Solution of Integral Equations of Fredholm Kind Involving Incomplete ℵ-Function, Generalized Extended Mittag-Leffler Function and S-Function

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ABSTRACT

The main objective of this paper is to solve Fredholm integral equations (IEs) that involve S-function, generalized extended Mittag-Leffler function (GEMLF), and incomplete ℵ-function as the kernel. These types of integral equations appear frequently in applied mathematics, particularly in mathematical physics, engineering, and finance. To solve these integral equations, we employ two powerful mathematical tools, namely fractional calculus (FC) and integral transforms. Specifically, we use the Weyl operator and Mellin transform to solve the integral equation associated with S-functions, GEMLF, and incomplete ℵ-functions. These techniques allow us to express the solution in a closed form, which is essential for practical applications. Moreover, we present several special cases of the solutions obtained, which provide additional insights into the behavior of the solutions. These results are significant for the study of integral equations, as they can be used to derive several known results. Furthermore, the techniques used in this study can be applied to other integral equations that involve different types of functions.

Keywords: Integral equations of Fredholm kind, S- function, generalized extended Mittag-Leffler function, incomplete ℵ- functions, Mellin inversion theorem, Weyl fractional integral operator, Mellin transform.

1. Introduction and Preliminaries

Integral equation is an essential tool in solving problems related to science and engineering. The equations are highly versatile and are used in a diverse range of fields. In the problems related to heat and mass transfer, these equations are used to model and predict the behavior of thermal and fluid systems, such as the flow of fluids through pipes and the transfer of heat in buildings. In scattering theory, these equations are used to study how particles or waves interact with each other and with their environment. In the kinetic theory of gases, they are used to describe the behavior of gases on a microscopic level, including the motion and collisions of individual gas molecules. In integral geometry, these equations are used to study how geometric shapes interact with each

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other and with their surroundings. In construction science, they are used to understand how materials behave under different conditions, to optimize the design and construction of buildings and other structures. Many researchers have done notable work in these fields [3–5,22,26–32,35,36].

Among the different types of integral equations, the Fredholm integral equation is particularly significant in the study of special functions. Incomplete special functions have a unique role in distribution theory, mathematical modeling, probability theory, and other fields. Its properties and applications have been extensively studied by many authors [1, 6, 9–12, 16, 17, 20, 33, 34].

A specific area of focus for mathematicians has been the study of Fredholm integral equations involving incomplete hypergeometric functions, incomplete I-functions, incomplete H-functions, and incomplete $\overline{H}$-functions as kernels. Singh *et. al.* [37] have done very novel work on applications of the fractional differential equations associated with integral operators involving $\aleph$-function in the kernel. Motivated by the work mentioned above, we have now turned our attention to investigating the Fredholm integral equation that involves the multiplication of incomplete $\aleph$-functions, GEMLF, and $\mathcal{S}$-function as the kernel. This research will advance our understanding of the properties and applications of these functions and their role in solving complex problems in various fields.

Definition 1: L. Euler [24] investigated the Gamma function as the extension of the factorial operation given below:

$$\Gamma(n+1) = n!. \tag{1.1}$$

The Gamma function is defined by a convergent improper integral as:

$$\Gamma(\theta) = \begin{cases} \int_0^\infty e^{-t} t^{\theta-1} dt, & (\Re(\theta) > 0) \\ \\ \frac{\Gamma(\theta+\omega)}{(\theta)_\omega}, & (\theta \in \mathbb{C} \setminus \mathbb{Z}_0^-; \omega \in \mathbb{N}_0). \end{cases} \tag{1.2}$$

where $(\theta)_\omega$ is the Pochhammer symbol [2] and is defined as:

$$(\theta)_\omega = \frac{\Gamma(\theta+\omega)}{\Gamma(\theta)} = \begin{cases} 1, & (\omega = 0; \theta \in \mathbb{C} \setminus \{0\}) \\ \theta(\theta+1)\ldots(\theta+k-1), & (\omega = k \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{1.3}$$

Definition 2: The incomplete gamma function [13] is widely applicable in various fields, including physics and medical sciences. The properties of the real incomplete gamma functions are commonly used in complex analysis.

The upper and lower incomplete gamma functions are defined as:

$$\gamma(u, x) = \int_0^x v^{u-1} e^{-v} dv \quad (\Re(u) > 0; x \geq 0), \tag{1.4}$$

and

$$\Gamma(u, x) = \int_x^\infty v^{u-1} e^{-v} dv \quad (x \geq 0; \Re(u) > 0), \tag{1.5}$$

where

$$\gamma(u, x) + \Gamma(u, x) = \Gamma(u) \quad (\Re(u) > 0). \tag{1.6}$$

Definition 3: Sdland *et. al.* [23] have introduced a new concept called the $\aleph$-function. This function has recently been expanded upon by Bansal *et. al.* [19], who have introduced the incomplete $\aleph$-function. This new function is a generalization of the original $\aleph$-function, which leads to further advancements in mathematical theory and applications.

$$\begin{aligned} {}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i;R}[z] &= {}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i,R}\left[z \left| \begin{matrix} (\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i} \\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i} \end{matrix} \right.\right] \\ &= \frac{1}{2\pi i}\int_{\mathcal{L}} \Phi(\nu,x) z^{-\nu} d\nu, \end{aligned} \tag{1.7}$$

where $z \neq 0$ and

$$\Phi(\nu,x) = \frac{\Gamma(1-\mathfrak{b}_1-\mathfrak{B}_1\nu,x)\prod_{j=1}^{M}\Gamma(\mathfrak{a}_j+\mathfrak{A}_j\nu)\prod_{j=2}^{N}\Gamma(1-\mathfrak{b}_j-\mathfrak{B}_j\nu)}{\sum_{i=1}^{R}\delta_i\left[\prod_{j=M+1}^{Q_i}\Gamma(1-\mathfrak{a}_{ji}-\mathfrak{A}_{ji}\nu)\prod_{j=N+1}^{P_i}\Gamma(\mathfrak{b}_{ji}+\mathfrak{B}_{ji}\nu)\right]}. \tag{1.8}$$

$$\begin{aligned} {}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i;R}[z] &= {}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i,R}\left[z \left| \begin{matrix} (\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i} \\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i} \end{matrix} \right.\right] \\ &= \frac{1}{2\pi i}\int_{\mathcal{L}} \Psi(\nu,x) z^{-\nu} d\nu, \end{aligned} \tag{1.9}$$

where where $z \neq 0$ and

$$\Psi(\nu,x) = \frac{\gamma(1-\mathfrak{b}_1-\mathfrak{B}_1\nu,x)\prod_{j=1}^{M}\Gamma(\mathfrak{a}_j+\mathfrak{A}_j\nu)\prod_{j=2}^{N}\Gamma(1-\mathfrak{b}_j-\mathfrak{B}_j\nu)}{\sum_{i=1}^{R}\delta_i\left[\prod_{j=M+1}^{Q_i}\Gamma(1-\mathfrak{a}_{ji}-\mathfrak{A}_{ji}\nu)\prod_{j=N+1}^{P_i}\Gamma(\mathfrak{b}_{ji}+\mathfrak{B}_{ji}\nu)\right]}. \tag{1.10}$$

The both incomplete $\aleph$-functions $\left({}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i;R}[z] \text{ and } {}^{(\gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i;R}[z]\right)$ given by Eq. (1.7) and Eq. (1.9) exist for all $x \geq 0$ with the following conditions:

- The contour $\mathcal{L}$ extends from $C - \iota\infty$ to $C + \iota\infty$ on the complex plane, $C \in \Re$.
- Poles of $\Gamma(1-\mathfrak{b}_j-\mathfrak{B}_j\zeta)$, $j = \overline{2,N}$ never match exactly with the poles of $\Gamma(\mathfrak{a}_j+\mathfrak{A}_j\zeta)$, $j = \overline{1,M}$.
- The parameters M, N, P_i, Q_i are non negative integers that satisfy $0 \leq N \leq P_i$, $0 \leq M \leq Q_i$ and $i = \overline{1,R}$.
- Parameters $\mathfrak{B}_\mathrm{j}, \mathfrak{A}_\mathrm{j}, \mathfrak{B}_\mathrm{ji}, \mathfrak{A}_\mathrm{ji}$ are positive real numbers and $\mathfrak{b}_\mathrm{j}, \mathfrak{a}_\mathrm{j}, \mathfrak{b}_\mathrm{ji}, \mathfrak{a}_\mathrm{ji}$ are complex numbers.
- All the poles of $\Phi(\zeta, y)$ and $\Psi(\zeta, y)$ are supposed to be simple, and the null product is considered as unity.
- $$\mathfrak{F}_i \geq 0, \quad |\arg(z)| < \frac{\pi}{2}\mathfrak{F}_i \quad \text{and} \quad \Re(\mathfrak{G}_i) + 1 < 0, \quad i = \overline{1,R}, \tag{1.11}$$

where

$$\mathfrak{F}_i = \sum_{j=1}^{N} \mathfrak{B}_j + \sum_{j=1}^{M} \mathfrak{A}_j - \left(\sum_{j=N+1}^{P_i} \mathfrak{B}_{ji} + \sum_{j=M+1}^{Q_i} \mathfrak{A}_{ji} \right), \tag{1.12}$$

$$\mathfrak{G}_i = \sum_{j=1}^{M} \mathfrak{a}_j - \sum_{j=1}^{N} \mathfrak{b}_j + \left(\sum_{j=M+1}^{Q_i} \mathfrak{B}_{ji} - \sum_{j=N+1}^{P_i} \mathfrak{A}_{ji} \right) + \frac{1}{2}\left(P_i - Q_i\right). \tag{1.13}$$

Definition 4: GEMLF is defined by Bansal *et al.* [18] as:

$$E_{\mu,\lambda}^{\phi;\rho}\left(y;\xi,\psi,\omega\right) = \sum_{m=0}^{\infty} \frac{\mathbf{B}_{\xi}^{\psi,\omega}(\phi+m,\rho-\phi)}{\mathbf{B}(\phi,\rho-\phi)} \frac{(\rho)_m}{\Gamma(\mu m+\lambda)} \frac{y^m}{(m)!}, \tag{1.14}$$

$$(\xi \geq 0, \Re(\rho) > \Re(\phi) > 0, \Re(\mu) > 0, \Re(\lambda) > 0).$$

Here $\mathbf{B}_p^{\psi,\omega}(\alpha,\beta)$ is generalized beta function [8].

Definition 5: The $\mathcal{S}$-function [7] is defined as follows:

$$\mathcal{S}_{(p,q)}^{\sigma,\eta,\epsilon,\tau,\kappa}\left[g_1, g_2, \ldots, g_p; h_1, h_2, \ldots, h_q; y\right] = \sum_{n=0}^{\infty} \frac{(g_1)_n (g_2)_n \cdots (g_p)_n (\epsilon)_{n\tau,\kappa}}{(h_1)_n (h_2)_n \cdots (h_q)_n \Gamma_\kappa(n\sigma+\eta)} \frac{y^n}{n!}, \tag{1.15}$$

where the κ-Pochhammer symbol [25] is defined as:

$$(\epsilon)_{n,\kappa} = \begin{cases} \frac{\Gamma_\kappa(\kappa n+\epsilon)}{\Gamma_\kappa(\epsilon)}, & \left(\kappa \in \Re, \epsilon \in \frac{\mathbb{C}}{\{0\}}\right) \\ \epsilon(\epsilon+\kappa)\ldots(\epsilon+(n-1)\kappa), & (n \in \mathbb{N}, \epsilon \in \mathbb{C}). \end{cases} \tag{1.16}$$

Definition 6: The Mellin transforms of incomplete $\aleph$-functions are investigated by Bansal *et. al.* [19] in following manner:

$$\mathfrak{M}\left\{ {}^{(\Gamma)}\aleph_{P_i,Q_i,\delta_i;R}^{M,N}\left[kz^\mu \left| \begin{matrix} \left(\mathfrak{b}_1,\mathfrak{B}_1,x\right), \left(\mathfrak{b}_j,\mathfrak{B}_j\right)_{2,N}, \left[\delta_i\left(\mathfrak{b}_{ji},\mathfrak{B}_{ji}\right)\right]_{N+1,P_i} \\ \left(\mathfrak{a}_j,\mathfrak{A}_j\right)_{1,M}, \left[\delta_i\left(\mathfrak{a}_{ji},\mathfrak{A}_{ji}\right)\right]_{M+1,Q_i} \end{matrix} \right. \right]; \mathfrak{p} \right\} = \frac{k^{-\mathfrak{p}/\mu}}{\mu} \Phi\left(\frac{\mathfrak{p}}{\mu}, x\right), \tag{1.17}$$

and

$$\mathfrak{M}\left\{ {}^{(\gamma)}\aleph_{P_i,Q_i,\delta_i;R}^{M,N}\left[kz^\mu \left| \begin{matrix} \left(\mathfrak{b}_1,\mathfrak{B}_1,x\right), \left(\mathfrak{b}_j,\mathfrak{B}_j\right)_{2,N}, \left[\delta_i\left(\mathfrak{b}_{ji},\mathfrak{B}_{ji}\right)\right]_{N+1,P_i} \\ \left(\mathfrak{a}_j,\mathfrak{A}_j\right)_{1,M}, \left[\delta_i\left(\mathfrak{a}_{ji},\mathfrak{A}_{ji}\right)\right]_{M+1,Q_i} \end{matrix} \right. \right]; \mathfrak{p} \right\} = \frac{k^{-\mathfrak{p}/\mu}}{\mu} \Psi\left(\frac{\mathfrak{p}}{\mu}, x\right), \tag{1.18}$$

where Φ and Ψ are defined by Eq.(1.8) and Eq. (1.10) respectively.

Definition 7: The Weyl fractional integral operator of order β [14] is defined as:

$$\mathcal{W}^{-\beta}\{\mathfrak{F}(\mathfrak{z})\} = \frac{1}{\Gamma(\beta)} \int_{\mathfrak{z}}^{\infty} (\mathfrak{t}-\mathfrak{z})^{\beta-1} \mathfrak{F}(\mathfrak{z}) d\mathfrak{t}, \qquad (\Re(\beta) > 0, \mathfrak{F} \in \mathcal{A}), \tag{1.19}$$

here $\mathcal{A}$ indicates the space of all functions $\mathfrak{F}$ defined on $\mathbb{R} = [0,\infty)$ [15].

2. Solution of Integral Equation of Fredholm Kind Involving Incomplete ℵ-function, GEMLF and $\mathcal{S}$-Function

In this section, we will be applying the Mellin transform method as well as the Weyl fractional integral operator to solve the Fredholm integral equation which involves incomplete ℵ-function, GEMLF, and $\mathcal{S}$-function. By utilizing these mathematical techniques, we aim to provide a comprehensive and precise solution to the problem.

Lemma 1. Let

(i) The parameters M, N, P_i, Q_i are non negative integers that satisfy $0 \le N \le P_i$, $0 \le M \le Q_i$ and $i = \overline{1, R}$.

(ii) $\Re(\alpha - s) > 0$; $\quad \Re(\mathfrak{G}_i) + 1 < 0 \quad (i = \overline{1, R})$ where $\mathfrak{G}_i$ is given by Eq. (1.13).

(iii) $x \ge 0$, $\beta > 0$ and $\alpha \in \mathbb{C}$.

(iv) $|\arg(C)| < \frac{\pi}{2}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is given by Eq. (1.12).

Then,

$$W^{s-\alpha}\Bigg\{u^{-\alpha} E^{\phi;\rho}_{\mu,\lambda}\Big(u;\xi,\psi,\omega\Big)\mathcal{S}^{\sigma,\eta,\epsilon,\tau,\kappa}_{(p,q)}\Big[g_1, g_2, \ldots, g_p; h_1, h_2, \ldots, h_q; u\Big]$$

$$\times\, {}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i,R}\left[C\left(\frac{y}{u}\right)^{\beta}\ \middle|\ \begin{matrix} (\mathfrak{b}_1, \mathfrak{B}_1, x), (\mathfrak{b}_j, \mathfrak{B}_j)_{2,N}, [\delta_i(\mathfrak{b}_{ji}, \mathfrak{B}_{ji})]_{N+1,P_i} \\ (\mathfrak{a}_j, \mathfrak{A}_j)_{1,M}, [\delta_i(\mathfrak{a}_{ji}, \mathfrak{A}_{ji})]_{M+1,Q_i} \end{matrix}\right]\Bigg\}$$

$$= u^{-s}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathbf{B}^{\psi,\omega}_{\xi}(\phi+m, \rho-\phi)}{\mathbf{B}(\phi, \rho-\phi)}\frac{(\rho)_m}{\Gamma(\mu m+\lambda)}\frac{(g_1)_n(g_2)_n\cdots(g_p)_n(\epsilon)_{n\tau,\kappa}}{(h_1)_n(h_2)_n\cdots(h_q)_n\Gamma_{\kappa}(n\sigma+\eta)}\frac{u^{m+n}}{m!n!}$$

$$\times\, {}^{(\Gamma)}\aleph^{M,N+1}_{P_i+1,Q_i+1,\delta_i,R}\left[C\left(\frac{y}{u}\right)^{\beta}\ \middle|\ \begin{matrix} (\mathfrak{b}_1, \mathfrak{B}_1, x), (1-s+m+n, \beta), (\mathfrak{b}_j, \mathfrak{B}_j)_{2,N}, [\delta_i(\mathfrak{b}_{ji}, \mathfrak{B}_{ji})]_{N+1,P_i} \\ (\mathfrak{a}_j, \mathfrak{A}_j)_{1,M}, [\delta_i(\mathfrak{a}_{ji}, \mathfrak{A}_{ji})]_{M+1,Q_i}, (1-\alpha+m+n, \beta) \end{matrix}\right]. \tag{2.1}$$

Proof. To attain the desired result, we commence our process by expressing the incomplete ℵ-function in terms of the Mellin Barne contour integral. Afterward, we proceed to expand the GEMLF and $\mathcal{S}$-function in series form and then change the order of integral and summation. At last, we apply the Weyl operator, interpret the result using the definition of the incomplete ℵ-function, and get the desired result. □

Lemma 2. Let

(i) The parameters M, N, P_i, Q_i are non negative integers that satisfy $0 \le N \le P_i$, $0 \le M \le Q_i$ and $i = \overline{1, R}$.

(ii) $\Re(\alpha - s) > 0$; $\quad \Re(\mathfrak{G}_i) + 1 < 0 \quad (i = \overline{1, R})$ where $\mathfrak{G}_i$ is given by Eq. (1.13).

(iii) $x \geq 0$, $\beta > 0$ and $\alpha \in \mathbb{C}$.

(iv) $|\arg(C)| < \frac{\pi}{2}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is given by Eq. (1.12).

Then,

$$W^{s-\alpha}\Bigg\{u^{-\alpha}E_{\mu,\lambda}^{\phi;\rho}\Big(u;\xi,\psi,\omega\Big)\mathcal{S}_{(p,q)}^{\sigma,\eta,\epsilon,\tau,\kappa}\Big[g,g_2,\ldots,g_p;h_1,h_2,\ldots,h_q;u\Big]$$

$$\times\,{}^{(\gamma)}\aleph_{P_i,Q_i,\delta_i,R}^{M,N}\left[C\left(\frac{y}{u}\right)^{\beta}\left|\begin{array}{c}(\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ \\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i}\end{array}\right.\right]\Bigg\}$$

$$=u^{-s}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathbf{B}_{\xi}^{\psi,\omega}(\phi+m,\rho-\phi)}{\mathbf{B}(\phi,\rho-\phi)}\frac{(\rho)_m}{\Gamma(\mu m+\lambda)}\frac{(g_1)_n(g_2)_n\ldots(g_p)_n(\epsilon)_{n\tau,\kappa}}{(h_1)_n(h_2)_n\ldots(h_q)_n\Gamma_{\kappa}(n\sigma+\eta)}\frac{u^{m+n}}{m!n!}$$

$$\times\,{}^{(\gamma)}\aleph_{P_i+1,Q_i+1,\delta_i,R}^{M,N+1}\left[C\left(\frac{y}{u}\right)^{\beta}\left|\begin{array}{c}(\mathfrak{b}_1,\mathfrak{B}_1,x),(1-s+m+n,\beta),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ \\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i},(1-\alpha+m+n,\beta)\end{array}\right.\right]. \tag{2.2}$$

Proof. To attain the desired result, we commence our process by expressing the incomplete $\aleph$-function in terms of the Mellin Barne contour integral. Afterward, we proceed to expand the GEMLF and $\mathcal{S}$-function in series form and then change the order of integral and summation. At last, we apply the Weyl operator, interpret the result using the definition of the incomplete $\aleph$-function, and get the desired result. □

Theorem 2.1 *Let*

(i) The parameters M, N, P_i, Q_i are non negative integers that satisfy $0 \leq N \leq P_i$, $0 \leq M \leq Q_i$ and $i = \overline{1, R}$

(ii) $\Re(\alpha - s) > 0$; $\Re(\mathfrak{G}_i) + 1 < 0$ $(i = \overline{1, R})$ *where $\mathfrak{G}_i$ is given by* (1.13)

(iii) $x \geq 0$, $\beta > 0$ and $\alpha \in \mathbb{C}$

Then, the relation given below holds :

$$\int_0^\infty u^{-s}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\boldsymbol{B}_\xi^{\psi,\omega}(\phi+m,\rho-\phi)}{\boldsymbol{B}(\phi,\rho-\phi)}\frac{(\rho)_m}{\Gamma(\mu m+\lambda)}\frac{(g_1)_n(g_2)_n\dots(g_p)_n(\epsilon)_{n\tau,\kappa}}{(h_1)_n(h_2)_n\dots(h_q)_n\Gamma_\kappa(n\sigma+\eta)}\frac{u^{m+n}}{m!n!}$$

$$\times\,{}^{(\Gamma)}\aleph^{M,N+1}_{P_i+1,Q_i+1,\delta_i,R}\left[C\left(\frac{y}{u}\right)^\beta\ \middle|\ \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(1-s+m+n,\beta),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i},(1-\alpha+m+n,\beta)\end{matrix}\right]g(u)du$$
$$=\int_0^\infty u^{-\alpha}E^{\phi;\rho}_{\mu,\lambda}\Big(u;\xi,\psi,\omega\Big)\mathcal{S}^{\sigma,\eta,\epsilon,\tau,\kappa}_{(p,q)}\Big[g_1,g_2,\dots,g_p;h_1,h_2,\dots,h_q;u\Big]$$

$$\times\,{}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i,R}\left[C\left(\frac{y}{u}\right)^\beta\ \middle|\ \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i}\end{matrix}\right]D^{s-\alpha}\{g(u)\}du, \tag{2.3}$$

provided that $\mathfrak{F}\in\mathcal{A}$ *and* $y>0$.

Proof. Let $\mathbb{I}$ refers to the left-hand side of Eq. (2.3), then

$$\mathbb{I}=\int_0^\infty u^{-s}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathbf{B}_\xi^{\psi,\omega}(\phi+m,\rho-\phi)}{\mathbf{B}(\phi,\rho-\phi)}\frac{(\rho)_m}{\Gamma(\mu m+\lambda)}\frac{(g_1)_n(g_2)_n\dots(g_p)_n(\epsilon)_{n\tau,\kappa}}{(h_1)_n(h_2)_n\dots(h_q)_n\Gamma_\kappa(n\sigma+\eta)}\frac{u^{m+n}}{m!n!}$$

$$\times\,{}^{(\Gamma)}\aleph^{M,N+1}_{P_i+1,Q_i+1,\delta_i,R}\left[C\left(\frac{y}{u}\right)^\beta\ \middle|\ \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(1-s+m+n,\beta),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i},(1-\alpha+m+n,\beta)\end{matrix}\right]g(u)du$$
$$=\int_0^\infty g(u)W^{s-\alpha}\Bigg\{u^{-\alpha}E^{\phi;\rho}_{\mu,\lambda}\Big(u;\xi,\psi,\omega\Big)\mathcal{S}^{\sigma,\eta,\epsilon,\tau,\kappa}_{(p,q)}\Big[g_1,g_2,\dots,g_p;h_1,h_2,\dots,h_q;u\Big]$$

$$\times\,{}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i,R}\left[C\left(\frac{y}{u}\right)^\beta\ \middle|\ \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i}\end{matrix}\right]\Bigg\}du. \qquad \text{(using Eq. (2.1))}$$

Using Eq. (1.19) and changing the order of integration, we obtain

$$\mathbb{I}=\int_0^\infty t^{-\alpha}E^{\phi;\rho}_{\mu,\lambda}\Big(t;\xi,\psi,\omega\Big)\mathcal{S}^{\sigma,\eta,\epsilon,\tau,\kappa}_{(p,q)}\Big[g_1,g_2,\dots,g_p;h_1,h_2,\dots,h_q;t\Big]$$

$$\times\,{}^{(\Gamma)}\aleph^{M,N}_{P_i,Q_i,\delta_i,R}\left[C\left(\frac{y}{t}\right)^\beta\ \middle|\ \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i}\end{matrix}\right]\left(\int_0^t\frac{(t-u)^{\alpha-s-1}}{\Gamma(\alpha-s)}g(u)du\right)dt.$$

Afterward, by utilizing Riemann-Liouville's fractional derivative [14], we get

$$\mathbb{I} = \int_0^\infty t^{-\alpha} E_{\mu,\lambda}^{\phi;\rho}\Big(t;\xi,\psi,\omega\Big)\mathcal{S}_{(p,q)}^{\sigma,\eta,\epsilon,\tau,\kappa}\Big[g_1,g_2,\ldots,g_p;h_1,h_2,\ldots,h_q;t\Big]$$

$$\times\, {}^{(\Gamma)}\aleph_{P_i,Q_i,\delta_i,R}^{M,N}\left[C\left(\frac{y}{t}\right)^\beta \,\middle|\, \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i}\end{matrix}\right] D^{s-\alpha}\{g(t)\}dt,$$

which is the right-hand side of Eq. (2.3). □

Theorem 2.2 *Let*

(i) The parameters M,N,P_i,Q_i *are non negative integers that satisfy* $0\le N\le P_i$, $0\le M\le Q_i$ *and* $i=\overline{1,R}$.

(ii) $\Re(\alpha-s)>0$; $\Re(\mathfrak{G}_i)+1<0$ $(i=\overline{1,R})$ *where* $\mathfrak{G}_i$ *is given by* (1.13).

(iii) $x\ge 0$, $\beta>0$ *and* $\alpha\in\mathbb{C}$.

Then, the relation given below holds :

$$\int_0^\infty u^{-s}\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{\boldsymbol{B}_\xi^{\psi,\omega}(\phi+m,\rho-\phi)}{\boldsymbol{B}(\phi,\rho-\phi)}\frac{(\rho)_m}{\Gamma(\mu m+\lambda)}\frac{(g_1)_n(g_2)_n\ldots(g_p)_n(\epsilon)_{n\tau,\kappa}}{(h_1)_n(h_2)_n\ldots(h_q)_n\Gamma_\kappa(n\sigma+\eta)}\frac{u^{m+n}}{m!n!}$$

$$\times\, {}^{(\gamma)}\aleph_{P_i+1,Q_i+1,\delta_i,R}^{M,N+1}\left[C\left(\frac{y}{u}\right)^\beta \,\middle|\, \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(1-s+m+n,\beta),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i},(1-\alpha+m+n,\beta)\end{matrix}\right] g(u)du$$

$$= \int_0^\infty u^{-\alpha} E_{\mu,\lambda}^{\phi;\rho}\Big(u;\xi,\psi,\omega\Big)\mathcal{S}_{(p,q)}^{\sigma,\eta,\epsilon,\tau,\kappa}\Big[g_1,g_2,\ldots,g_p;h_1,h_2,\ldots,h_q;u\Big]$$

$$\times\, {}^{(\gamma)}\aleph_{P_i,Q_i,\delta_i,R}^{M,N}\left[C\left(\frac{y}{u}\right)^\beta \,\middle|\, \begin{matrix}(\mathfrak{b}_1,\mathfrak{B}_1,x),(\mathfrak{b}_j,\mathfrak{B}_j)_{2,N},[\delta_i(\mathfrak{b}_{ji},\mathfrak{B}_{ji})]_{N+1,P_i}\\ (\mathfrak{a}_j,\mathfrak{A}_j)_{1,M},[\delta_i(\mathfrak{a}_{ji},\mathfrak{A}_{ji})]_{M+1,Q_i}\end{matrix}\right] D^{s-\alpha}\{g(u)\}du, \tag{2.4}$$

provided that $\mathfrak{F}\in\mathcal{A}$ *and* $y>0$.

Proof. The proof of this theorem follows a similar process to that of Theorem 2.1. □

3. Conclusions

Our research yields significant implications across a wide range of fields. Our methodology involves the solution of an integral equation of Fredholm kind, which includes S-function, generalized extended Mittag-Leffler function (GEMLF), and incomplete $\aleph$-function in the kernel. Specifically, we have discovered that a vast array of results as derived by authors [12, 21, 34, 35], can be obtained by setting specific values for different parameters of the S-function, generalized extended Mittag-Leffler function (GEMLF), and incomplete $\aleph$-function. As a result, the outcomes presented in this article have the potential to contribute to numerous advancements in science and engineering by providing valuable insights into the behavior of special functions relevant to these fields.

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