A study of a coupled system of nonlinear second-order ordinary differential equations with nonlocal integral multi-strip boundary conditions on an arbitrary domain

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Abstract

In this paper, we study a nonlinear system of second order ordinary differential equations with nonlocal integral multi-strip coupled boundary conditions. Leray-Schauder alternative criterion, Schauder fixed point theorem and Banach contraction mapping principle are employed to obtain the desired results. Examples are constructed for the illustration of the obtained results. We emphasize that our results are new and enhance the literature on boundary value problems of coupled systems of ordinary differential equations. Several new results appear as special cases of our work.

Keywords: System of ordinary differential equations; integral boundary condition; multi-strip; existence; fixed point.
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1 Introduction

This paper is concerned with the following coupled system of nonlinear second-order ordinary differential equations:

$$\begin{cases} u''(t) = f(t, u(t), v(t)), \ t \in [a, b], \\ v''(t) = g(t, u(t), v(t)), \ t \in [a, b], \end{cases} \tag{1.1}$$

supplemented with the nonlocal integral multi-strip coupled boundary conditions of the form:

$$\begin{cases} \displaystyle\int_a^b u(s)ds = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} v(s)ds + \lambda_1, \int_a^b u'(s)ds = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} v'(s)ds + \lambda_2, \\ \displaystyle\int_a^b v(s)ds = \sum_{j=1}^m \sigma_j \int_{\xi_j}^{\eta_j} u(s)ds + \lambda_3, \int_a^b v'(s)ds = \sum_{j=1}^m \delta_j \int_{\xi_j}^{\eta_j} u'(s)ds + \lambda_4, \end{cases} \tag{1.2}$$

where $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $a < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < b$, and $\gamma_j, \ \rho_j, \ \sigma_j$ and $\delta_j \ \in \mathbb{R}^+ \ (j = 1, 2, \ldots, m), \ \lambda_i \in \mathbb{R} \ (i = 1, 2, 3, 4)$.

Mathematical modeling of several real world phenomena lead to the occurrence of nonlinear boundary value problems of differential equations. During the past few decades, the topic of boundary value problems has evolved as an important and interesting area of investigation in view of its extensive applications in diverse disciplines such as fluid mechanics, mathematical physics, etc. For application details, we refer the reader to the text [1], while some recent works on boundary value problems of ordinary differential equations can be found in the papers ([2]-[5]).

Much of the literature on boundary value problems involve classical boundary conditions. However, these conditions cannot cater the complexities of the physical and chemical processes occurring within the domain. In order to cope with this situation, the concept of nonlocal boundary conditions was introduced. Such conditions relate the boundary values of the unknown function to its values at some interior positions of the domain. For a detailed account of nonlocal nonlinear boundary value problems, for instance, see ([6]-[16]) and the references cited therein.

Computational fluid dynamics (CFD) technique are directly concerned with the boundary data [1]. However, the assumption of circular cross-section in the fluid flow problems is not justifiable in many situations. The concept of integral boundary conditions played a key role in resolving this issue as such conditions can be applied to arbitrary shaped structures. Integral boundary conditions are also found to be quite useful in the study of thermal and hydrodynamic problems. In fact, one can find numerous applications of integral boundary conditions in the fields like chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. ([17]-[20]). For some recent results on boundary value problems integral boundary conditions, we refer the reader to a series of articles ([21]-[32]) and the references cited therein.

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Motivated by the importance of nonlocal and integral boundary conditions, we introduce a new kind of coupled integral boundary conditions (1.2) and solve a nonlinear coupled system of second-order ordinary differential equations (1.1) equipped with these conditions. Our main results rely on Leray-Schauder alternative and Banach contraction mapping principle.

The rest of the paper is organized as follows. In Section 2, we present an auxiliary lemma. The main results for the problem (1.1) and (1.2) are discussed in Section 3. We also construct examples illustrating the obtained results. The paper concludes with some interesting observations.

2 An auxiliary lemma

The following lemma plays a key role in defining the solution for the problem (1.1) − (1.2).

Lemma 2.1 *For $f_1, g_1 \in C([a,b], \mathbb{R})$, the solution of the linear system of differential equations*

$$\begin{aligned} u''(t) &= f_1(t), \ t \in [a,b], \\ v''(t) &= g_1(t), \ t \in [a,b], \end{aligned} \tag{2.1}$$

subject to the boundary conditions (1.2) *is equivalent to the system of integral equations*

$$\begin{aligned} u(t) &= \int_a^t (t-s) f_1(s) ds \\ &-\frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{1}{2} A_1 (b-a)(b-s) + L_1 + (b-a) A_2 (t-a)\Big] (b-s) f_1(s) ds \\ &+ \int_a^b \Big[\frac{1}{2} A_1 (b-s) \sum_{j=1}^m \gamma_j (\eta_j - \xi_j) + L_2 + A_2 (t-a) \sum_{j=1}^m \rho_j (\eta_j - \xi_j)\Big] \\ &\times (b-s) g_1(s) ds \Big\} + \frac{1}{A_3}\Big\{ \sum_{j=1}^m \int_{\xi_j}^{\eta_j} \int_a^s \Big[\gamma_j A_1 (b-a)(s-p) + \rho_j L_1 \\ &+ \rho_j (b-a) A_2 (t-a)\Big] g_1(p) dp ds + \sum_{j=1}^m \int_{\xi_j}^{\eta_j} \int_a^s \Big[\sigma_j A_1 \sum_{j=1}^m \gamma_j (\eta_j - \xi_j)(s-p) \\ &+ \delta_j L_2 + \delta_j A_2 (t-a) \sum_{j=1}^m \rho_j (\eta_j - \xi_j)\Big] f_1(p) dp ds \Big\} + \Omega_1(t), \end{aligned} \tag{2.2}$$

$$v(t) = \int_a^t (t-s) g_1(s) ds - \frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{A_1\Big((b-a)^2 - A_2\Big)}{2\sum_{j=1}^m \gamma_j (\eta_j - \xi_j)} (b-s)$$

$$
\begin{aligned}
&+L_3 + A_2(t-a)\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big](b-s)f_1(s)ds \\
&+\int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s) + L_4 + A_2(b-a)(t-a)\Big](b-s)g_1(s)ds\Big\} \\
&+\frac{1}{A_3}\Big\{\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s \Big[(s-p)\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)} + \rho_j L_3 \\
&+\delta_j A_2(t-a)\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big]g_1(p)dpds \\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j A_1(b-a)(s-p) + \delta_j L_4 + \delta_j A_2(b-a)(t-a)\Big]f_1(p)dpds\Big\} \\
&+\Omega_2(t),
\end{aligned} \tag{2.3}
$$

where

$$
A_1 = (b-a)^2 - \Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)^2\Big), \tag{2.4}
$$

$$
A_2 = (b-a)^2 - \Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)^2\Big), \quad A_3 = A_1A_2 \neq 0, \tag{2.5}
$$

$$
\begin{aligned}
L_1 &= (b-a)\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)\Big(\sum_{j=1}^{m}\sigma_j + \sum_{j=1}^{m}\delta_j\Big) \\
&\quad - \frac{(b-a)^4}{2} - \frac{(b-a)^2}{2}\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)^2\Big),
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
L_2 &= \sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)\Big(\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)^2\Big) + (b-a)^2\Big) \\
&\quad - \frac{(b-a)^3}{2}\sum_{j=1}^{m}(\eta_j-\xi_j)(\rho_j+\gamma_j),
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
L_3 &= \sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)\Big((b-a)^2(\sigma_j+\delta_j) - A_2\delta_j\Big) \\
&\quad + \frac{(b-a)^3}{2}\Big[\frac{A_2-(b-a)^2}{\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)} - \sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big],
\end{aligned} \tag{2.8}
$$

$$
L_4 = \frac{(b-a)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)\Big[\sigma_j\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)^2
$$

$$+(b-a)^2 - A_2\Big] - \frac{(b-a)^4}{2} + \frac{(b-a)^2}{2\sum_{j=1}^m \gamma_j}\sum_{j=1}^m \rho_j\Big(A_2 - (b-a)^2\Big), \tag{2.9}$$

$$\Omega_1(t) = \frac{1}{A_3}\Big\{A_1(b-a)\lambda_1 + \Big[L_1 + A_2(b-a)(t-a)\Big]\lambda_2 + A_1\sum_{j=1}^m \gamma_j(\eta_j - \xi_j)\lambda_3 + \Big[L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big]\lambda_4\Big\}, \tag{2.10}$$

$$\Omega_2(t) = \frac{1}{A_3}\Big\{\frac{A_1\Big((b-a)^2 - A_2\Big)}{\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}\lambda_1 + \Big[L_3 + A_2(t-a)\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\Big]\lambda_2 + A_1(b-a)\lambda_3 + \Big[L_4 + A_2(b-a)(t-a)\Big]\lambda_4\Big\}. \tag{2.11}$$

Proof. Integrating the linear system (2.1) twice from a to t, we get

$$u(t) = c_1 + c_2(t-a) + \int_a^t (t-s)f_1(s)ds, \tag{2.12}$$

$$v(t) = c_3 + c_4(t-a) + \int_a^t (t-s)g_1(s)ds, \tag{2.13}$$

where c_1, c_2, c_3 and c_4 are arbitrary real constants.
Using the boundary conditions (1.2) in (2.12) and (2.13), together with notations (2.4), we obtain

$$(b-a)c_1 + \frac{(b-a)^2}{2}c_2 - \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)c_3 - \sum_{j=1}^m \gamma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)c_4 = -\int_a^b \frac{(b-s)^2}{2}f_1(s)ds + \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j}\int_a^s (s-p)g_1(p)dpds + \lambda_1, \tag{2.14}$$

$$(b-a)c_2 - \sum_{j=1}^m \rho_j(\eta_j-\xi_j)c_4 = -\int_a^b (b-s)f_1(s)ds + \sum_{j=1}^m \rho_j\int_{\xi_j}^{\eta_j}\int_a^s g_1(p)dpds + \lambda_2, \tag{2.15}$$

$$-\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)c_1 - \sum_{j=1}^m \sigma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)c_2 + (b-a)c_3 + \frac{(b-a)^2}{2}c_4 = -\int_a^b \frac{(b-s)^2}{2}g_1(s)ds + \sum_{j=1}^m \sigma_j\int_{\xi_j}^{\eta_j}\int_a^s (s-p)f_1(p)dpds + \lambda_3, \tag{2.16}$$

$$-\sum_{j=1}^m \delta_j(\eta_j-\xi_j)c_2 + (b-a)c_4 = -\int_a^b (b-s)g_1(s)ds + \sum_{j=1}^m \delta_j\int_{\xi_j}^{\eta_j}\int_a^s f_1(p)dpds + \lambda_4. \tag{2.17}$$

Solving the equations (2.15) and (2.17) for c_2 and c_4, we find that

$$\begin{aligned} c_2 &= \frac{1}{A_1}\Big[-\int_a^b (b-s)\Big((b-a)f_1(s)+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)g_1(s)\Big)ds \\ &+\sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j}\int_a^s \Big((b-a)g_1(p)+\sum_{j=1}^m \delta_j(\eta_j-\xi_j)f_1(p)\Big)dpds \\ &+(b-a)\lambda_2+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\lambda_4\Big], \end{aligned} \tag{2.18}$$

$$\begin{aligned} c_4 &= \frac{1}{A_1}\Big[-\int_a^b (b-s)\Big(\sum_{j=1}^m \delta_j(\eta_j-\xi_j)f_1(s)+(b-a)g_1(s)\Big)ds \\ &+\sum_{j=1}^m \delta_j \int_{\xi_j}^{\eta_j}\int_a^s \Big(\sum_{j=1}^m \rho_j(\eta_j-\xi_j)g_1(p)+(b-a)f_1(p)\Big)dpds \\ &+\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big]. \end{aligned} \tag{2.19}$$

Using (2.18) and (2.19) in (2.14) and (2.16) and then solving the resulting equations for c_1 and c_3, we obtain

$$\begin{aligned} c_1 &= \frac{1}{A_3}\Big\{-\int_a^b \Big[\frac{1}{2}(b-a)A_1(b-s)+L_1\Big](b-s)f_1(s)ds \\ &-\int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+L_2\Big](b-s)g_1(s)ds \\ &+\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[A_1\gamma_j(b-a)(s-p)+\rho_j L_1\Big]g_1(p)dpds \\ &+\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[A_1\sum_{j=1}^m \gamma_j\sigma_j(\eta_j-\xi_j)(s-p)+\delta_j L_2\Big]f_1(p)dpds \\ &+A_1(b-a)\lambda_1+L_1\lambda_2+A_1\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)\lambda_3+L_2\lambda_4\Big\}, \end{aligned}$$

$$\begin{aligned} c_3 &= \frac{1}{A_3}\Big\{-\int_a^b \Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}(b-s)+L_3\Big](-s)f_1(s)ds \\ &-\int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s)+L_4\Big](b-s)g_1(s)ds \end{aligned}$$

$$+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}\sum_{j=1}^m\gamma_j(s-p)+\rho_jL_3\Big]g_1(p)dpds$$
$$+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4\Big]f_1(p)dpds$$
$$+\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}\lambda_1+L_3\lambda_2+A_1(b-a)\lambda_3+L_4\lambda_4\Big\}.$$

Inserting the values of c_1, c_2, c_3 and c_4 in (2.12) and (2.13), we get the solutions (2.2) and (2.3). The converse follows by direct computation. This completes the proof. □

3 Main results

Let us introduce the space $\mathcal{X}=\{u(t)|u(t)\in C([a,b])\}$ equipped with norm $\|u\|=\sup\{|u(t)|, t\in[a,b]\}$. Obviously $(\mathcal{X},\|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{X}\times\mathcal{X},\|u,v\|)$ is a Banach space with norm $\|(u,v)\|=\|u\|+\|v\|$ for $(u,v)\in\mathcal{X}\times\mathcal{X}$.

By Lemma 2.1, we define an operator $\mathcal{T}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ as

$$\mathcal{T}(u,v)(t):=(\mathcal{T}_1(u,v)(t),\mathcal{T}_2(u,v)(t)),$$

where

$$\begin{aligned}\mathcal{T}_1(u,v)(t) &= \int_a^t(t-s)f(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)\\ &+L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\ &-\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\ &\times(b-s)g(s,u(s),v(s))ds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p) \qquad (3.1)\\ &+\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\ &+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\ &+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t),\end{aligned}$$

$$\mathcal{T}_2(u,v)(t) = \int_a^t(t-s)g(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}(b-s)$$

$$
\begin{aligned}
&+L_3+A_2(t-a)\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)+L_4+A_2(b-a)(t-a)\Big](b-s)\\
&\times g(s,u(s),v(s))ds+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[(s-p)\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\\
&+\rho_jL_3+\delta_jA_2(t-a)\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4+\delta_jA_2(b-a)(t-a)\Big]\\
&\times f(p,u(p),v(p))dpds\Big\}+\Omega_2(t).
\end{aligned}
\tag{3.2}
$$

In order to prove our main results, we need the following assumptions.

(H_1) There exist real constants $m_i, n_i \geq 0,\ (i=1,2)$ and $m_0>0,\ n_0>0$ such that $\forall u,v\in\mathbb{R}$, we have

$$|f(t,u,v)| \leq m_0+m_1|u|+m_2|v|,$$
$$|g(t,u,v)| \leq n_0+n_1|u|+n_2|v|.$$

(H_2) There exist nonnegative functions $\alpha(t),\ \beta(t)\in L(0,1)$ and $u,v\in\mathbb{R}$, such that

$$|f(t,u,v)| \leq \alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2},\ \epsilon_1,\epsilon_2>0,\ 0<p_1,p_2<1,$$

$$|g(t,u,v)| \leq \beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2},\ d_1,d_2>0,\ 0<l_1,l_2<1.$$

(H_3) There exist ℓ_1 and ℓ_2 such that for all $t\in[a,b]$ and $u_i,v_i\in\mathbb{R},\ i=1,2$, we have

$$|f(t,u_1,v_1)-f(t,u_2,v_2)| \leq \ell_1(|u_1-u_2|+|v_1-v_2|),$$

$$|g(t,u_1,v_1)-g(t,u_2,v_2)| \leq \ell_2(|u_1-u_2|+|v_1-v_2|).$$

For the sake of convenience in the forthcoming analysis, we set

$$
\begin{aligned}
q_1 \;=\;& \frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^4}{2}\\
&+|A_1|\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)
\end{aligned}
$$

$$+|A_2|(b-a)\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \tag{3.3}$$

$$\begin{aligned}\bar{q}_1 &= \frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\\ &+|A_1|(b-a)\sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\ &+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}q_2 &= \frac{1}{|A_3|}\Big\{\Big|\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)}\Big|\frac{(b-a)^3}{6}+|L_3|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^3}{2}\\ &\times\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)+|A_1|(b-a)\sum_{j=1}^{m}\sigma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\ &+\sum_{j=1}^{m}\delta_j|L_4|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\ &+|A_2|(b-a)^2\sum_{j=1}^{m}\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\bar{q}_2 &= \frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_4|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^4}{2}\\ &+\Big|\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\Big|\sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\ &+\sum_{j=1}^{m}\rho_j|L_3|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\ &+|A_2|(b-a)\Big(\sum_{j=1}^{m}\delta_j\Big)\Big(\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\},\end{aligned} \tag{3.6}$$

$$\bar{\lambda}_1=\sup_{t\in[a,b]}|\Omega_1(t)|,\quad \bar{\lambda}_2=\sup_{t\in[a,b]}|\Omega_2(t)|. \tag{3.7}$$

Moreover, we set

$$Q_1=q_1+q_2,\quad Q_2=\bar{q}_1+\bar{q}_2,\quad \bar{\lambda}=\bar{\lambda}_1+\bar{\lambda}_2, \tag{3.8}$$

where q_i, $\bar{q}_i$ and $\bar{\lambda}_i$ (i=1,2) are given in the equations (3.3) – (3.7) and

$$Q_0=\min\{1-(Q_1m_1+Q_2n_1),1-(Q_1m_2+Q_2n_2)\},\ m_i,n_i\geq 0\ (i=1,2). \tag{3.9}$$

3.1 Existence of solutions

In this subsection, we discuss the existence of solutions for the problem (1.1)-(1.2) by using standard fixed poit theorems.

Lemma 3.1 *(Leray-Schauder alternative [33]). Let $T : K \to K$ be a completely continuous operator (i.e., a map that restricted to any bounded set in K is compact). Let $\omega(T) = \{x \in K : x = \varphi T(x)$ for some $0 < \varphi < 1\}$. Then either the set $\omega(T)$ is unbounded, or T has at least one fixed point.*

Theorem 3.2 *Assume that condition (H_1) holds. In addition it is assumed that*

$$Q_1 m_1 + Q_2 n_1 < 1 \quad \text{and} \quad Q_1 m_2 + Q_2 n_2 < 1, \tag{3.10}$$

where Q_1 and Q_2 are given by (3.8). Then there exist at least one solution for problem $(1.1) - (1.2)$ on $[a, b]$

Proof. First of all, we show that the operator $\mathcal{T} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is completely continuous. Notice that the operator $\mathcal{T}$ is continuous as the functions f and g are continuous. Let $\Upsilon \subset \mathcal{X} \times \mathcal{X}$ be bounded. Then there exist positive constants κ_f and κ_g such that $|f(t, u(t), v(t))| \le \kappa_f$, $|g(t, u(t), v(t))| \le \kappa_g$, $\forall (u, v) \in \Upsilon$. Then, for any $(u, v) \in \Upsilon$, we can obtain

$$\begin{aligned}
|\mathcal{T}_1(u,v)(t)| = \quad & \sup_{t\in[a,b]} \Big| \int_a^t (t-s) f(s,u(s),v(s))ds - \frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s) \\
& + L_1 + (b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds \\
& + \int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times (b-s)g(s,u(s),v(s))ds\Big\} + \frac{1}{A_3}\Big\{\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j A_1(b-a)(s-p) \\
& + \rho_j L_1 + \rho_j(b-a)A_2(t-a)\Big] g(p,u(p),v(p))dpds \\
& + \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j A_1 \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p) + \delta_j L_2 \\
& + \delta_j A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] f(p,u(p),v(p))dpds\Big\} + \Omega_1(t)\Big| \\
& \le \kappa_f \Big\{\frac{(b-a)^2}{2} + \frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^4}{6} + |L_1|\frac{(b-a)^2}{2} + |A_2|\frac{(b-a)^4}{2} \\
& + |A_1|\Big(\sum_{j=1}^m \gamma_j\Big)\Big(\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big]\Big\}\\
&+\kappa_g\Big\{\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\\
&+|A_1|(b-a)\sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\}+\bar{\lambda}_1\\
&\le \kappa_f q_1+\kappa_g\bar{q}_1+\bar{\lambda}_1,
\end{aligned}
$$

which implies that

$$\|\mathcal{T}_1(u,v)\|\le \kappa_f q_1+\kappa_g\bar{q}_1+\bar{\lambda}_1.$$

Similarly, it can be found that

$$\|\mathcal{T}_2(u,v)\|\le \kappa_f q_2+\kappa_g\bar{q}_2+\bar{\lambda}_2.$$

Consequently, we get $\|\mathcal{T}(u,v)(t)\|\le \kappa_f Q_1+\kappa_g Q_2+\bar{\lambda}$ (Q_1 , Q_2 and $\bar{\lambda}$ are given by (3.8)), which implies that the operator $\mathcal{T}$ is uniformly bounded. Next, we show that $\mathcal{T}$ is equicontinuous. For $t_1,t_2\in[a,b]$ with $t_1<t_2$, we have

$$
\begin{aligned}
&|\mathcal{T}_1(u,v)(t_2)-\mathcal{T}_1(u,v)(t_1)|\\
\le\ &\kappa_f\Big|\int_a^{t_1}\Big[(t_2-s)-(t_1-s)\Big]ds+\int_{t_1}^{t_2}(t_2-s)ds\Big|\\
&+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\int_a^b(b-a)(b-s)ds+\Big(\sum_{j=1}^{m}\delta_j\Big)\int_{\xi_j}^{\eta_j}\int_a^s\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)dpds\Big]\\
&+\kappa_g\Big[\int_a^b\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)(b-s)ds+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\rho_j(b-a)dpds\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\lambda_4\Big\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \kappa_f\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big]+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\frac{(b-a)^3}{2}\\
&+\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\kappa_g\Big[\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\frac{(b-a)^2}{2}+(b-a)\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\lambda_4\Big\}\to 0 \text{ independent of } u \text{ and } v \text{ as } (t_2-t_1)\to 0.
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
&|\mathcal{T}_2(u,v)(t_2)-\mathcal{T}_2(u,v)(t_1)|\\
&\leq \kappa_g\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big]+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\frac{(b-a)^3}{6}\\
&+(b-a)\sum_{j=1}^m\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\kappa_g\Big[\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^m\delta_j\Big)\Big(\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big\}\to 0 \text{ independent of } u \text{ and } v \text{ as } (t_2-t_1)\to 0.
\end{aligned}$$

Finally, we will verify that the set $\omega=\{(u,v)\in\mathcal{X}\times\mathcal{X}|(u,v)=\varphi\mathcal{T}(u,v),0<\varphi<1\}$ is bounded. Let $(u,v)\in\omega$. Then $(u,v)=\varphi\mathcal{T}(u,v)$ and for any $t\in[a,b]$, we have

$$u(t)=\varphi\mathcal{T}_1(u,v)(t),\quad v(t)=\varphi\mathcal{T}_2(u,v)(t).$$

Then

$$\begin{aligned}
|u(t)| &\leq q_1(m_0+m_1\|u\|+m_2\|v\|)+\bar{q}_1(n_0+n_1\|u\|+n_2\|v\|)+\bar{\lambda}_1\\
&= q_1m_0+\bar{q}_1n_0+(q_1m_1+\bar{q}_1n_1)\|u\|+(q_1m_2+\bar{q}_1n_2)\|v\|+\bar{\lambda}_1,
\end{aligned}$$

and

$$\begin{aligned}
|v(t)| &\leq q_2(m_0+m_1\|u\|+m_2\|v\|)+\bar{q}_2(n_0+n_1\|u\|+n_2\|v\|)+\bar{\lambda}_2\\
&= q_2m_0+\bar{q}_2n_0+(q_2m_1+\bar{q}_2n_1)\|u\|+(q_2m_2+\bar{q}_2n_2)\|v\|+\bar{\lambda}_2.
\end{aligned}$$

Hence, we have

$$\|u\|+\|v\| \leq (q_1+q_2)m_0+(\bar{q}_1+\bar{q}_2)n_0+[(q_1+q_2)m_1+(\bar{q}_1+\bar{q}_2)n_1]\|u\|$$

$$+[(q_1+q_2)m_2+(\bar{q}_1+\bar{q}_2)n_2]\|v\|+\bar{\lambda}_1+\bar{\lambda}_2,$$

which, in view of (3.9) and (3.10), yields

$$\|(u,v)\| \le \frac{Q_1 m_0 + Q_2 n_0 + \bar{\lambda}}{Q_0},$$

for any $t \in [a,b]$, which proves that the set ω is bounded. Hence, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problem (1.1) – (1.2) has at least one solution on $[a,b]$. This completes the proof. □

Next, we apply Schauder fixed point theorem to prove the existence of solutions for the problem (1.1)-(1.2) by imposing the the sub-growth condition on the nonlinear functions involved in the problem.

Theorem 3.3 *Assume that* (H_2) *holds. Then, there exist at least one solution on* $[a,b]$ *for the problem* (1.1) – (1.2).

Proof. Define a set Y in the Banach space $\mathcal{X}\times\mathcal{X}$ by

$$Y=\{(u,v)\in\mathcal{X}\times\mathcal{X} : \|(u,v)\|\le y\},$$

where

$$y \geqslant \max\{7\bar{\lambda}, 7Q_1\alpha(t), 7Q_2\beta(t), (7Q_1\epsilon_1)^{\frac{1}{1-p_1}}, (7Q_1\epsilon_2)^{\frac{1}{1-p_2}}, (7Q_2 d_1)^{\frac{1}{1-l_1}}, (7Q_2 d_2)^{\frac{1}{1-l_1}}\}.$$

In order to show that $\mathcal{T}: Y \to Y$. We have

$$\begin{aligned}|\mathcal{T}_1(u,v)(t)| = \quad & \sup_{t\in[a,b]}\Big|\int_a^t (t-s)f(s,u(s),v(s))ds - \frac{1}{A_3}\Big\{\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)\\ & +L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\ & +\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\ & \times(b-s)g(s,u(s),v(s))ds\Big\}+\frac{1}{A_3}\Big\{\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p)\\ & +\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\ & +\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\ & +\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t)\Big|\end{aligned}$$

$$\leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_1+\bar{\lambda}_1,$$

which implies that

$$\|\mathcal{T}_1(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_1+\bar{\lambda}_1.$$

Analogously, we have

$$\|\mathcal{T}_2(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_2+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_2+\bar{\lambda}_2.$$

In consequence,

$$\|\mathcal{T}(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)Q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)Q_2+\bar{\lambda} \leq y,$$

where Q_1 , Q_2 and $\bar{\lambda}$ are given by (3.8). Therefore, we conclude that $\mathcal{T}: Y \to Y$, where $\mathcal{T}_1(u,v)(t)$ and $\mathcal{T}_2(u,v)(t)$ are continuous on $[a,b]$.

Now we prove that $\mathcal{T}$ is completely continuous operator by fixing that

$$G=\max_{t\in[a,b]}|f(t,u(t),v(t))|, \quad H=\max_{t\in[a,b]}|g(t,u(t),v(t))|.$$

Letting $t,\tau \in [a,b]$ with $a<t<\tau<b$ and $(u,v)\in Y$, we get

$$\begin{aligned}
&|\mathcal{T}_1(u,v)(\tau)-\mathcal{T}_1(u,v)(t)|\\
&\leq G\Big[(\tau-t)(t-a)+\frac{(\tau-t)^2}{2}\Big]+\frac{(\tau-t)}{|A_1|}\Big\{G\Big[\frac{(b-a)^3}{2}\\
&+\Big(\sum_{j=1}^m \rho_j\Big)\Big(\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+H\Big[\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\frac{(b-a)^2}{2}+(b-a)\sum_{j=1}^m \rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\lambda_4\Big\} \to 0 \text{ as } (\tau-t)\to 0.
\end{aligned}$$

In a similar manner, one can obtain

$$\begin{aligned}
&|\mathcal{T}_2(u,v)(\tau)-\mathcal{T}_2(u,v)(t)|\\
&\leq H\Big[(\tau-t)(t-a)+\frac{(\tau-t)^2}{2}\Big]+\frac{(\tau-t)}{|A_1|}\Big\{G\Big[\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\frac{(b-a)^3}{6}\\
&+(b-a)\sum_{j=1}^m \delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]
\end{aligned}$$

$$+H\Big[\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^{m}\delta_j\Big)\Big(\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]$$
$$+\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big\}\to 0 \text{ as } (\tau-t)\to 0.$$

Thus the operator $\mathcal{T}Y\subset Y$ is equicontinuous and uniformaly bounded set. Hence $\mathcal{T}$ is a completely continuous operator. So, by Schauder fixed point theorem, there exist a solution to the problem $(1.1)-(1.2)$. □

3.2 Uniqueness of solutions

Here we establish the uniqueness of solutions for the problem $(1.1)-(1.2)$ by means of Banach's contraction mapping principle.

Theorem 3.4 *Assume that* (H_3) *holds and that*

$$Q_1\ell_1+Q_2\ell_2<1, \tag{3.11}$$

where Q_1 *and* Q_2 *are given by* (3.8). *Then the problem* $(1.1)-(1.2)$ *has a unique solution on* $[a,b]$.

Proof. Define $\sup_{t\in[a,b]}|f(t,0,0)|=N_1, \sup_{t\in[a,b]}|g(t,0,0)|=N_2$ and

$$r\geq\frac{Q_1N_1+Q_2N_2+\bar{\lambda}}{1-(Q_1\ell_1+Q_2\ell_2)}.$$

Then we show that $\mathcal{T}B_r\subset B_r$, where $B_r=\{(u,v)\in\mathcal{X}\times\mathcal{X}:\|(u,v)\|\leq r\}$. For any $(u,v)\in B_r, t\in[a,b]$, we find that

$$\begin{aligned}|f(s,u(s),v(s))| &= |f(s,u(s),v(s))-f(s,0,0)+f(s,0,0)|\\ &\leq |f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|\\ &\leq \ell_1(\|u\|+\|v\|)+N_1\leq\ell_1\|(u,v)\|+N_1\leq\ell_1r+N_1,\end{aligned}$$

and

$$\begin{aligned}|g(s,u(s),v(s))| &= |g(s,u(s),v(s))-g(s,0,0)+g(s,0,0)|\\ &\leq |g(s,u(s),v(s))-g(s,0,0)|+|g(s,0,0)|\\ &\leq \ell_2(\|u\|+\|v\|)+N_2\leq\ell_2\|(u,v)\|+N_2\leq\ell_2r+N_2.\end{aligned}$$

Then, for $(u,v)\in B_r$, we obtain

$$|\mathcal{T}_1(u,v)(t)|\leq \sup_{t\in[a,b]}\Big|\int_a^t(t-s)f(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)$$

$$
\begin{aligned}
& +L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds \\
& -\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big] \\
& \times(b-s)g(s,u(s),v(s))ds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p) \\
& +\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds \\
& +\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2 \\
& +\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t)\Big| \\
& \le[\ell_1r+N_1]\times\Big\{\frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2} \\
& +|A_2|\frac{(b-a)^4}{2}+\sum_{j=1}^m\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big) \\
& +|A_1|\Big(\sum_{j=1}^m\gamma_j\Big)\Big(\sum_{j=1}^m\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big) \\
& +|A_2|(b-a)\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\} \\
& +[\ell_2r+N_2]\times\Big\{\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2} \\
& +|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^m\rho_j(\eta_j-\xi_j)+|A_1|(b-a) \\
& \times\sum_{j=1}^m\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)+\sum_{j=1}^m\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big) \\
& +|A_2|(b-a)^2\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}+\bar{\lambda}_1 \\
\le\ & q_1(\ell_1r+N_1)+\bar{q}_1(\ell_2r+N_2)+\bar{\lambda}_1.
\end{aligned}
$$

Hence

$$\|\mathcal{T}_1(u,v)\|\le q_1(\ell_1r+N_1)+\bar{q}_1(\ell_2r+N_2)+\bar{\lambda}_1.$$

Likewise, we find that

$$\|\mathcal{T}_2(u,v)\|\le q_2(\ell_1r+N_1)+\bar{q}_2(\ell_2r+N_2)+\bar{\lambda}_2.$$

From the above estimates, it follows that that $\|\mathcal{T}(u,v)\| \le r$.

Next we show that the operator $\mathcal{T}$ is a contraction. For $(u_1,v_1),\ (u_2,v_2) \in \mathcal{X}\times\mathcal{X}$, we have

$$
\begin{aligned}
& |\mathcal{T}_1(u_1,v_1)(t) - \mathcal{T}_1(u_2,v_2)(t)| \\
\le\ & \sup_{t\in[a,b]} \Big\{ \int_a^t (t-s)\Big|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\Big|ds \\
& + \frac{1}{|A_3|}\Big\{ \int_a^b \Big[\frac{1}{2}|A_1|(b-a)(b-s) + L_1 + (b-a)|A_2|(t-a)\Big](b-s) \\
& \times\Big|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\Big|ds \\
& + \int_a^b \Big[\frac{1}{2}|A_1|(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + L_2 + |A_2|(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times(b-s)\Big|g(s,u_1(s),v_1(s)) - g(s,u_2(s),v_2(s))\Big|ds\Big\} \\
& + \frac{1}{|A_3|}\Big\{ \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j|A_1|(b-a)(s-p) + \rho_j L_1 + \rho_j(b-a)|A_2|(t-a)\Big] \\
& \times\Big|g(p,u_1(p),v_1(p)) - g(p,u_2(p),v_2(p))\Big|dpds \\
& + \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j|A_1|\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p) + \delta_j L_2 + \delta_j|A_2|(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times\Big|f(p,u_1(p),v_1(p)) - f(p,u_2(p),v_2(p))\Big|dpds\Big\} \\
\le\ & \ell_1(|u_1-u_2| + |v_1-v_2|)\times\Big\{\frac{(b-a)^2}{2} + \frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^4}{6} + |L_1|\frac{(b-a)^2}{2} \\
& + |A_2|\frac{(b-a)^4}{2} + |A_1|\Big(\sum_{j=1}^m \gamma_j\Big)\Big(\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big) \\
& + \sum_{j=1}^m \delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big) \\
& + |A_2|(b-a)\Big(\sum_{j=1}^m \rho_j\Big)\Big(\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\} \\
& + \ell_2(|u_1-u_2| + |v_1-v_2|)\times\Big\{\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + |L_2|\frac{(b-a)^2}{2} \\
& + |A_2|\frac{(b-a)^3}{2}\sum_{j=1}^m \rho_j(\eta_j-\xi_j) + |A_1|(b-a)\sum_{j=1}^m \gamma_j\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big)
\end{aligned}
$$

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$$+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\}$$
$$\leq\ (\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|),$$

which yields

$$\|\mathcal{T}_1(u_1,v_1)-\mathcal{T}_1(u_2,v_2)\|\leq(\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|).$$

Similarly,

$$\|\mathcal{T}_2(u_1,v_1)-\mathcal{T}_2(u_2,v_2)\|\leq(\ell_1 q_2+\ell_2\bar{q}_2)(|u_1-u_2|+|v_1-v_2|).$$

So, it follows from the above inequalities that

$$\|\mathcal{T}(u_1,v_1)-\mathcal{T}(u_2,v_2)\|\leq(Q_1\ell_1+Q_2\ell_2)(\|u_1-u_2\|+\|v_1-v_2\|),$$

where Q_1 and Q_2 are given by (3.8). By the given assumption (3.11), it follows that the operator $\mathcal{T}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{T}$ has a fixed point, which corresponds to a unique solution of the problem (1.1)-(1.2) on $[a,b]$. □

Example 3.5 *Consider the following second order system of ordinary differential equations*

$$\begin{cases} u''(t)=\dfrac{1}{10+t^2}\Big(\dfrac{|u|}{1+|u|}+v(t)\Big)+e^{-t}, & t\in[2,3],\\ v''(t)=\dfrac{1}{3\sqrt{32+t^2}}\Big(u(t)+\tan^{-1}v(t)\Big)+\cos(t-2), & t\in[2,3],\end{cases}\tag{3.12}$$

subject to the boundary conditions

$$\begin{cases} \displaystyle\int_2^3 u(s)ds=\sum_{j=1}^{3}\gamma_j\int_{\xi_j}^{\eta_j}v(s)ds+2, \int_2^3 u'(s)ds=\sum_{j=1}^{3}\rho_j\int_{\xi_j}^{\eta_j}v'(s)ds+1,\\ \displaystyle\int_2^3 v(s)ds=\sum_{j=1}^{3}\sigma_j\int_{\xi_j}^{\eta_j}u(s)ds+\frac{3}{2}, \int_2^3 v'(s)ds=\sum_{j=1}^{3}\delta_j\int_{\xi_j}^{\eta_j}u'(s)ds+\frac{1}{2},\end{cases}\tag{3.13}$$

where $a=2,b=3,m=3,\lambda_1=2,\lambda_2=1,\lambda_3=3/2,\lambda_4=1/2,\gamma_1=2/5,\gamma_2=21/40,\gamma_3=13/20,\rho_1=1/3,\rho_2=1/2,\rho_3=2/3,\sigma_1=3/7,\sigma_2=5/7,\sigma_3=1,\delta_1=3/8,\delta_2=5/8,\delta_3=7/8,\xi_1=15/7,\eta_1=16/7,\xi_2=17/7,\eta_2=18/7,\xi_3=19/7,\eta_3=20/7$.

Using the given data, we find that $\ell_1=\frac{1}{7},\ell_2=\frac{1}{9}$, $A_1\approx0.827806\neq0, A_2\approx0.793367\neq0, A_3\approx0.656754, |L_1|=0.03337, |L_2|\approx0.225389, |L_3|\approx0.027121, |L_4|\approx0.185097$, $q_1\approx1.963984, q_2\approx1.422591, \bar{q}_1\approx1.290164$ and $\bar{q}_2\approx1.851349$. Also $Q_1\ell_1+Q_2\ell_2\approx0.832853<1$ (Q_1 and Q_2 are given by (3.8)). Thus, all the conditions of Theorem 3.4 are satisfied. Hence it follows by the conclusion of Theorem 3.4 that the problem (3.12) – (3.13) has a unique solution on $[2,3]$.

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4 Conclusions

The salient features of this work includes *(i)* considering a coupled system of nonlinear ordinary differential equations on an arbitrary domain *(ii)* a new kind of integral multi-strip coupled boundary conditions. The results obtained for the given problem are new and significantly contribute to the existing literature on the topic. As a special case, our results correspond to the uncoupled integral boundary conditions of the form:

$$\int_a^b u(s)ds = \lambda_1, \int_a^b u'(s)ds = \lambda_2; \int_a^b v(s)ds = \lambda_3, \int_a^b v'(s)ds = \lambda_4,$$

if we take all $\gamma_j = 0, \rho_j = 0, \sigma_j = 0, \delta_j = 0$ $(j = 1, \ldots, m)$ in the results of this paper.

References

[1] L. Zheng, X. Zhang, *Modeling and analysis of modern fluid problems. Mathematics in Science and Engineering*, Elsevier/Academic Press, London, 2017.

[2] F.T. Akyildiz, H. Bellout, K. Vajravelu, R.A. Van Gorder, Existence results for third order nonlinear boundary value problems arising in nano boundary layer fluid flows over stretching surfaces, *Nonlinear Anal. Real World Appl.* **12** (2011), 2919-2930.

[3] A. Abid, P. Eloe, Positive solutions of boundary value problems for ordinary differential equations with dependence on higher order derivatives, *Comm. Appl. Nonlinear Anal.* **24** (2017), 101-108.

[4] M. Ashordia, On boundary value problems for systems of nonlinear generalized ordinary differential equations, *Czechoslovak Math. J.* **67** (2017), 579-608.

[5] R. Liu, Z. Wu, Well-posedness of a class of two-point boundary value problems associated with ordinary differential equations, *Adv. Difference Equ.* (2018), **2018:54**.

[6] M. Greguš, F. Neumann, F. M. Arscott, Three-point boundary value problems in differential equations, *Proc. London Math. Soc.* **3** (1964), 459-470.

[7] V.A. Il'in, E.I. Moiseev, Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differential Equations* **23** (1987), 803-810.

[8] C.P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, *J. Math. Anal. Appl.* **168** (1998), 540-551.

[9] M.R. Grossinho, F.M. Minhos, Existence result for some third order separated boundary value problems, *Nonlinear Anal.* **47** (2001), 2407-2418.

[10] P.W. Eloe, B. Ahmad, Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions, *Appl. Math. Lett.* **18** (2005), 521-527.

[11] S. Clark, J. Henderson, Uniqueness implies existence and uniqueness criterion for non local boundary value problems for third-order differential equations, *Proc. Amer. Math. Soc.* **134** (2006), 3363-3372.

[12] J.R.L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: A unified approach, *J. London Math. Soc.* **74** (2006), 673-693.

[13] J.R. Graef, J.R.L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal.* **71** (2009), 1542-1551.

[14] L. Wang, M. Pei, W. Ge, Existence and approximation of solutions for nonlinear second-order four-point boundary value problems, *Math. Comput. Modelling* **50** (2009), 1348-1359.

[15] Y. Sun, L. Liu, J. Zhang, R.P. Agarwal, Positive solutions of singular three-point boundary value problems for second-order differential equations, *J. Comput. Appl. Math.* **230** (2009), 738-750.

[16] M. Feng, X. Zhang, W. Ge, Existence theorems for a second order nonlinear differential equation with nonlocal boundary conditions and their applications, *J. Appl. Math. Comput.* **33** (2010), 137-153.

[17] J.R. Cannon, The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.* **21** (1963), 155-160.

[18] N.I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition (Russian), *Diff. Uravn.* **13** (1977), 294-304.

[19] C. Taylor, T. Hughes, C. Zarins, Finite element modeling of blood flow in arteries, *Comput. Methods Appl. Mech. Engrg.* **158** (1998), 155-196.

[20] F. Nicoud, T. Schfonfeld, Integral boundary conditions for unsteady biomedical CFD applications, *Int. J. Numer. Meth. Fluids* **40** (2002), 457-465.

[21] S.K. Ntouyas, Nonlocal Initial and Boundary Value Problems: A survey, *Handbook on Differential Equations: Ordinary Differential Equations*, Edited by A. Canada, P. Drabek and A. Fonda, Elsevier Science B. V., 2005, 459-555.

[22] Z. Yang, Positive solutions of a second order integral boundary value problem, *J. Math. Anal. Appl.* **321** (2006), 751-765.

[23] J.R.L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA, Nonlinear Differ. Equ. Appl.* **15** (2008), 45-67.

[24] B. Ahmad, A. Alsaedi, B.S. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. *Nonlinear Anal. Real World Appl.* **2008**, *9*, 1727–1740.

[25] A. Boucherif, Second-order boundary value problems with integral boundary conditions, *Nonlinear Anal. TMA.* **70** (2009), 364-371.

[26] B. Ahmad, S.K. Ntouyas, H.H. Alsulami, Existence results for n-th order multipoint integral boundary-value problems of differential inclusions, *Electron. J. Differential Equations* 2013, No. 203, 13 pp.

[27] Y. Li, H. Zhang, Positive solutions for a nonlinear higher order differential system with coupled integral boundary conditions, *J. Appl. Math.* 2014, Art. ID 901094, 7 pp.

[28] B. Ahmad, A. Alsaedi, A. Assolami, Relationship between lower and higher order anti-periodic boundary value problems and existence results, *J. Comput. Anal. Appl.* **16** (2014), 210-219.

[29] J. Henderson, Smoothness of solutions with respect to multi-strip integral boundary conditions for nth order ordinary differential equations, *Nonlinear Anal. Model. Control* **19** (2014), 396-412.

[30] I.Y. Karaca, F.T. Fen, Positive solutions of nth-order boundary value problems with integral boundary conditions, *Math. Model. Anal.* **20** (2015), 188-204.

[31] B. Ahmad, A. Alsaedi, N. Al-Malki, On higher-order nonlinear boundary value problems with nonlocal multipoint integral boundary conditions, *Lith. Math. J.* **56** (2016), 143-163.

[32] M. Boukrouche, D.A. Tarzia, A family of singular ordinary differential equations of the third order with an integral boundary condition, *Bound. Value Probl.* (2018), **2018:32**.

[33] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2005.