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QUADRATIC FUNCTIONAL INEQUALITY IN MODULAR SPACES AND ITS STABILITY

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ABSTRACT. In this paper, we prove the generalized Hyers-Ulam stability for the following functional inequality

$\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \geq \rho(k[f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y)])$

in modular spaces without $\triangle_2$-conditions.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [16]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [4] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias' theorem was obtained by Găvruta [3] by replasing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. The more abstract generalization was given by Nakano [11] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*([1], [6], [7], [8], [9], [12], [15], [18]). This idea was refined and generalized by Musielak and Orlicz [10] in 1959.

Recently, Sadeghi [14] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and Wongkum, Chaipunya, and Kumam [17] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

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In this paper, we prove the generalized Hyers-Ulam stability for the following quadratic functional equation

$$\begin{aligned} &\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \\ &\geq \rho(k[f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y)]) \end{aligned} \tag{1.1}$$

in modular spaces without $\triangle_2$-conditions by using a fixed point theorem.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}, \mathbb{C}$, or $\mathbb{N})$.
(1) A generalized functional $\rho : X \longrightarrow [0,\infty]$ is called *a modular* if
(M1) $\rho(x) = 0$ if and only if $x = 0$,
(M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
(M3) $\rho(z) \leq \rho(x) + \rho(y)$ whenever z is a convex combination of x and y.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in V$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is *convex.*

For any modular ρ on X, the modular space X_ρ is defined by

$$X_\rho = \{x \in X \mid \rho(\lambda x) \to 0 \ as\ \lambda \to 0\}$$

and the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\Big\{\lambda > 0 \mid \rho\Big(\frac{x}{\lambda}\Big) \leq 1\Big\}.$$

Let X_ρ be a modular space and $\{x_n\}$ a sequence in X_ρ. Then (i) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, (ii) $\{x_n\}$ is called *ρ-convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, and (iii) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence is ρ-convergent to a point in K.

A modular space X_ρ is said to *satisfy the $\triangle_2$-condition* if there exists $k \geq 2$ such that $X_\rho(2x) \leq kX_\rho(x)$ for all $x \in X$.

Example 1.2. ([9], [11], [12]) A convex function ζ defined on the interval $[0,\infty)$, nondecreasing and continuous, such that $\zeta(0) = 0, \zeta(\alpha) > 0$ for $\alpha > 0$, $\zeta(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. Let (Ω, Σ, μ) be a measure space and $L^0(\mu)$ the set of all measurable real valued (or complex valued) functions on Ω. Deine the Orlicz modular ρ_ζ on $L^0(\mu)$ by the formula $\rho_\zeta(f) = \int_\Omega \zeta(|f|)d\mu$. The associated modular space with respect to this modular is called an Orlicz space, and will be denoted by (L^ζ, Ω, μ) or briefly L^ζ. In other words,

$$L^\zeta = \{f \in L^0(\mu) \mid \rho_\zeta(\lambda f) < \infty \ \text{ for some } \ \lambda > 0\}.$$

It is known that the Orlicz space L^ζ is ρ_ζ-complete. Moreover, $(L^\zeta, \|\cdot\|_{\rho_\zeta})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\zeta}$ is defined as follows

$$\|f\|_{\rho_\zeta} = \inf\Big\{\lambda > 0 \ \Big|\ \int_\Omega \zeta\Big(\frac{|f|}{\lambda}\Big)d\mu \leq 1\Big\}.$$

Further, if μ is the Lebesgue measure on $\mathbb{R}$ and $\zeta(t) = e^t - 1$, then ρ_ζ does not satisfy the $\triangle_2$-condition.

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For a modular space X_ρ, a nonempty subset C of X_ρ, and a mapping $T : C \longrightarrow C$, *the orbit of* T *at* $x \in C$ is the set

$$\mathbb{O}(x) = \{x, Tx, T^2x, \cdots\}.$$

If $\delta_\rho(x) = \sup\{\rho(u - v) \mid u, v \in \mathbb{O}(x)\} < \infty$, then one says that T has *a bounded orbit at* x.

Lemma 1.3. [5] *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ-complete subset. If $T : C \longrightarrow C$ is a ρ-contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \le L\rho(x - y), \ \ \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $w \in C$.

For any modular ρ on X and any linear space V, we define a set $\mathbb{M}$

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and the generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \le c\psi(x, 0), \ \forall x \in V\},$$

where $\psi : V^2 \longrightarrow [0, \infty)$ is a mapping. The proof of the following lemma is similar to the proof of Lemma 10 in [17].

Lemma 1.4. *Let V be a linear space, X_ρ a ρ-complete modular space where ρ is convex lower semi-continuous and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\psi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\psi(ax, ax) \le a^2 L\psi(x, x) \tag{1.2}$$

for all $x, y \in V$ and some a and L with $a \ge 2$ and $0 \le L < 1$. Then we have the following :
(1) $\mathbb{M}$ is a linear space,
(2) $\widetilde{\rho}$ is a convex modular, and
(3) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete, and
(4) $\widetilde{\rho}$ is lower semi-continuous.

2. Solutions of (1.1)

In this section, we consider solutions of (1.1).
For any $f : V \longrightarrow X_\rho$, let

$$A_f(x, y) = k[f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y)]$$

and

$$B_f(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y).$$

Lemma 2.1. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ an even mapping with $f(0) = 0$. Suppose that $ka^2 \ge 1$ and $b^2 > a^2$. Then f is a quadratic mapping if and only if f is a solution of (1.1).*

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Proof. Since $k \neq 0$ and f is even, we have

$$f(ax) = a^2 f(x), \;\; f(bx) = b^2 f(x) \tag{2.1}$$

for all $x \in V$. Putting $y = ay$ in (1.1), by (2.1), we have

$$\begin{aligned} &\rho(f(x+ay) + f(x-ay) - 2f(x) - 2a^2 f(y)) \\ &\geq \rho(ka^2[f(x+by) + f(x-by) - 2f(x) - 2b^2 f(y)]) \end{aligned} \tag{2.2}$$

for all $x, y \in V$ and letting $y = \frac{y}{a}$ in (2.2), by (2.1), we have

$$\rho(B_f(x,y)) \geq \rho(ka^2[f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)]) \tag{2.3}$$

for all $x, y \in V$,where $p = \frac{b}{a}$. Since ρ is convex and $ka^2 \geq 1$, by (2.3),

$$\rho(B_f(x,y)) \geq ka^2 \rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \tag{2.4}$$

for all $x, y \in V$. Letting $x = py$ in (2.3), by (2.1), we have

$$\begin{aligned} &\rho(f(px+y) + f(px-y) - 2p^2 f(x) - 2f(y)) \\ &\geq kb^2 \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{2.5}$$

for all $x, y \in V$, because ρ is convex and $b^2 > a^2$. Interchanging x and y in (2.5), we have

$$\rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \geq kb^2 \rho(B_f(x,y)) \tag{2.6}$$

for all $x, y \in V$. By (M4), (2.4), and (2.6), we have

$$\begin{aligned} &\rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \\ &\geq k^2a^2b^2 \rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \end{aligned} \tag{2.7}$$

for all $x, y \in V$. Since $k^2a^2b^2 > 1$, by (2.7) and (M1), we get

$$f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y) = 0$$

for all $x, y \in V$ and hence f is a quadratic mapping. The converse is trivial. $\square$

Theorem 2.2. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $ka^2 \geq 2$ and $b^2 > a^2$. Then f is a quadratic mapping if and only if f is a solution of (1.1).*

Proof. By (1.1), we have

$$\begin{aligned} \rho(A_{f_o}(x,y)) &\leq \frac{1}{2}\rho(A_f(x,y)) + \frac{1}{2}\rho(A_f(-x,-y)) \\ &\leq \frac{1}{2}\rho(B_f(x,y)) + \frac{1}{2}\rho(B_f(-x,-y)) \\ &\leq \frac{1}{2}\rho(2B_{f_o}(x,y)) + \frac{1}{2}\rho(2B_{f_e}(x,y)) \end{aligned} \tag{2.8}$$

for all $x, y \in V$ and similarly, we have

$$\rho(A_{f_e}(x,y)) \leq \frac{1}{2}\rho(2B_{f_o}(x,y)) + \frac{1}{2}\rho(2B_{f_e}(x,y)) \tag{2.9}$$

for all $x, y \in V$. Letting $x = 0$ in (2.8), by (M4), we have

$$\frac{1}{2}\rho(4f_o(y)) \geq \rho(2kb^2 f_o(y)) \geq \frac{kb^2}{2}\rho(4f_o(y)) \tag{2.10}$$

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for all $y \in V$, because ρ is convex and $kb^2 > 2$. Since $kb^2 > 1$, by (2.10) and (M1), we have $f_o(y) = 0$ for all $y \in V$ and hence by (2.9), we have

$$\rho(A_{f_e}(x, y)) \le \rho(2B_{f_e}(x, y)) \tag{2.11}$$

for all $x, y \in V$. Since $ka^2 \ge 2$ and $b^2 > a^2$, by Lemma 2.1 and (2.11), $2f_e$ is a quadratic mapping and since $f_o(x) = 0$ for all $x \in X$, f is a quadratic mapping. □

For $k = 1$ in Theorem 2.2, we have the following corollary:

Corollary 2.3. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $b^2 > a^2 \ge 2$. The f is quadratic if and only if*

$$\rho(B_f(x, y)) \ge \rho(f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y))$$

for all $x, y \in V$.

Corollary 2.4. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $ka^2 \ge 2$ and $b^2 > a^2$. Then the following are equivalent*
(1) *f is quadratic,*
(2) *f satisfies (1.1), and*
(3) *f satisfies the following*

$$\rho(rB_f(x, y)) \ge \rho(rA_f(x, y))$$

for all $x, y \in V$ and all real number r.

3. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this section, we assume that every modular is convex and lower semicontinuous. In this section, we will prove the generalized Hyers-Ulam stability for (1.1).

Lemma 3.1. *Let ρ be a convex modular on X and t a real number with $2 \le t$. Then*

$$\rho\Big(\frac{1}{t}x + \frac{1}{t}y\Big) \le \frac{1}{t}\rho(x) + \frac{1}{t}\rho(y)$$

for all $x, y \in X$.

Proof. Since ρ is a convex modular on X, we have

$$\rho\Big(\frac{1}{t}x + \frac{1}{t}y\Big) \le \frac{1}{t}\rho(x) + \Big(1 - \frac{1}{t}\Big)\rho\Big(\frac{1}{t-1}y\Big) \le \frac{1}{t}\rho(x) + \frac{1}{t}\rho(y)$$

for all $x, y \in X$, because $2 \le t$. □

Theorem 3.2. *Let ρ be a modular on X, V a linear space, X_ρ a ρ-complete modular space and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $a \ge 2$, $k \ge a^2$, and $b^2 > a^2$. Let $\phi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\phi(ax, ay) \le a^2 L\phi(x, y) \tag{3.1}$$

for all $x, y \in V$ and some L with $0 < L < 1$ and

$$\rho(rA_f(x, y)) \le \rho(rB_f(x, y)) + |r|\phi(x, y) \tag{3.2}$$

for all $x, y \in V$ and all real number r. Then there exists a unique quadratic mapping $Q : V \longrightarrow X_\rho$ such that

$$\rho\Big(Q(x) - \frac{1}{a^2}f(x)\Big) \le \frac{1}{ka^4(1 - L)}\phi(x, 0) \tag{3.3}$$

for all $x \in V$.

Proof. By Lemma 1.4, $\widetilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $Tg(x) = \dfrac{1}{a^2}g(ax)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \le c$ for some nonnegative real number c. Then by (3.1), we have

$$\rho(Tg(x) - Th(x)) \le \frac{1}{a^2}\rho(g(ax) - h(ax)) \le Lc\phi(x, 0)$$

for all $x \in V$ and so $\widetilde{\rho}(Tg - Th) \le L\widetilde{\rho}(g - h)$. Hence T is a $\widetilde{\rho}$-contraction. Since $2k > 1$, by (3.2), for $r = 1$, we get

$$\rho\Big(f(ax) - a^2 f(x)\Big) \le \frac{1}{2k}\rho(2kf(ax) - 2ka^2 f(x)) \le \frac{1}{2k}\phi(x, 0) \tag{3.4}$$

for all $x \in X$. Since $a \ge 2$, by (3.4),

$$\rho(Tf(x)-f(x)) = \rho\Big(\frac{1}{a^2}f(ax)-f(x)\Big) \le \frac{1}{a^2}\rho(f(ax)-a^2 f(x)) \le \frac{1}{2ka^2}\phi(x, 0) \tag{3.5}$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{a^2}f$. By Lemma 3.1 and (3.5), for any nonnegative integer n, we obtain

$$\begin{aligned}\rho\Big(\frac{1}{a}T^n f(x) - \frac{1}{a}f(x)\Big) &\le \frac{1}{a}\rho\Big(T^n f(x) - \frac{1}{a^2}f(ax)\Big) + \frac{1}{a}\rho\Big(\frac{1}{a^2}f(ax) - f(x)\Big)\\ &\le \frac{1}{a^2}\rho\Big(\frac{1}{a}T^{n-1}f(ax) - \frac{1}{a}f(ax)\Big) + \frac{1}{2ka^3}\phi(x, 0)\end{aligned}$$

for all $x \in V$ and by (3.1), we have

$$\rho\Big(\frac{1}{a}T^n f(x) - \frac{1}{a}f(x)\Big) \le \frac{1}{2ka^3}\sum_{i=0}^{n-1} L^i\phi(x, 0) \le \frac{1}{2ka^3(1 - L)}\phi(x, 0) \tag{3.6}$$

for all $x \in V$ and all $n \in \mathbb{N}$. By Lemma 3.1 and (3.6), we get

(3.7)
$$\rho\Big(\frac{1}{a^2}T^n f(x) - \frac{1}{a^2}T^m f(x)\Big) = \rho\Big(\frac{1}{a^2}T^n f(x) - \frac{1}{a^2}T^m f(x)\Big) \le \frac{1}{ka^4(1 - L)}\phi(x, 0)$$

for all $x \in V$ and all nonnegative integers n, m. Hence T has a bounded orbit at $\frac{1}{a^2}f$.

By Lemma 1.3, there is a $Q \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n \frac{1}{a^2}f\}$ is $\widetilde{\rho}$-convergent to Q. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$0 \le \widetilde{\rho}(TQ - Q) \le \liminf_{n\to\infty} \widetilde{\rho}\Big(TQ - T^{n+1}\frac{1}{a^2}f\Big) \le \liminf_{n\to\infty} L\widetilde{\rho}\Big(Q - T^n\frac{1}{a^2}f\Big) = 0$$

and hence Q is a fixed point of T in $\mathbb{M}_{\widetilde{\rho}}$. Since $a \ge 2$, there is a a natural number t with $k < a^{t-6}$ and $2kb^2 < a^{t-3}$ and hence we have

$$\begin{aligned}&\rho\Big(\frac{1}{a^t}\Big[A_Q(x, y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big)\\ &\le \frac{k}{a^t}\rho\Big(Q(ax + by) - \frac{1}{a^{2n+2}}f(a^{n+1}x + a^n by)\Big) + \frac{2k}{a^{t-2}}\rho\Big(Q(x) - \frac{1}{a^{2n+2}}f(a^n x)\Big)\\ &+ \frac{k}{a^t}\rho\Big(Q(ax - by) - \frac{1}{a^{2n+2}}f(a^{n+1}x - a^n by)\Big) + \frac{2kb^2}{a^t}\rho\Big(Q(y) - \frac{1}{a^{2n+2}}f(a^n y)\Big)\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \rho\Big(\frac{1}{a^t}\Big[A_Q(x, y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big) = 0 \tag{3.8}$$

for all $x, y \in V$, because $\{\frac{1}{a^{2n+2}} f\}$ is $\widetilde{\rho}$-convergent to Q. Similarly, we have

$$\lim_{n\to\infty} \rho\Big(\frac{1}{a^t}\Big[B_Q(x,y) - \frac{1}{a^{2n+2}} B_f(a^n x, a^n y)\Big]\Big) = 0 \tag{3.9}$$

for all $x, y \in V$. Since $a^2 \le k$, by (3.2), we have

$$\begin{aligned}
&\rho\Big(\frac{1}{ka^{t+1}} A_Q(x,y)\Big)\\
&\le \frac{1}{a}\rho\Big(\frac{1}{ka^t}\Big[A_Q(x,y) - \frac{1}{a^{2n+2}} A_f(a^n x, a^n y)\Big]\Big) + \frac{1}{a}\rho\Big(\frac{1}{a^{2n+t+4}} A_f(a^n x, a^n y)\Big)\\
&\le \frac{1}{a^3}\rho\Big(\frac{1}{a^t}\Big[A_Q(x,y) - \frac{1}{a^{2n+2}} A_f(a^n x, a^n y)\Big]\Big) + \frac{1}{a}\rho\Big(\frac{1}{a^{2n+t+4}} B_f(a^n x, b^n y)\Big)\\
&\quad + \frac{1}{a^{2n+t+5}}\phi(a^n x, a^n y)\\
&\le \frac{1}{a^3}\rho\Big(\frac{1}{a^t}\Big[A_Q(x,y) - \frac{1}{a^{2n+2}} A_f(a^n x, a^n y)\Big]\Big) + \frac{1}{a^2}\rho\Big(\frac{1}{a^{t+1}} B_Q(x,y)\Big)\\
&+ \frac{1}{a^3}\rho\Big(\frac{1}{a^t}\Big[\frac{1}{a^{2n+2}} B_f(a^n x, a^n y) - B_Q(x,y)\Big]\Big) + \frac{L^n}{a^{t+5}}\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, by (3.8) and (3.9), we get

$$\rho\Big(\frac{1}{ka^{t+1}} A_Q(x,y)\Big) \le \rho\Big(\frac{1}{a^{t+1}} B_Q(x,y)\Big)$$

for all $x, y \in V$. By Corollary 2.3, Q is a quadratic mapping. Moreover, since ρ is lower semi-continuous, by (3.7), we have (3.3). □

Corollary 3.3. *Let X and Y be normed spaces and ϵ, θ, and p real numbers with $\epsilon \ge 0$, $\theta \ge 0$, and $0 < p < 1$. Suppose that $a \ge 2$, $k \ge a^2$, and $b^2 > a^2$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|A_f(x,y)\| \le \|B_f(x,y)\| + \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

for all $x, y \in X$. Then there is a quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{k(a^2 - a^{2p})}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

Proof. Let $\rho(z) = \|z\|$ for all $y \in Y$ and $\phi(x,y) = \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$ for all $x, y \in V$. Then ρ is a convex modular on a normed space Y, $Y = Y_\rho$, and $\phi(ax, ay) \le a^{2p}\phi(x,y)$ for all $x, y \in V$. By Theorem 3.2, we have the results. □

Using Example 1.1, we get the following example.

Example 3.4. Let θ, and p be real numbers with $\theta \ge 0$ and $0 < p < 1$. Suppose that $a \ge 2$, $k \ge a^2$, and $b^2 > a^2$. Let ζ be an Orlicz function and L^ζ the Orlicz space. Let $f : V \longrightarrow L^\zeta$ be a mapping such that $f(0) = 0$ and

$$\int_\Omega \zeta(rA_f(x,y))d\mu \le \int_\Omega \zeta(rB_f(x,y))d\mu + |r|\theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

for all $x, y \in X$ and all real number r. Then there is a quadratic mapping $Q : X \longrightarrow Y$ such that

$$\int_\Omega \zeta\Big(\Big|Q(x) - \frac{1}{a^2} f(x)\Big|\Big)d\mu \le \frac{\theta}{ka^2(a^2 - a^{2p})}\|x\|^{2p}$$

for all $x \in X$.

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