sum_{i=0}^{N} x a_i(N,x)(1-t)^{-i}\right) F(t,x) + \left(\sum_{i=1}^{N+1} i a_{i-1}(N,x)(1-t)^{-i}\right) F(t,x). \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1,x)(1-t)^{-i}\right) F(t,x). \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned} &\left(\sum_{i=0}^{N} x a_i(N,x)(1-t)^{-i}\right) F(t,x) + \left(\sum_{i=1}^{N+1} i a_{i-1}(N,x)(1-t)^{-i}\right) F(t,x) \\ &= \left(\sum_{i=0}^{N+1} a_i(N+1,x)(1-t)^{-i}\right) F(t,x).. \end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned} a_0(N+1,x) &= x a_0(N,x), \\ a_{N+1}(N+1,x) &= (N+1) a_N(N,x), \end{aligned} \tag{2.8}$$

and

$$a_i(N+1,x) = x a_i(N,x) + i a_{i-1}(N,x), (1 \le i \le N). \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F = F^{(0)} = a_0(0,x) F(t,x) = F(t,x). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,x) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} &(1-t)^{-1} F(t,x) + x F(t,x) \\ &= \sum_{i=0}^{1} a_i(1,x)(1-t)^{-i} F(t,x) \\ &= a_0(1,x) F(t,x) + a_1(1,x)(1-t)^{-1} F(t,x). \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,x) = x, \quad a_1(1,x) = 1. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x) = xa_0(N,x) = x^2 a_0(N-1,x) = \cdots = x^{N+1},$$

and

$$a_{N+1}(N+1,x) = (N+1)a_N(N,x) = \cdots = (N+1)!. \tag{2.14}$$

For $i = 1, 2, 3$ in (2.9), we get

$$\begin{aligned} a_1(N+1,x) &= \sum_{k=0}^{N} x^k a_0(N-k,x), \\ a_2(N+1,x) &= 2\sum_{k=0}^{N-1} x^k a_1(N-k,x), \text{ and} \\ a_3(N+1,x) &= 3\sum_{k=0}^{N-2} x^k a_2(N-k,x). \end{aligned}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x) = i\sum_{k=0}^{N-i+1} x^k a_{i-1}(N-k,x). \tag{2.15}$$

Now, we give explicit expressions for $a_i(N+1,x)$. By (2.14) and (2.15), we get

$$a_1(N+1,x) = \sum_{k_1=0}^{N} x^{k_1} a_0(N-k_1,x) = x^N(N+1),$$

$$a_2(N+1,x) = 2\sum_{k_1=0}^{N-1} x^{k_1} a_1(N-k_1,x) = 2!\sum_{k_1=0}^{N-1} x^{N-1}(N-k_1),$$

and

$$\begin{aligned} a_3(N+1,x) &= 3\sum_{k_2=0}^{N-2} x^{k_2} a_2(N-k_2,x) \\ &= 3!\sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-k_2-2} x^{N-k_2-2}(N-k_2-k_1-1). \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} a_i(N+1,x) = i!\sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-k_{i-1}-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1} & x^{N-k_i-\cdots-k_2-i+1} \\ \times (N-k_{i-1}-k_{i-2}-\cdots-k_2-k_1-i+2). & \end{aligned} \tag{2.16}$$

Note that, here the matrix $a_i(j,x)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{N+1} \\ 0 & 1! & 2x & \cdot & \cdots & (N+1)x^N \\ 0 & 0 & 2! & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 3! & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N+1)! \end{pmatrix}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2, \dots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x) \left(\frac{1}{1-t} \right)^i \right) F$$

has a solution

$$F = F(t,x) = \left(\frac{1}{1-t} \right) e^{xt},$$

where

$$\begin{aligned} &a_0(N,x) = x^N, \\ &a_N(N,x) = N!, \\ &a_i(N,x) = i! \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-k_{i-1}-i} \cdots \sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} x^{N-k_{i-1}-\cdots-k_2-i} \\ &\qquad \times (N - k_{i-1} - k_{i-2} - \cdots - k_2 - k_1 - i + 1), \\ &(1 \le i \le N). \end{aligned}$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t,x) = \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} e_{k+N}(x) t^k. \tag{2.17}$$

From Theorem 1, (1.2), and (2.17), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} e_{k+N}(x) t^k &= \left(\sum_{i=0}^{N} a_i(N,x) \left(\frac{1}{1-t} \right)^i \right) F \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\frac{1}{1-t} \right)^{i+1} e^{xt} \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\sum_{k=0}^{\infty} G_k^{(i)}(x) t^k \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} a_i(N,x) G_k^{(i)}(x) \right) t^k. \end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \dots,$ and $N = 0, 1, 2, \dots$, we have

$$e_{k+N}(x) = \frac{k!}{(k+N)!} \sum_{i=0}^{N} a_i(N,x) G_k^{(i)}(x), \tag{2.19}$$

where

$$\begin{aligned} &a_0(N,x) = x^N, \\ &a_N(N,x) = N!, \\ &a_i(N,x) = i! \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-k_{i-1}-i} \cdots \sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} x^{N-k_{i-1}-\cdots-k_2-i} \\ &\qquad \times (N - k_{i-1} - k_{i-2} - \cdots - k_2 - k_1 - i + 1), \\ &(1 \le i \le N). \end{aligned}$$

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \ldots$, we have

$$e_N(x) = \frac{1}{N!} \sum_{i=0}^{N} a_i(N, x) G_0^{(i)}(x).$$

For $N = 1, 2, \ldots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N, x) \left(\frac{1}{1-t} \right)^i \right) F$$

has a solution

$$F = F(t, x) = \left(\frac{1}{1-t} \right) e^{xt}.$$

Here is a plot of the surface for this solution.

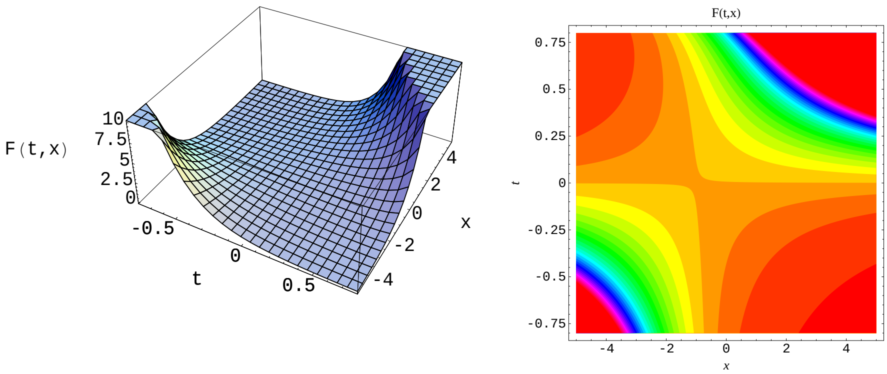

Figure 1: The surface for the solution $F(t, x)$

In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

3. Zeros of the truncated exponential polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the truncated exponential polynomials $e_n(x)$. By using computer, the truncated exponential polynomials $e_n(x)$ can be determined explicitly. We display the shapes of the truncated exponential polynomials $e_n(x)$ and investigate the zeros of the truncated exponential polynomials $e_n(x)$. We investigate the beautiful zeros of the truncated exponential polynomials $e_n(x)$ by using a computer. We plot the zeros of the $e_n(x)$ for $n = 20, 30, 40, 50$ and $x \in \mathbb{C}$(Figure 2). In Figure 2(top-left), we choose $n = 20$. In Figure 2(top-right), we choose $n = 30$. In Figure 2(bottom-left), we choose $n = 40$. In Figure 2(bottom-right), we choose $n = 50$.

Stacks of zeros of $e_n(x)$ for $1 \le n \le 40$, forming a 3D structure are presented(Figure 3). In Figure 3(top-left), we plot stacks of zeros of $e_n(x)$ for $1 \le n \le 40$. In Figure 3(top-right), we draw x and y axes but no z axis in three dimensions. In Figure 3(bottom-left), we draw y and z axes but no x axis in three dimensions. In Figure 3(bottom-right), we draw x and z axes but no y axis in three dimensions.

Our numerical results for approximate solutions of real zeros of the truncated exponential polynomials $e_n(x)$ are displayed(Tables 1, 2).

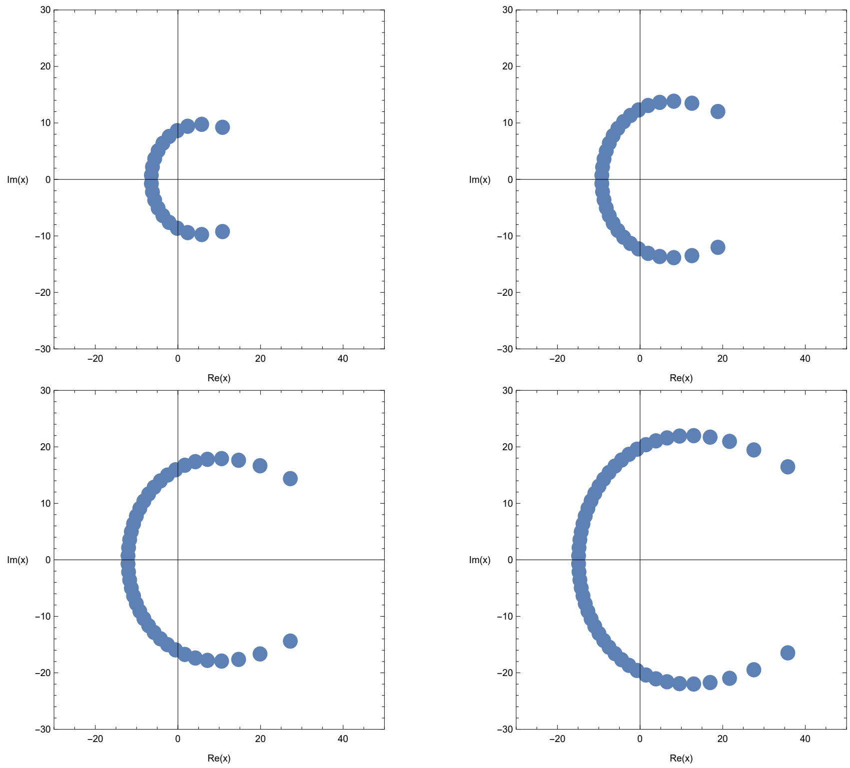

Figure 2: Zeros of $e_n(x)$

Table 1. Numbers of real and complex zeros of $e_n(x)$

degree n	real zeros	complex zeros
1	1	0
2	0	2
3	1	2
4	0	4
5	1	4
6	0	6
7	1	6
8	0	8
9	1	8
10	0	10
11	1	10
12	0	12
13	1	12
14	0	14

How many zeros does $e_n(x)$ have? We are not able to decide if $e_n(x)$ has n distinct solutions(see Table 1, Table 2). We would also like to know the number of complex zeros $C_{e_n(x)}$ of $e_n(x), Im(x) \neq 0$. Since n is the degree of the polynomial $e_n(x)$, the number of real zeros $R_{e_n(x)}$ lying on the real line $Im(x) = 0$ is then $R_{e_n(x)} = n - C_{e_n(x)}$, where $C_{e_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{e_n(x)}$ and $C_{e_n(x)}$.

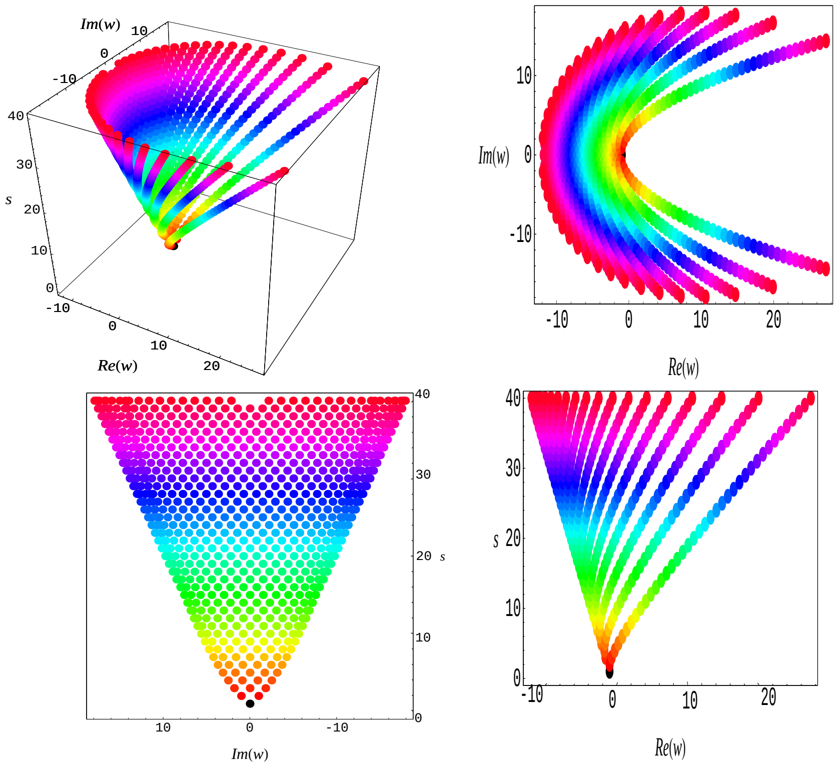

Figure 3: Stacks of zeros of $e_n(x)$ for $1 \leq n \leq 40$

Conjecture 5. Prove that $e_n(x) = 0$ has n distinct solutions.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n. Since n is the degree of the polynomial $e_n(x)$, the number of real zeros $R_{e_n(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{e_n(x)} = n - C_{e_n(x)}$, where $C_{e_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{e_n(x)}$ and $C_{e_n(x)}$.

Conjecture 6. Prove that the numbers of complex zeros $C_{e_n(x)}$ of $e_n(x), Im(x) \neq 0$ is

$$C_{e_n(x)} = 2\left[\frac{n}{2}\right],$$

where [ ] denotes taking the integer part.

Conjecture 7. For $n \in \mathbb{N}_0$, if $n \equiv 1 \pmod 2$, then $R_{e_n(x)} = 1$, if $n \equiv 0 \pmod 2$, then $R_{e_n(x)} = 0$.

The plot of real zeros of the truncated exponential polynomials $e_n(x)$ for $1 \leq n \leq 50$ structure are presented(Figure 4). It is expected that $e_n(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions (see Figure 2, Figure 3, Figure 4). For $a \in \mathbb{R}$, we expect that $e_n(x), x \in \mathbb{C}$, has not $Re(x) = a$ reflection symmetry analytic complex functions. We observe a remarkable regular structure of the complex roots of the truncated exponential polynomials $e_n(x)$. We also hope to verify a remarkable regular structure of the complex roots of the truncated exponential polynomials $e_n(x)$(Table 1). Next, we calculated an approximate solution satisfying $e_n(x) = 0, x \in \mathbb{C}$. The

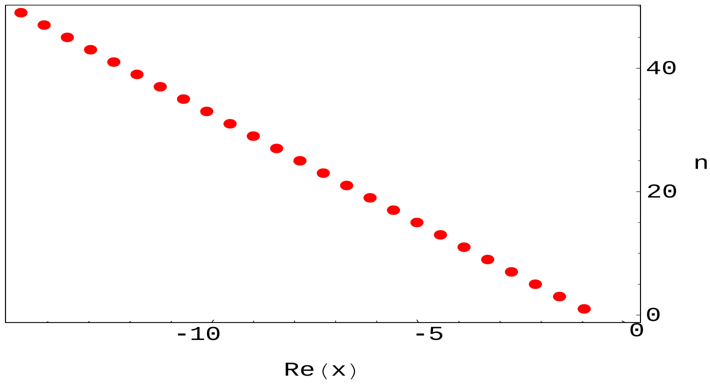

Figure 4: Real zeros of $e_n(x)$ for $1 \leq n \leq 50$

results are given in Table 2.

Table 2. Approximate solutions of $e_n(x) = 0, x \in \mathbb{C}$

degree n	x
1	-1.0000
2	$-1.0000 - 1.0000i, \quad -1.0000 + 1.0000i$
3	$-1.5961, \quad -0.7020 - 1.8073i, \quad -0.7020 + 1.8073i$
4	$-1.7294 - 0.8890i, \quad -1.7294 + 0.8890i$ $-0.2706 - 2.5048i, \quad -0.2706 + 2.5048i$
5	$-2.1806, \quad -1.6495 - 1.6939i, \quad -1.6495 + 1.6939i$ $0.2398 - 3.1283i, \quad 0.2398 + 3.1283i$
6	$-2.3618 - 0.8384i, \quad -2.3618 + 0.8384i, \quad -1.4418 - 2.4345i$ $-1.4418 + 2.4345i, \quad 0.8036 - 3.6977i, \quad 0.8036 + 3.6977i$

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