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FURTHER INEQUALITIES FOR HEINZ OPERATOR MEAN

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Abstract. In this paper we obtain some new inequalities for Heinz operator mean.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0,1]$

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean,* and

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean* [13]. When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

Define the *Heinz operator mean* by

$$H_\nu (A, B) := \frac{1}{2} (A \sharp_\nu B + A \sharp_{1-\nu} B).$$

The following interpolatory inequality is obvious

$$A \sharp B \leq H_\nu (A, B) \leq A \nabla B \tag{1.1}$$

for any $\nu \in [0,1]$.

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{1.3}$$

The function K is decreasing on $(0,1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

In the recent paper [1] we have obtained the following additive and multiplicative reverse of Young's inequality

$$0 \leq (1-\nu) a + \nu b - a^{1-\nu} b^\nu \leq \nu (1-\nu) (a-b) (\ln a - \ln b) \tag{1.4}$$

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and

$$1 \leq \frac{(1-\nu) a + \nu b}{a^{1-\nu} b^{\nu}} \leq \exp\left[4\nu (1-\nu) \left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{1.5}$$

for any $a, b > 0$ and $\nu \in [0,1]$, where K is Kantorovich's constant.

The operator version of (1.4) is as follows [1]:

Theorem 1. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$0 \leq A\nabla_\nu B - A\sharp_\nu B \leq \nu (1-\nu) (h-1) \ln hA \tag{1.6}$$

and, in particular

$$0 \leq A\nabla B - A\sharp B \leq \frac{1}{4} (h-1) \ln hA. \tag{1.7}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$0 \leq A\nabla_\nu B - A\sharp_\nu B \leq \nu (1-\nu) \frac{h-1}{h} \ln hA \tag{1.8}$$

and, in particular

$$0 \leq A\nabla B - A\sharp B \leq \frac{1}{4} \frac{h-1}{h} \ln hA. \tag{1.9}$$

The operator version of (1.5) is [1]:

Theorem 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions*

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;

we have

$$A\nabla_\nu B \leq \exp\left[4\nu (1-\nu) (K(h) - 1)\right] A\sharp_\nu B \tag{1.10}$$

and, in particular

$$A\nabla B \leq \exp\left[K(h) - 1\right] A\sharp B. \tag{1.11}$$

For other recent results on geometric operator mean inequalities, see [2]-[12], [14] and [16]-[17].

We recall that *Specht's ratio* is defined by [15]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.12}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In the recent paper [6] we obtained amongst other the following result for the Heinz operator mean of A, B that are positive invertible operators that satisfy the condition $mA \leq B \leq MA$ for some constants $M > m > 0$,

$$\omega(m,M) A\sharp B \leq H_\nu(A,B) \leq \Omega(m,M) A\sharp B, \tag{1.13}$$

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where

$$\Omega(m,M):=\begin{cases} S\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \\ \max\left\{S\left(m^{|2\nu-1|}\right), S\left(M^{|2\nu-1|}\right)\right\} \text{ if } m\leq 1\leq M, \\ \\ S\left(M^{|2\nu-1|}\right) \text{ if } 1<m, \end{cases}$$

and

$$\omega(m,M):=\begin{cases} S\left(M^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } M<1, \\ \\ 1 \text{ if } m\leq 1\leq M, \\ \\ S\left(m^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } 1<m. \end{cases}$$

Motivated by the above results we establish in this paper some new additive and multiplicative reverse inequalities for the Heinz operator mean.

2. Additive Reverse Inequalities for Heinz Mean

We have the following generalization of Theorem 1:

Theorem 3. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that*

$$mA\leq B\leq MA. \tag{2.1}$$

Then for any $\nu\in[0,1]$ we have

$$(0\leq)\, A\nabla_{\nu}B-A\sharp_{\nu}B\leq\nu(1-\nu)\,\Omega(m,M)\,A \tag{2.2}$$

where

$$\Omega(m,M):=\begin{cases} (m-1)\ln m \textit{ if } M<1, \\ \\ \max\left\{(m-1)\ln m,(M-1)\ln M\right\} \textit{ if } m\leq 1\leq M, \\ \\ (M-1)\ln M \textit{ if } 1<m. \end{cases} \tag{2.3}$$

In particular, we have

$$(0\leq)\, A\nabla B-A\sharp B\leq\frac{1}{4}\Omega(m,M)\,A. \tag{2.4}$$

Proof. We consider the function $D:(0,\infty)\to[0,\infty)$ defined by $D(x)=(x-1)\ln x$. We have that $D'(x)=\ln x+1-\frac{1}{x}$ and $D''(x)=\frac{x+1}{x^2}$ for $x\in(0,\infty)$. This shows that the function is convex on $(0,\infty)$, monotonic decreasing on $(0,1)$ and monotonic increasing on $[1,\infty)$ with the minimum 0 realized in $x=1$.

From the inequality (1.4) we have

$$(0\leq)\,(1-\nu)+\nu x-x^{\nu}\leq\nu(1-\nu)\,D(x)$$

for any $x>0$, $\nu\in[0,1]$ and hence

$$(0\leq)\,(1-\nu)I+\nu X-X^{\nu}\leq\nu(1-\nu)\max_{m\leq x\leq M}D(x) \tag{2.5}$$

for the positive operator X that satisfies the condition $0<mI\leq X\leq MI$ for $0<m<M$ and $\nu\in[0,1]$.

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If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (2.5) we get

$$(1-\nu) I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \leq \nu (1-\nu) \max_{m\leq x\leq M} D(x) \tag{2.6}$$

Now, if we multiply (2.6) in both sides with $A^{1/2}$ we get

$$\begin{aligned}(0\leq)\,&(1-\nu) A + \nu B - A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu} A^{1/2} \\ &\leq \nu (1-\nu) \max_{m\leq x\leq M} D(x) A\end{aligned} \tag{2.7}$$

for any $\nu \in [0,1]$.

Finally, since

$$\max_{m\leq x\leq M} D(x) = \begin{cases} (m-1)\ln m \text{ if } M<1, \\ \\ \max\{(m-1)\ln m, (M-1)\ln M\} \text{ if } m\leq 1\leq M, \\ \\ (M-1)\ln M \text{ if } 1<m, \end{cases}$$

then by (2.7) we get the desired result (2.2). □

Corollary 1. *With the assumptions of Theorem 3 we have*

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\Omega(m,M) A. \tag{2.8}$$

Proof. From (2.2) we have by replacing ν with $1-\nu$ that

$$(0\leq)\, A\nabla_{1-\nu}B - A\sharp_{1-\nu}B \leq \nu(1-\nu)\Omega(m,M) A. \tag{2.9}$$

Adding (2.2) with (2.9) and dividing by 2 we get (2.8). □

Corollary 2. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)(h-1)\ln hA. \tag{2.10}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\left(\frac{h-1}{h}\right)\ln hA. \tag{2.11}$$

Proof. If the condition (i) is valid, then we have

$$I < \frac{M'}{m'}I = h'I \leq X \leq hI = \frac{M}{m}I,$$

which, by (2.8) gives the desired result (2.10).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \leq X \leq \frac{1}{h'}I < I,$$

which, by (2.8) gives

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\left(\frac{1}{h}-1\right)\ln\frac{1}{h}$$

that is equivalent to (2.11). □

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Theorem 4. *With the assumptions of Theorem 3 we have*

$$(0 \leq) H_\nu (A, B) - A\sharp B \leq \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right) A, \tag{2.12}$$

where the function $D : (0, \infty) \to [0, \infty)$ *is defined by* $D(x) = (x - 1) \ln x$ *(see the proof of Theorem 3).*

Proof. From the inequality (1.4) we have for $\nu = \frac{1}{2}$

$$(0 \leq) \frac{c + d}{2} - \sqrt{cd} \leq \frac{1}{4} (c - d) (\ln c - \ln d) \tag{2.13}$$

for any c, $d > 0$.

If we take in (2.13) $c = a^{1-\nu} b^\nu$ and $d = a^\nu b^{1-\nu}$ then we get

$$\frac{a^{1-\nu} b^\nu + a^\nu b^{1-\nu}}{2} - \sqrt{ab} \leq \frac{1}{4} \left(a^{1-\nu} b^\nu - a^\nu b^{1-\nu}\right) \left(\ln a^{1-\nu} b^\nu - \ln a^\nu b^{1-\nu}\right) \tag{2.14}$$

for any a, $b > 0$ and $\nu \in [0, 1]$.

This inequality is of interest in itself.

Now, if we take in (2.14) $a = 1$ and $b = x$, then we get

$$\begin{aligned} 0 &\leq \frac{x^\nu + x^{1-\nu}}{2} - \sqrt{x} \leq \frac{1}{4} \left(x^\nu - x^{1-\nu}\right) \left(\ln x^\nu - \ln x^{1-\nu}\right) \\ &= \frac{2\nu - 1}{4} \left(x^\nu - x^{1-\nu}\right) \ln x = \frac{1}{4x^{1-\nu}} \left(x^{2\nu-1} - 1\right) \ln x^{2\nu-1} \\ &= \frac{1}{4x^{1-\nu}} D\left(x^{2\nu-1}\right) \end{aligned} \tag{2.15}$$

for any $x > 0$ and $\nu \in [0, 1]$.

Now, if $x \in [m, M] \subset (0, \infty)$, then by (2.15) we get the upper bound

$$(0 \leq) \frac{x^\nu + x^{1-\nu}}{2} - \sqrt{x} \leq \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right).$$

Using the continuous functional calculus, we then have

$$(0 \leq) \frac{X^\nu + X^{1-\nu}}{2} - X^{1/2} \leq \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right) \tag{2.16}$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2} B A^{-1/2} \leq MI$ and by taking $X = A^{-1/2} B A^{-1/2}$ in (2.16) we get

$$\begin{aligned} 0 &\leq \frac{\left(A^{-1/2} B A^{-1/2}\right)^\nu + \left(A^{-1/2} B A^{-1/2}\right)^{1-\nu}}{2} - \left(A^{-1/2} B A^{-1/2}\right)^{1/2} \\ &\leq \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right) \end{aligned} \tag{2.17}$$

for any $\nu \in [0, 1]$.

Now, if we multiply (2.17) in both sides with $A^{1/2}$ we get the desired result (2.12). □

Corollary 3. *Let* A, B *be as in Corollary 2.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$*, then*

$$(0 \leq) H_\nu(A,B) - A\sharp B \leq \frac{1}{4(h')^{1-\nu}} \begin{cases} \left(h^{2\nu-1}-1\right)\ln h^{2\nu-1} & \text{if } \nu \in \left[\frac{1}{2},1\right], \\ \left((h')^{2\nu-1}-1\right)\ln (h')^{2\nu-1} & \text{if } \nu \in \left[0,\frac{1}{2}\right). \end{cases} \tag{2.18}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$*, then*

$$(0 \leq) H_\nu(A,B) - A\sharp B \leq \frac{1}{4}h^{1-\nu} \begin{cases} \left(h^{-2\nu+1}-1\right)\ln h^{-2\nu+1} & \text{if } \nu \in \left[\frac{1}{2},1\right], \\ \left((h')^{-2\nu+1}-1\right)\ln (h')^{-2\nu+1} & \text{if } \nu \in \left[0,\frac{1}{2}\right). \end{cases} \tag{2.19}$$

Proof. If the condition (i) is valid, then we have

$$I < \frac{M'}{m'}I = h'I \leq X \leq hI = \frac{M}{m}I,$$

which, by (2.12) gives

$$0 \leq H_\nu(A,B) - A\sharp B \leq \frac{1}{4(h')^{1-\nu}} \max_{x\in[h',h]} D\left(x^{2\nu-1}\right)A. \tag{2.20}$$

Observe that, if $\nu \in \left[\frac{1}{2},1\right]$, then

$$\max_{x\in[h',h]} D\left(x^{2\nu-1}\right) = D\left(h^{2\nu-1}\right) = \left(h^{2\nu-1}-1\right)\ln h^{2\nu-1}.$$

If $\nu \in \left[0,\frac{1}{2}\right]$, then

$$\max_{x\in[h',h]} D\left(x^{2\nu-1}\right) = D\left((h')^{2\nu-1}\right) = \left((h')^{2\nu-1}-1\right)\ln (h')^{2\nu-1}.$$

By (2.20) we get the desired result (2.18).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \leq X \leq \frac{1}{h'}I < I,$$

which, by (2.12) gives

$$0 \leq H_\nu(A,B) - A\sharp B \leq \frac{1}{4\left(\frac{1}{h}\right)^{1-\nu}} \max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} D\left(x^{2\nu-1}\right)A. \tag{2.21}$$

If $\nu \in \left[\frac{1}{2},1\right]$, then

$$\max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} D\left(x^{2\nu-1}\right) = D\left(\left(\frac{1}{h}\right)^{2\nu-1}\right) = D\left((h)^{-2\nu+1}\right).$$

If $\nu \in \left[0,\frac{1}{2}\right]$, then

$$\max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} D\left(x^{2\nu-1}\right) = D\left(\left(\frac{1}{h'}\right)^{2\nu-1}\right) = D\left((h')^{-2\nu+1}\right).$$

By (2.21) we get the desired result (2.19). $\square$

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3. Multiplicative Reverse Inequalities for Heinz Mean

We have the following generalization of Theorem 2:

Theorem 5. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that the condition (2.1) is valid. Then for any $\nu\in[0,1]$ we have*

$$A\nabla_{\nu}B\leq A\sharp_{\nu}B\exp\left[4\nu\left(1-\nu\right)\left(\digamma\left(m,M\right)-1\right)\right] \tag{3.1}$$

where

$$\digamma\left(m,M\right):=\begin{cases}K\left(m\right)\ \ \text{if } M<1,\\ \\ \max\left\{K\left(m\right),K\left(M\right)\right\}\ \ \text{if } m\leq1\leq M,\\ \\ K\left(M\right)\ \ \text{if } 1<m,\end{cases}$$

In particular, we have

$$A\nabla B\leq A\sharp B\exp\left[\digamma\left(m,M\right)-1\right]. \tag{3.2}$$

Proof. From the inequality (1.5) we have for $a=1$ and $b=x$ that

$$\begin{aligned}\left(1-\nu\right)+\nu x&\leq x^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(K\left(\frac{1}{x}\right)-1\right)\right]\\&=x^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right]\end{aligned} \tag{3.3}$$

for any $x>0$ and hence

$$\begin{aligned}\left(1-\nu\right)I+\nu X&\leq X^{\nu}\max_{m\leq x\leq M}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right]\\&=X^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{m\leq x\leq M}K\left(x\right)-1\right)\right]\end{aligned} \tag{3.4}$$

for any operator X with the property that $0<mI\leq X\leq MI$ and for any $\nu\in[0,1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI\leq A^{-1/2}BA^{-1/2}\leq MI$ and by taking $X=A^{-1/2}BA^{-1/2}$ in (3.4) we get

$$\begin{aligned}&\left(1-\nu\right)I+\nu A^{-1/2}BA^{-1/2}\\&\leq\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\max_{m\leq x\leq M}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right]\\&=\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{m\leq x\leq M}K\left(x\right)-1\right)\right]\end{aligned} \tag{3.5}$$

for any $\nu\in[0,1]$.

Now, if we multiply (3.5) in both sides with $A^{1/2}$ we get

$$\begin{aligned}&\left(1-\nu\right)A+\nu BA\\&\leq A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}\max_{m\leq x\leq M}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right]\\&=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}\exp\left[4\nu\left(1-\nu\right)\left(\max_{m\leq x\leq M}K\left(x\right)-1\right)\right]\end{aligned} \tag{3.6}$$

for any $\nu\in[0,1]$.

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Since

$$\max_{m\leq x\leq M} K\left(x\right)=\begin{cases} K\left(m\right) \text{ if } M<1, \\ \\ \max\left\{K\left(m\right),K\left(M\right)\right\} \text{ if } m\leq 1\leq M, \\ \\ K\left(M\right) \text{ if } 1<m, \end{cases}$$

then by (3.6) we get the desired result (3.1). □

Corollary 4. *With the assumptions of Theorem 5 we have*

$$A\nabla B\leq \exp\left[4\nu\left(1-\nu\right)\left(F\left(m,M\right)-1\right)\right]H_{\nu}\left(A,B\right). \tag{3.7}$$

Corollary 5. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*
(i) $0<mI\leq A\leq m'I<M'I\leq B\leq MI$;
(ii) $0<mI\leq B\leq m'I<M'I\leq A\leq MI$;
we have

$$A\nabla B\leq \exp\left[4\nu\left(1-\nu\right)\left(K\left(h\right)-1\right)\right]H_{\nu}\left(A,B\right). \tag{3.8}$$

We also have:

Theorem 6. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that the condition (2.1) is valid. Then for any $\nu\in[0,1]$ we have*

$$H_{\nu}\left(A,B\right)\leq \exp\left[\Theta_{\nu}\left(m,M\right)-1\right]A\sharp B \tag{3.9}$$

where

$$\Theta_{\nu}\left(m,M\right):=\begin{cases} K\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \\ \max\left\{K\left(m^{|2\nu-1|}\right),K\left(M^{|2\nu-1|}\right)\right\} \text{ if } m\leq 1\leq M, \\ \\ K\left(M^{|2\nu-1|}\right) \text{ if } 1<m. \end{cases} \tag{3.10}$$

Proof. From the inequality (1.5) we have for $\nu=\frac{1}{2}$

$$\frac{\frac{c+d}{2}}{\sqrt{cd}}\leq \exp\left(K\left(\frac{c}{d}\right)-1\right) \tag{3.11}$$

for any $c,d>0$.

If we take in (3.11) $c=a^{1-\nu}b^{\nu}$ and $d=a^{\nu}b^{1-\nu}$ then we get

$$\frac{a^{1-\nu}b^{\nu}+a^{\nu}b^{1-\nu}}{2}\leq \exp\left(K\left(\left(\frac{a}{b}\right)^{1-2\nu}\right)-1\right)\sqrt{ab} \tag{3.12}$$

for any a, $b>0$ for any $\nu\in[0,1]$.

This is an inequality of interest in itself.

If we take in (2.19) $a=x$ and $b=1$, then we get

$$\frac{x^{1-\nu}+x^{\nu}}{2}\leq \exp\left(K\left(x^{1-2\nu}\right)-1\right)\sqrt{x}, \tag{3.13}$$

for any $x>0$.

Now, if $x\in[m,M]\subset(0,\infty)$ then by (2.20) we have

$$\frac{x^{1-\nu}+x^{\nu}}{2}\leq \sqrt{x}\exp\left(\max_{x\in[m,M]}K\left(x^{1-2\nu}\right)-1\right) \tag{3.14}$$

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for any $x \in [m, M]$.

If $\nu \in \left(0, \frac{1}{2}\right)$, then

$$\max_{x\in[m,M]} K\left(x^{1-2\nu}\right) = \begin{cases} K\left(m^{1-2\nu}\right) \text{ if } M < 1, \\ \\ \max\left\{K\left(m^{1-2\nu}\right), K\left(M^{1-2\nu}\right)\right\} \text{ if } m \le 1 \le M, \\ \\ K\left(M^{1-2\nu}\right) \text{ if } 1 < m. \end{cases}$$

If $\nu \in \left(\frac{1}{2}, 1\right)$, then

$$\begin{aligned} \max_{x\in[m,M]} K\left(x^{1-2\nu}\right) &= \max_{x\in[m,M]} K\left(x^{2\nu-1}\right) \\ &= \begin{cases} K\left(m^{2\nu-1}\right) \text{ if } M < 1, \\ \\ \max\left\{K\left(m^{2\nu-1}\right), K\left(M^{2\nu-1}\right)\right\} \text{ if } m \le 1 \le M, \\ \\ K\left(M^{2\nu-1}\right) \text{ if } 1 < m. \end{cases} \end{aligned}$$

Therefore, by (3.14) we have

$$\frac{x^{1-\nu}+x^{\nu}}{2} \le \exp\left[\Theta\left(m, M\right) - 1\right]\sqrt{x} \tag{3.15}$$

for any $x \in [m, M] \subset (0, \infty)$ and for any $\nu \in [0, 1]$.

If X is an operator with $mI \le X \le MI$, then by (3.15) we have

$$\frac{X^{1-\nu}+X^{\nu}}{2} \le \exp\left[\Theta\left(m, M\right) - 1\right] X^{1/2}.$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \le A^{-1/2}BA^{-1/2} \le MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (3.15) we get

$$\begin{aligned} &\frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu} + \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\right] \\ &\le \exp\left[\Theta\left(m, M\right) - 1\right]\left(A^{-1/2}BA^{-1/2}\right)^{1/2}. \end{aligned} \tag{3.16}$$

Now, if we multiply (3.16) in both sides with $A^{1/2}$ we get the desired result (3.9). $\square$

Finally, we have

Corollary 6. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*

(i) $0 < mI \le A \le m'I < M'I \le B \le MI$,

(ii) $0 < mI \le B \le m'I < M'I \le A \le MI$,

we have for $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$ that

$$H_{\nu}\left(A, B\right) \le \exp\left[K\left(h^{|2\nu-1|}\right) - 1\right] A\sharp B, \tag{3.17}$$

where $\nu \in [0, 1]$.

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