J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Global Dynamics of Monotone Second Order Difference Equation

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Abstract. We investigate the global character of the difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

with several period-two solutions, where f is decreasing in the first variable and is increasing in the second variable. We show that the boundaries of the basins of attractions of different locally asymptotically stable equilibrium solutions or period-two solutions are in fact the global stable manifolds of neighboring saddle or non-hyperbolic equilibrium solutions or period-two solutions. We illustrate our results with the complete study of global dynamics of a certain rational difference equation with quadratic terms.

Keywords. asymptotic stability, attractivity, bifurcation, difference equation, global, local stability, period two;

AMS 2000 Mathematics Subject Classification: 37B25, 37D10, 39A20, 39A30.

1 Introduction and Preliminaries

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$ be such that $f(I \times I) \subseteq \mathcal{K}$ where $\mathcal{K} \subseteq I$ is a compact set. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{1}$$

where f is a continuous and decreasing in the first variable and increasing in the second variable. The following result gives a general information about global behavior of solutions of Equation (1).

Theorem 1 *([4])*
Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which is non-decreasing in first and non-increasing in second argument. Then for every solution of Equation (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of (1) converges to either an equilibrium or period-two solution or to the singular point on the boundary. Consequently, most important question becomes determining the basins of attraction of these solutions as well as the unbounded solutions. The answer to this question follows from an application of the theory of monotone maps in the plane which will be presented in Preliminaries.

In [1, 2, 3] authors consider difference equation (1) with several equilibrium solutions as well as the period-two solutions and determine the basins of attraction of different equilibrium solutions and the period-two solutions. In this paper we consider Equation (1) which has up to two equilibrium solutions and up to two minimal period-two solutions which are in South-East ordering. More precisely, we will give sufficient conditions for the precise description of the basins of attraction of different equilibrium solutions and period-two solutions. The results can be immediately extended to the case of any number of the equilibrium solutions and the period-two solutions by replicating our main results.

This paper is organized as follows. In the rest of this section we will recall several basic results on competitive systems in the plane from [7, 15, 16, 17] which are included for completeness of presentation. Our main results about some global dynamics scenarios for monotone systems in the plane and their application to global dynamics of

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Equation (1) are given in section 2. As an application of the results from section 2 in section 3 the global dynamics of difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{2}$$

with all non-negative parameters and initial conditions is presented. All global dynamic scenarios for Equation (1) will be illustrated in the case of Equation (2), which global dynamics can be shortly described as the sequence of exchange of stability bifurcations between an equilibrium and one or two period-two solutions.

We now give some basic notions about monotone maps in the plane.

Definition 2 *Let R be a subset of $\mathbb{R}^2$ with nonempty interior, and let $T : R \to R$ be a map (i.e., a continuous function). Set $T(x, y) = (f(x, y), g(x, y))$. The map T is* competitive *if $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. If both f and g are nondecreasing in x and y, we say that T is* cooperative. *If T is competitive (cooperative), the associated system of difference equations*

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots, \quad (x_{-1}, x_0) \in R \tag{3}$$

is said to be competitive (cooperative). The map T and associated difference equations system are said to be strongly competitive *(*strongly cooperative*) if the adjectives non-decreasing and non-increasing are replaced by increasing and decreasing.*

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. Now we can show that a map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is *cooperative* and a map monotone with respect to the South-East ordering is *competitive*.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at $x = (x_1, x_2)$ and numbered in a counterclockwise direction, for example, $Q_1(x) = \{\mathbf{y} = (y_1, y_2) \in R^2 : \ x_1 \leq y_1, \ x_2 \leq y_2\}$. Basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The fixed point $A(x, y)$ of the map T is said to be *non-hyperbolic point of stable type* if one of the roots of characteristic equation evaluated in A is 1 or -1 and the second root is in $(-1, 1)$.

The next four results, from [16, 17], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [7, 19] and in [18].

Theorem 3 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Theorem 4 *For the curve $\mathcal{C}$ of Theorem 3 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 3 reduces just to $|\lambda| < 1$. This follows from a change of variables [19] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 5 *Assume the hypotheses of Theorem 3, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 3. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\}, \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{4}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\operatorname{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\operatorname{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in int\, Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in int\, Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 6 *In addition to the hypotheses of part (B) of Theorem 5, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 3 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Remark 7 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (1) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (1) is a strictly competitive map on $I \times I$, see [16].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (1) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots.$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [16].

Remark 8 The characteristic equation of Equation (1) at an equilibrium solution $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{5}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 3-6 depends on the nonexistence of minimal period-two solution.

2 Main Results

In this section we present some global dynamics scenarios which feasibility will be illustrated in Section 3.

Theorem 9 *Consider the competitive map T generated by the system* (3) *on a rectangular region $\mathcal{R}$ with nonempty interior. Suppose T has no minimal period-two solutions in $\mathcal{R}$, is strongly competitive on int $\mathcal{R}$, is C^2 in a neighborhood of any fixed point and b. of Theorem 3 holds.*

(a) *Assume that T has a saddle fixed points E_1, E_3 and locally asymptotically stable fixed point E_2, such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3$, and E_0, which is South-west corner of the region $\mathcal{R}$ is either repeller or singular point. Furthermore assume that $E_1 \preceq_{se} E_0 \preceq_{se} E_3$ and that the ray through E_0 and E_1 (resp. E_0 and E_2) is stable manifold of E_1 (resp. E_2). If T has no period-two solutions then every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2.*

(b) *Assume that T has locally asymptotically stable fixed points E_1, E_3 and a saddle fixed point E_2, such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3$, and E_0, which is South-west corner of the region $\mathcal{R}$ is either repeller or singular point. Furthermore assume that $E_1 \preceq_{se} E_0 \preceq_{se} E_3$ and that the ray through E_0 and E_1 (resp. E_0 and E_3) is attracted to E_1 (resp. E_3). If T has no period-two solutions then every solution which starts below (resp. above) the stable manifold $\mathcal{W}^s(E_2)$ converges to E_1 (resp. E_3).*

(c) *Assume that T has exactly five fixed points $E_1, \ldots, E_5$, $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$ where E_1, E_3, E_5 are saddle points, and E_2, E_4 are locally asymptotically stable points. Assume that E_0, which is South-west corner of the region $\mathcal{R}$, is either repeller or singular point such that $E_1 \preceq_{se} E_0 \preceq_{se} E_5$ and that the ray through E_0 and E_1 (resp. E_0 and E_5) is part of the basin of attraction of E_1 (resp. E_5). If T has no period-two solutions then every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2 while every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_3)$ and $\mathcal{W}^s(E_5)$ converges to E_4.*

(d) *Assume that T has exactly five fixed points $E_1, \ldots, E_5$, $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$ where E_1, E_3, E_5 are locally asymptotically stable points, and E_2, E_4 are saddle points. Assume that E_0, which is South-west corner of the region $\mathcal{R}$, is either repeller or singular point such that $E_1 \preceq_{se} E_0 \preceq_{se} E_5$ and that the ray through E_0 and E_1 (resp. E_0 and E_5) is part of the basin of attraction of E_1 (resp. E_5). If T has no period-two solutions then every solution which starts below (resp. above) the stable manifold $\mathcal{W}^s(E_4)$ (resp. $\mathcal{W}^s(E_2)$) converges to E_5 (resp. E_1). Every solution which starts between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ converges to E_3.*

Proof.

(a) The existence of the global stable and unstable manifolds of the saddle point equilibria E_1 and E_3 is guaranteed by Theorems 3 - 6. In view of uniqueness of these manifolds we have that $\mathcal{W}^s(E_1)$ has end points in E_0 and $(0, \infty)$ while $\mathcal{W}^s(E_3)$ has end points in E_0 and $(\infty, 0)$. Furthermore $\mathcal{W}^u(E_1)$ and $\mathcal{W}^u(E_3)$ have end points in E_2. Now, by Corollary 2 in [16] every solution which starts in the interior of the ordered interval $[[E_1, E_2]]$ is attracted to E_2 and similarly every solution which starts in the interior of the ordered interval $[[E_2, E_3]]$ is attracted to E_2. Furthermore, for every $(x_0, y_0) \in [[E_1, E_3]] \setminus ([[E_1, E_2]] \cup [[E_2, E_3]] \cup \{E_0\})$ one can find the points $(x_l, y_l) \in [[E_1, E_2]]$ and $(x_u, y_u) \in [[E_1, E_2]]$ such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$ and so $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u)), n \geq 1$, which implies that $T^n((x_0, y_0)) \to E_2$. Finally, for every $(x_0, y_0) \in \mathcal{R} \setminus ([[E_1, E_3]] \cup \{E_0\}))$ one can find the points $(x_L, y_L) \in \mathcal{W}^u(E_1), (x_U, y_U) \in \mathcal{W}^u(E_3)$ such that $(x_L, y_L) \preceq_{se} (x_0, y_0) \preceq_{se} (x_U, y_U)$ which implies that $T^n((x_0, y_0))$ will eventually enter $[[E_1, E_3]]$ and so it will converge to E_2.

(b) The existence of the stable and unstable manifolds of the saddle point equilibrium E_2 is guaranteed by Theorems 3-6. The endpoints of the unstable manifold are E_1 and E_3. First one can assume that the initial point $(x_0, y_0) \in [[E_1, E_2]] \setminus \{E_0\}$. In view of Corollary 2 in [16] the interior of $[[E_1, E_2]]$ is subset of the basin of attraction of E_1. If the initial point $(x_0, y_0) \notin [[E_1, E_2]]$ but it is between$\mathcal{W}^s(E_1)$ and the ray through E_0 and E_1 then one can find te points (x_l, y_l) the ray through E_0 and E_1 and $(x_u, y_u) \in \mathcal{W}^s(E_1)$ such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$ and so $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u)), n \geq 1$, which means $T^n((x_0, y_0))$ will eventually enter $[[E_1, E_2]]$ and so $T^n((x_0, y_0)) \to E_2$.

The proof when the initial point (x_0, y_0) is below $\mathcal{W}^s(E_2)$ is similar.

(c) The proof is similar to the one in case (a) and will be ommitted. This dynamic scenario is a replication of dynamic scenario in (a).

(d) The proof is similar to the one in case (b) and will be ommitted. This dynamic scenario is exactly replication of dynamic scenario in (b).

■

In the case of Equation (1) we have the following results which are direct application of Theorem 9.

Theorem 10 *Consider Equation* (1) *and assume that f is decreasing in first and increasing in the second variable on the set $(a,b)^2$, where a is either the repeller or a singular point of f, such that f is C^2 in a neighborhood of any fixed point.*

(a) Assume that Equation (1) *has locally asymptotically stable equilibrium solutions $\bar{x} > a$ and the unique saddle point minimal period-two solution $\{P_1, Q_1\}$, $P_1 \preceq_{se} (a,a) \preceq_{se} Q_1$. Assume that the stable manifold of P_1 (resp. Q_1) is the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1). Then the equilibrium $\bar{x}$ is globally asymptotically stable for all $x_{-1}, x_0 > a$.*

(b) Assume that Equation (1) *has the saddle equilibrium solution $\bar{x} > a$ and the unique locally asymptotically stable minimal period-two solution $\{P_1, Q_1\}$, $P_1 \preceq_{se} (a,a) \preceq_{se} Q_1$. Assume that the stable manifold of P_1 (resp. Q_1) is the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1). Then the period-two solution $\{P_1, Q_1\}$ attracts all initial points off the global stable manifold $\mathcal{W}^s(E(\bar{x},\bar{x}))$.*

(c) Assume that Equation (1) *has a saddle equilibrium solution $\bar{x} > a$. Assume that Equation* (1) *has two minimal period-two solutions $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ such that $P_1 \preceq_{se} P_2 \preceq_{se} E(\bar{x},\bar{x}) \preceq_{se} Q_2 \preceq_{se} Q_1$, where $\{P_2, Q_2\}$ is locally asymptotically stable and $\{P_1, Q_1\}$ is a saddle point and assume that the global stable manifold of P_1 (resp. Q_1) is the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1). Then every solution which starts off the union of global stable manifolds $\mathcal{W}^s(E(\bar{x},\bar{x})) \cup \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ converges to the period-two solution $\{P_2, Q_2\}$.*

(d) Assume that Equation (1) *has locally asymptotically stable equilibrium solution $\bar{x} > a$. Asume that Equation* (1) *has two minimal period-two solutions $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ such that $P_1 \preceq_{se} P_2 \preceq_{se} E(\bar{x},\bar{x}) \preceq_{se} Q_2 \preceq_{se} Q_1$, where $\{P_1, Q_1\}$ is locally asymptotically stable and $\{P_2, Q_2\}$ is a saddle point. If the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1) is a part of the basin of attraction of $\{P_1, Q_1\}$ then every solution which starts between the stable manifolds $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ converges to $\bar{x}$ while every solution which starts below $\mathcal{W}^s(Q_2)$ (resp. above $\mathcal{W}^s(P_2)$) converges to the period-two solution $\{P_1, Q_1\}$.*

Proof.

(a) In view of Remark 7 the second iterate T^2 of the map T associated with Equation (1) is strictly competitive. Applying Theorem 9 part (a) to T^2, where we set $E_1 = P_1, E_2 = (\bar{x},\bar{x}), E_3 = Q_1$ we complete the proof.

(b) The proof follows from Theorem 9 part (b) applied to T^2, where we set $E_1 = P_1, E_2 = (\bar{x},\bar{x}), E_3 = Q_1$ and observation that locally asymptotically stable fixed point (resp. saddle point) for T has the same character for T^2.

(c) The proof is similar to the proof in case (a) and will be ommitted.

(d) The proof follows from Theorem 9 part (d) applied to T^2, where we set $E_1 = P_1, E_2 = P_2, E_3 = (\bar{x},\bar{x}), E_4 = Q_2, E_5 = Q_1$ and the observation that locally asymptotically stable fixed point (resp. saddle point) for T has the same character for T^2.

■

Remark 11 The term "saddle point" in formulation of statements of Theorems 9 and 10 can be replaced by the term "non-hyperbolic point of stable type". Results related to Theorem 9 were obtained in [1, 2] and the results related to Theorem 10 were obtained in [6, 9, 10]. Furthermore Cases (b) and (c) of Theorem 9 can be extended to the case when we have any odd number of the equilibrium points which alternate its stability between two types: locally asymptotically stable and saddle points or non-hyperbolic equilibrium points of the stable type. The transition from Case (a) to Case (b) and from Case (c) to Case (d) in Theorem 9 is an exchange of stability bifurcation, while in the case of Theorem 10 these two bifurcations are two global period doubling bifurcations.

3 Case study: Equation $x_{n+1} = \frac{\gamma x_{n-1}}{Ax_n^2+Bx_nx_{n-1}+Cx_{n-1}}$

We investigate global behavior of Equation (2), where the parameters γ, A, B, C are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. Equation (2) is a special case of equations

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \dots \tag{6}$$

and

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots \tag{7}$$

The comprehensive linearized stability analysis of Equation (6) was given in [9] and some special cases were considered in [10]. Some special cases of Equation (7) have been considered in the series of papers [5, 6, 11, 12, 19]. Describing the global dynamics of Equation (7) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, ... \tag{8}$$

Equation (2) has 0 as a singular point and the first quadrant as the region $\mathcal{R}$.

3.1 Local stability analysis

By using the substitution $y_n = \frac{C}{\gamma} x_n$ Equation (2) is reduced to the equation

$$x_{n+1} = \frac{x_{n-1}}{A' x_n^2 + B' x_n x_{n-1} + x_{n-1}}, n = 0, 1, ... \tag{9}$$

where $A' = \frac{\gamma^2}{C^2} A$ and $B' = \frac{\gamma^2}{C^2} B$. In the sequel we consider Equation (9) where A' and B' will be replaced with A and B respectively.

First, we notice that under the conditions on parameters all solutions of Equation (9) are in interval $(0, 1]$ and that 0 is a singular point.

Equation (9) has the unique positive equilibrium $\bar{x}$ given by

$$\bar{x} = \frac{-1+\sqrt{1+4(A+B)}}{2(A+B)}. \tag{10}$$

The partial derivatives associated to Equation (9) at the equilibrium $\bar{x}$ are

$$f'_x = \frac{-y(2Ax+By)}{(Ax^2+Bxy+y)^2}\bigg|_{\bar{x}} = -\frac{4(2A+B)}{\left(1+\sqrt{1+4A+4B}\right)^2}, \quad f'_y = \frac{Ax^2}{(Ax^2+Bxy+y)^2}\bigg|_{\bar{x}} = \frac{4A}{\left(1+\sqrt{1+4A+4B}\right)^2}.$$

Characteristic equation associated to Equation (9) at the equilibrium is

$$\lambda^2 + \frac{4(2A+B)}{\left(1+\sqrt{1+4A+4B}\right)^2}\lambda - \frac{4A}{\left(1+\sqrt{1+4A+4B}\right)^2} = 0.$$

By applying the linearized stability Theorem, see [13], we obtain the following result.

Theorem 12 *The unique positive equilibrium solution $\bar{x}$ of Equation (9) is:*

i) locally asymptotically stable when $B + 3A > 4A^2$;

ii) a saddle point when $B + 3A < 4A^2$;

iii) a non-hyperbolic point of stable type (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{4A} < 1$) when $B + 3A = 4A^2$.

In the next lemma we prove the existence of period two solutions of Equation (9).

Lemma 13 *Equation (9) has the minimal period-two solution $\{(0, 1), (1, 0)\}$ and the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$, where*

$$\phi = \frac{A-\sqrt{(A-B)(A(-3+4A)-B}-B}{2A(A-B)} \text{ and } \psi = \frac{A+\sqrt{(A-B)(A(-3+4A)-B}-B}{2A(A-B)} \tag{11}$$

if and only if

$$\frac{3}{4} < A < 1 \text{ and } B + 3A < 4A^2 \quad \text{or} \quad A > 1 \text{ and } B + 3A > 4A^2.$$

Proof. A minimal period-two solution is a positive solution of the following system

$$\left\{ \begin{array}{r} x+(B-A)y-1=0 \\ -Axy+y=0. \end{array} \right. \tag{12}$$

where $\phi+\psi=x$ and $\phi\psi=y$. The system (12) has two solutions $x=1,\ y=0$ and

$$x=\frac{1}{A},\quad y=\frac{A-1}{A(B-A)}.$$

For second solution we have that $x, y, x^2-4y>0$ if and only if

$$\frac{3}{4}<A<1 \text{ and } B+3A<4A^2 \quad \text{or} \quad A>1 \text{ and } B+3A>4A^2.$$

Now, ϕ and ψ are solution of equation

$$t^2-\frac{1}{A}t-\frac{A-1}{A(B-A)}=0,$$

and the proof is complete. ■

The following theorem describes the local stability nature of the period-two solutions.

Theorem 14 *Consider Equation* (9).

i) The minimal period two solution $\{(0,1),(1,0)\}$ *is:*

- *a) locally asymptotically stable when* $A>1$;
- *b) a saddle point when* $A<1$;
- *c) a non-hyperbolic point of the stable type when* $A=1$.

ii) The minimal period two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$, *given by* (11) *is:*

- *a) locally asymptotically stable when* $\frac{3}{4}<A<1$ *and* $B+3A<4A^2$;
- *b) a saddle point when* $A>1$ *and* $B+3A>4A^2$.

iii) If $A=B=1$ *the minimal period two solution* $\{\phi, 1-\phi\}$ $(0<\phi<1)$ *is non-hyperbolic.*

Proof. In order to prove this theorem, we associate the second iterate map to Equation (9). We have

$$T^2\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

where

$$g(u,v)=\frac{u}{Av^2+Buv+u},\quad h(u,v)=\frac{v}{v+\frac{Au^2}{\left(Av^2+Buv+u\right)^2}+\frac{Buv}{Av^2+Buv+u}}.$$

The Jacobian of the map T^2 has the following form

$$J_{T^2}\begin{pmatrix} \phi \\ \psi \end{pmatrix}=\begin{pmatrix} g'_u(\phi,\psi) & g'_v(\phi,\psi) \\ h'_u(\phi,\psi) & h'_v(\phi,\psi) \end{pmatrix}$$

where

$$g'_u=\frac{Av^2}{(Av^2+Buv+u)^2},\quad g'_v=-\frac{u(Bu+2Av)}{(Av^2+Buv+u)^2},$$

$$h'_u=-\frac{Av^3(u+Buv+Av^2)(Buv(1+Bv)+A(2u+Bv^3))}{(A^2v^5+u^2v(1+Bv)(1+B+Bv)+Au(u+v^3(2+B+2Bv))^2},$$

$$h'_v=\frac{u(u+Buv+Av^2)(B^2u^2v^2(1+Bv)+A^2v^2(5u+2Bv^3)+Au(u+3Buv+Bv^3(2+3Bv^3)))}{(A^2v^5+u^2v(1+Bv)(1+B+Bv)+Au(u+v^3(2+B+2Bv))^2}.$$

Set

$$\mathcal{S}=g'_u(\phi,\psi)+h'_v(\phi,\psi)$$

and

$$\mathcal{D}=g'_u(\phi,\psi)h'_v(\phi,\psi)-g'_v(\phi,\psi)h'_u(\phi,\psi).$$

After some lengthy calculation one can see that:

i) for the minimal period-two solution $\{(0,1),(1,0)\}$ we have

$$\mathcal{S} = \frac{1}{A} \text{ and } \mathcal{D} = 0$$

and applying the linearized stability Theorem [13] we obtain that the minimal period-two solution $\{(0,1),(1,0)\}$ of Equation (9) is:

a) locally asymptotically stable when $A > 1$;

b) a saddle point when $A < 1$;

c) a non-hyperbolic point of the stable type when $A = 1$.

ii) For the positive minimal period two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ we have

$$\mathcal{S} = \frac{6A^4+A(B-2)B-B^2-3A^3(3+2B)+A^2(4+B(6+B))}{A^2(A-B)^2}, \quad \mathcal{D} = \frac{(A-1)^2}{(A-B)^2}.$$

Applying the linearized stability Theorem [13] we obtain that the minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ of Equation (9) is:

a) locally asymptotically stable when $\frac{3}{4} < A < 1$ and $B + 3A < 4A^2$;

b) a saddle point when $A > 1$ and $B + 3A > 4A^2$.

iii) If $A = B = 1$ then

$$\mathcal{S} = 1 + \phi^2(1-\phi)^2, \quad \mathcal{D} = \phi^2(1-\phi)^2$$

from which the proof follows.

■

3.2 Global results and basins of attraction

In this section we present global dynamics results for Equation (9).

Theorem 15 *If $B + 3A > 4A^2$ and $0 < A < 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ given by* (10) *which is locally asymptotically stable and the minimal period-two solution $\{P(0,1), Q(1,0)\}$ which is a saddle point. Furthermore, the global stable manifold of the period-two solution $\{P,Q\}$ is given by $\mathcal{W}^s(\{P,Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are the coordinate axes. The basin of attraction $\mathcal{B}(E) = \{(x,y) : x \geq 0, y \geq 0\}$. More precisely*

i) If $(u_0,v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P, and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q.

ii) If $(u_0,v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q, and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P.

iii) If $(u_0,v_0) \in \mathcal{R} \setminus (\mathcal{W}^s(P) \cup \mathcal{W}^s(Q))$ (the region between $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$) then the sequence $\{(u_n,v_n)\}$ is attracted to $E(\overline{x},\overline{x})$.

See Figure 1 for visual illustration.

Proof. The proof is direct application of Theorem 10 part (a). ■

Theorem 16 *If $B + 3A > 4A^2$ and $A = 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ which is locally asymptotically stable and the minimal period-two solution, $\{P(0,1), Q(1,0)\}$ which is a non-hyperbolic point of stable type. Furthermore, the global stable manifold of the period-two solution $\{P,Q\}$ is given by $\mathcal{W}^s(\{P,Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are the coordinate axes. The global dynamics is given in Theorem 15.*

Proof. In view of Remark 11 the proof is direct application of Theorem 10 part (a).

■

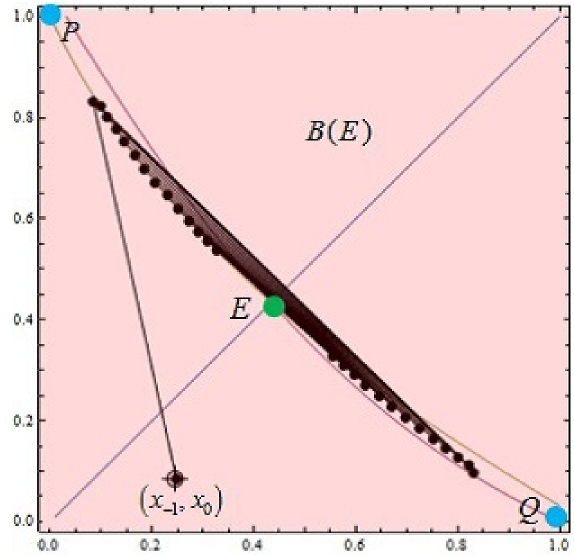

Figure 1: Visual illustration of Theorem 15.

Theorem 17 *If $B+3A>4A^2$ and $A>1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ which is locally asymptotically stable and two minimal period-two solutions $\{P_1(0,1),Q_1(1,0)\}$ which is locally asymptotically stable and $\{P_2(\phi,\psi),Q_2(\psi,\phi)\}$ given by (11), which is a saddle point. Furthermore, the global stable manifold of the period-two solution $\{P_2,Q_2\}$ is given by $\mathcal{W}^s(\{P_2,Q_2\})=\mathcal{W}^s(P_2)\cup\mathcal{W}^s(Q_2)$ where $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ are continuous increasing curves, that divide the first quadrant into two connected components, namely*

$\mathcal{W}_1^+ := \{x\in\mathcal{R}\setminus\mathcal{W}^s(P_2) : \exists y\in\mathcal{W}^s(P_2) \text{ with } y\preceq_{se} x\}$ *and* $\mathcal{W}_1^- := \{x\in\mathcal{R}\setminus\mathcal{W}^s(P_2) : \exists y\in\mathcal{W}^s(P_2) \text{ with } x\preceq_{se} y\}$

$\mathcal{W}_2^+ := \{x\in\mathcal{R}\setminus\mathcal{W}^s(Q_2) : \exists y\in\mathcal{W}^s(Q_2) \text{ with } y\preceq_{se} x\}$ *and* $\mathcal{W}_2^- = \{x\in\mathcal{R}\setminus\mathcal{W}^s(Q_2) : \exists y\in\mathcal{W}^s(Q_2) \text{ with } x\preceq_{se} y\}$

respectively such that the following statements are true.

i) If $(u_0,v_0)\in\mathcal{W}^s(P_2)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P_2 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q_2.

ii) If $(u_0,v_0)\in\mathcal{W}^s(Q_2)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q_2 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P_2.

iii) If $(u_0,v_0)\in\mathcal{W}_1^-$ (the region above $\mathcal{W}^s(P_2)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to Q_1.

iv) If $(u_0,v_0)\in\mathcal{W}_2^+$ (the region below $\mathcal{W}^s(Q_2)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to P_1.

v) If $(u_0,v_0)\in\mathcal{W}_1^+\cap\mathcal{W}_2^-$ (the region between $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$) then the sequence $\{(u_n,v_n)\}$ is attracted to $E(\overline{x},\overline{x})$.

Shortly the basin of attraction of E is the region between $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ while the rest of the first quadrant without $\mathcal{W}^s(P_2)\cup\mathcal{W}^s(Q_2)\cup(0,0)$ is the basin of attraction of $\{P_1,Q_1\}$.

See Figure 2 for visual illustration.

Proof. The proof is direct application of Theorem 10 part (d).

■

Theorem 18 *If $B+3A<4A^2$ and $\frac{3}{4}<A<1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ which is a saddle point and minimal period-two solution $\{P_1(0,1),Q_1(1,0)\}$ which is a saddle point and $\{P_2(\phi,\psi),Q_2(\psi,\phi)\}$, given by (11) which is locally asymptotically stable. Furthermore, there exists a set $\mathcal{C}_E$ which is an invariant subset of the basin of attraction of E. The set $\mathcal{C}_E$ is a graph of a strictly increasing continues function of the first variable*

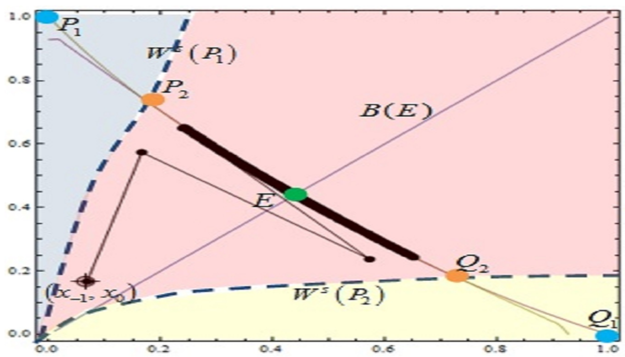

Figure 2: Visual illustration of Theorem 17.

on $(0,\infty)$ *interval and separates* $\mathcal{R}\setminus(0,0)$, *where* $\mathcal{R}=(0,\infty)\times(0,\infty)$ *into two connected and invariant components* $\mathcal{W}^-(\overline{x},\overline{x})$ *and* $\mathcal{W}^+(\overline{x},\overline{x})$. *The global stable manifold of the period-two solution* $\{P_1,Q_1\}$ *is given by* $\mathcal{W}^s(\{P_1,Q_1\})=\mathcal{W}^s(P_1)\cup\mathcal{W}^s(Q_1)$ *where* $\mathcal{W}^s(P_1)$ *and* $\mathcal{W}^s(Q_1)$ *are continuous nondecreasing curves which represent the coordinate axes. The basin of attraction of* $\{P_2,Q_2\}$ *is the first quadrant without* $\mathcal{W}^s(P_1)\cup\mathcal{W}^s(Q_1)\cup(0,0)\cup\mathcal{C}_E$. *More precisely*

i) Every initial point (u_0,v_0) *in* $\mathcal{C}_E$ *is attracted to* E.

ii) If $(u_0,v_0)\in\mathcal{W}^s(P_1)$ *then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *is attracted to* P_1 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *is attracted to* Q_1.

iii) If $(u_0,v_0)\in\mathcal{W}^s(Q_1)$ *then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *is attracted to* Q_1 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *is attracted to* P_1.

iv) If $(u_0,v_0)\in\mathcal{W}^-(\overline{x},\overline{x})$ *(the region between* $\mathcal{C}_E$ *and* $\mathcal{W}^s(P_1)$*) then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *is attracted to* P_2 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *tends to* Q_2.

v) If $(u_0,v_0)\in\mathcal{W}^+(\overline{x},\overline{x})$ *(the region between* $\mathcal{C}_E$ *and* $\mathcal{W}^s(Q_1)$*) then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *tends to* Q_2 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *tends to* P_2.

See Figure 3 for visual illustration.

Proof. Theorem 12 implies that there exists a unique equilibrium solution $E(\overline{x},\overline{x})$ which is a saddle point and Theorem 14 implies that minimal period-two solution $\{P_1(0,1),Q_1(1,0)\}$ is a saddle point and $\{P_2(\phi,\psi),Q_2(\psi,\phi)\}$ is locally asymptotically stable. Now the proof is direct application of Theorem 10 part (c). ■

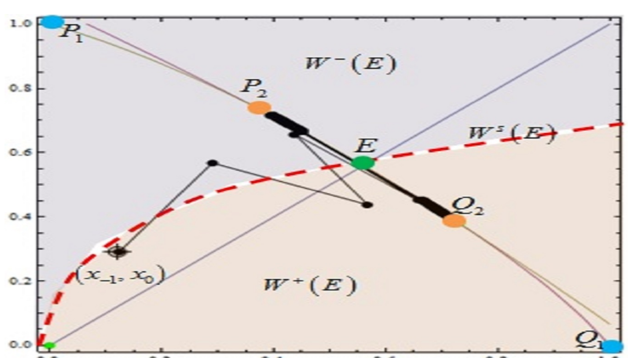

Figure 3: Visual illustration of Theorem 18.

Theorem 19 *If* $B+3A<4A^2$ *and* $A=1$ *then Equation (9) has a unique equilibrium solution* $E(\overline{x},\overline{x})$, *which is a saddle point and the minimal period-two solution* $\{P_1(0,1),Q_1(1,0)\}$ *which is a non-hyperbolic point of stable type.*

Furthermore, the global stable manifold $\mathcal{W}^s(E)$ is continuous increasing curve which divides first quadrant and the global stable manifold of the period-two solution $\{P_1, Q_1\}$ is given by $\mathcal{W}^s(\{P_1, Q_1\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ where $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ are the coordinate axes. The basin of attraction $\mathcal{B}(\{P_1, Q_1\}) = \{(x, y) : x \geq 0, y \geq 0\} \setminus (\mathcal{W}^s(E) \cup (0, 0))\}$. More precisely

i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E.

i) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q_1.

See Figure 4 for visual illustration.

Proof. From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x})$ which is a saddle point. Theorem 14 implies that the period-two solution $\{P, Q\}$ is a non-hyperbolic point. In view of Remark 11 the proof is direct application of Theorem 10 part (b). ∎

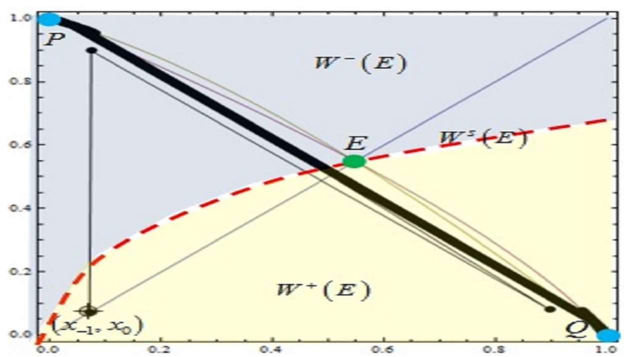

Figure 4: Visual illustration of Theorem 19.

Theorem 20 *If $B + 3A < 4A^2$ and $A > 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x}, \overline{x})$ which is a saddle point and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is locally asymptotically stable. The global behavior is the same as in Theorem 19.*

Proof. The proof is direct application of Theorem 10 part (b). ∎

Theorem 21 *Assume that $B + 3A = 4A^2$.*

a) If $\frac{3}{4} < A < 1$ then Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x})$ which is a non-hyperbolic point of stable type and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is a saddle point. Then every initial point (u_0, v_0) in $\mathcal{R}$ is attracted to E.

b) If $A > 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x}, \overline{x})$ which is a non-hyperbolic point of the stable type and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is locally asymptotically stable. The global behavior is the same as in Theorem 19.

c) If $A = 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x}, \overline{x})$ and infinitely many minimal period-two solution $\{P(\phi, 1 - \phi), Q(1 - \phi, \phi)\}$ $(0 < \phi < 1)$ which are a non-hyperbolic points of stable type.

i) There exists a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E

ii) For every minimal period-two solution $\{P(\phi, 1-\phi), Q(1-\phi, \phi)\}$ $(0 < \phi < 1)$ there exists the global stable manifold given by $\mathcal{W}^s(\{P, Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are continuous increasing curves. If $(u_0, v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to P and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to Q. If $(u_0, v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to Q and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to P. The union of these stable manifolds and $\mathcal{C}_E$ foliates the first quadrant without the singular point $(0, 0)$.

See Figure 5 for visual illustration.

Proof.

a) From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x},\overline{x}) = (\frac{1}{2A}, \frac{1}{2A})$ which is non-hyperbolic of stable type. From Theorem 14 Equation (9) has a unique minimal period-two solution $\{P_1(0,1), Q_1(1,0)\}$ which is a saddle point. In view of Remark 11 the proof is direct application of Theorem 10 part (a).

b) From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x},\overline{x}) = (\frac{1}{2A}, \frac{1}{2A})$, which is non-hyperbolic of stable type. From Theorem 14 Equation (9) has a unique minimal period-two solution $\{P_1(0,1), Q_1(1,0)\}$ which is locally asymptotically stable point. In view of Remark 11 the proof is direct application of Theorem 10 part (b).

c) From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x},\overline{x}) = (\frac{1}{2A}, \frac{1}{2A})$ which is non-hyperbolic. All conditions of Theorem 5 are satisfied, which yields the existence a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E. The proof of the statement ii) follows from Theorems 3, 5, 14 and Theorem 5 in [8].

■

Remark 22 The global dynamics of Equation (9) can be described in the language of bifurcation theory as follows: when $B + 3A \neq 4A^2$, then the period-doubling bifurcation happens when A is passing through the value 1 in such a way that for $A > 1$ new interior period-two solution emerges and exchange stability with already existing period-two solution on the boundary. Another bifurcation happens when $B + 3A < 4A^2$ in which case the stability of the unique equilibrium changes from local attractor to the saddle point. Finally, there is a bifurcation at another critical value $B + 3A = 4A^2$ when A is passing through the critical value 1, which is one of exchange stability between the unique equilibrium and unique period-two solution, with specific dynamics at $A = 1$, when there is an infinite number of period-two solutions which basins of attraction filled up the first quadrant without the origin. See [16] for similar results.

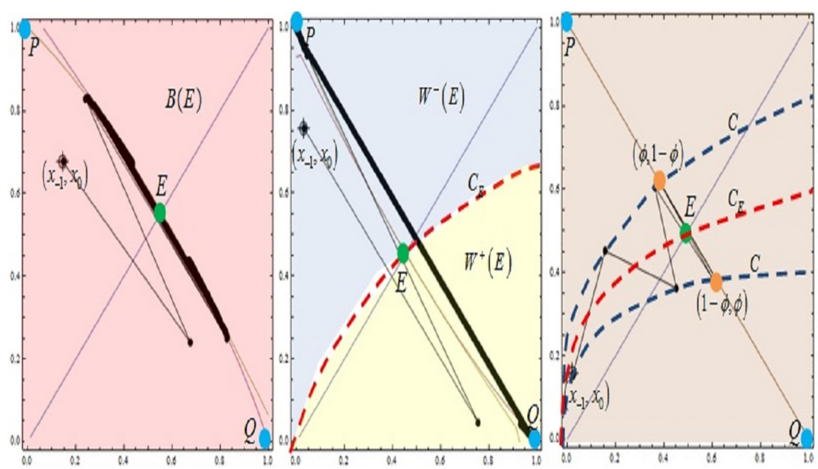

Figure 5: Visual illustration of Theorem 21.

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