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Numerical study of the space fractional Burger's equation by using Lax-Friedrichs-implicit scheme

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Abstract

This paper deals with the numerical solution of space fractional Burger's equation using the implicit finite difference scheme and Lax-Friedrichs-implicit finite difference scheme respectively. The Riemann-Liouville based fractional derivative (non-integer order) is fitted for the diffusion term of fractional order $1.0 < \alpha \leq 2.0$. The Mathematical induction is used to estimate a stability of both the implicit and Lax-Friedrichs-implicit schemes. The study shows that the implicit based scheme is stable and the results are good in agreement with the exact solution. Finally, the significance of space fractional order with respect to the solution is discussed. It is noted that the solution of space fractional Burger's equation get affected by changing the space fractional order.

Key words: Lax-Friedrichs, implicit scheme, fractional calculus, finite difference method

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1 Introduction

Fractional Calculus plays an important role in various fields of science and engineering. Examples include ground water flow modeling, electric circuit design, quantum mechanics, optics, plasma model, dengue fever transmission dynamics and atmospheric CO_2 dynamics model [1, 2, 3, 4, 5]. Due to its wide applications, solving techniques of those fractional equations are extensively improved by the researchers. For instance, Goswami et al. [6] used Homotopy perturbation Sumudu transform for solving time-fractional regularized long wave equations. Later, they used the techniques to find the solutions for the time fractional Schrdinger equations and fractional equal width equations (describes the

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hydro-magnetic waves) [7, 8]. Also, Goswami et al. [9] made a mixed approach of Homotopy perturbation and Laplace transform to solve the fifth order KdV equations in order to illustrate the plasma's magneto-acoustic waves. Recently, Hashmi et al. [10] used B-spline method to solve the fractional telegraph equation and quoted that the scheme is efficient. In numerical methods there are numerous methods including finite difference method (FDM), finite element method, finite volume method etc. Out of this methods FDM is a pioneering tool used among the investigators. In the present investigation, we establish two schemes namely the implicit FDM and Lax-Friedrichs implicit FDM for the space fractional Burger's equation (SFBE).

Fractional calculus application gives a real system better than integer-order. The Burger's equations arises in various domain such as fluid and gas dynamics, theory of shock waves, traffic flow, etc [11, 12, 13, 14]. Many researchers have applied various analytical techniques, numerical algorithms/schemes for extracting the solution for the Burger's equation. The exact solution and explicit FDM solutions for the 1-D Burger's equation was surveyed by Kutluay et al. [15]. Aksan and Ozdes [16] constructed variational method for solving the Burger's equation. Inan and Bahadir [17] converted non-linear Burger's equation into linear using Hopf-Cole transformation and obtained Numerical solution (NS) using explicit exponential FDM. Pandey et al. [18] coupled Hopf-Cole transformation and Douglas FDM to get the NS with accuracy of second order in time and fourth order in space.

Zhang et al.[19] used the implicit FDM to solve the fractional convection-diffusion equation. It is found that the NS is unconditionally stable. Sousa [20] obtained the NS for the fractional advection diffusion equation using explicit-central difference FDM, explicit-upwind FDM and Lax-Wendroff FDM. The study consider Riemann-Liouville fractional derivative for space fractional and Caputo fractional derivative for the time derivative. The result shows that all the explicit FDM schemes are stable under restricted conditions. Later, Sousa [21] presented the explicit-Lax-Wendroff method for the Riemann-Liouville derivative based space fractional advection diffusion equation. The study illustrates that the scheme is second order accurate and conditionally stable. Bekir and Gnerb [22] and Das et al. [23] used (G'/G) expansion method to solve the modified Riemann-Liouville derivative based fractional Burger's equation. Esen and Tasbozan[24] solved the time fractional Burger's equation by applying the B-spline quadratic Galerkin method. Moreover, Esen and Tasbozan[25] used finite element method based cubic B-spline for the time fractional Burger's equation. They also compared the NS with the various exact solutions (ES) and found that the scheme is stable and accurate. Rawashdeh [26] proposed a new scheme named the fractional reduced differential transform to solve the TFBE. It is noted that the proposed scheme is accurate and good comparable with the ES. Yokus [27] studied the FDM based NS with respect to the fractional derivatives such as Caputo, shifted Grunwald and Riemann-Liouville and obtained the solutions using the software Mathematica 11. Saad and Eman[28] have applied the variational iteration method (VIM) for the Riemann-Liouville based fractional Burger's equation and compared the results with the ES.

In this work, we propose a numerical solution based on implicit FDM scheme and Lax-Friedrichs FDM scheme to solve a non-linear SFBE. Generally, the Lax-Friedrichs method is used for achieving the solutions for a hyperbolic based PDE's [29]. In general, an implicit scheme is the most well-known schemes for approximating the PDEs. This paper presents an approximation based on Lax-Friedrichs-implicit FDM to non-linear SFBE with appropriate initial/boundary conditions. The stability of a proposed scheme is analysed along with the numerical results.

2 Mathematical equation

Time fractional Burgers' equation was discussed in the articles [16, 24, 25, 26]. Following their study, we consider the non-linear SFBE as,

$$\frac{\partial u(x,t)}{\partial t}+u\frac{\partial u(x,t)}{\partial x}=\mu\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}},(x,t)\in[a,b]\times(0,T_{max}] \tag{2.1}$$

included with initial values

$$u(x,0)=u_0(x) \tag{2.2}$$

and respective boundary values

$$u(0,t)=h_1(t);u(1,t)=h_2(t),t\in[0,T] \tag{2.3}$$

where $\mu>0$ is kinematic viscosity, $u_0(x)$, $h_1(t)$ and $h_2(t)$ are specified boundaries. $u(x)$ is unknown functional.

To solve the SFBE in this work, let us consider the Riemann-Liouville fractional derivatives [20, 21, 30].

$$({}_0D_x^{\alpha})u(x,t)=\frac{1}{\Gamma(r-\alpha)}\frac{d^r}{dx^r}\int_L^x\frac{u(t)}{(x-t)^{\alpha-r+1}}dt,\alpha>0 \tag{2.4}$$

where $\Gamma\left(.\right)$ is the Gamma function.

For space fractional derivative $({}_0D_x^{\alpha})u\left(x,t\right)$, we taken the Grunwald and shifted-Grunwald formula at level t_{n+1} [31].

$$\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=\frac{1}{h^{\alpha}}\sum_{j=0}^{i+1}g_j^{\alpha}\ u_{i-j+1}^{k+1}+O\left(h\right) \tag{2.5}$$

where $g_j^{\alpha}=\dfrac{\alpha\left(\alpha-1\right)....(\alpha-k+1)}{j!}$,

We can express, $\qquad g_0^{\alpha}=1,\ldots,\ g_j^{\alpha}=\left(1-\dfrac{\alpha}{j}\right)g_{j-1}^{\alpha},\ j=1,2,3,\ldots.$

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2.1 Implicit scheme

The implicit scheme is one of the more accurate scheme for a non-linear Burger's equation [32]. Here, we consider a same for SFBE due to its stability than the explicit scheme [21, 31, 33, 34].

Let $u(x_i, t_k)$ is denoted as u_i^k. Define, $t_k = k\tau, \quad k = 0, 1, 2, ..., n;\ x_i = ih, \quad i = 0, 1, 2, ..., m$. Here, $h = L/m$ is the step size on space and $\tau = T/n$ is the step size on time respectively. Now,let us consider the nonlinear term, $u^{k+1}u_x^{k+1}$ by denoting it on Taylor expansion using the explicit time layer. We approximate the equation (2.1) by using an implicit FDM and approximated Riemann-Liouville derivatives equation (2.5) in space fractional viscous terms as follows.

$$\begin{aligned}\frac{(u_i^{k+1} - u_i^k)}{\tau} + \frac{u_i^k}{2}\left(\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h}\right) + \frac{u_i^{k+1}}{2}\left(\frac{u_{i+1}^k - u_{i-1}^k}{2h}\right)& \\ = \frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1}&\end{aligned} \tag{2.6}$$

$$\begin{aligned}\left(u_i^{k+1} - u_i^k\right) + \tau\frac{u_i^k}{2}\left(\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h}\right) + \tau u_i^{k+1}\left(\frac{u_{i+1}^k - u_{i-1}^k}{4h}\right)& \\ = \tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1} g_j^\alpha u_{i-j+1}^{k+1}&\end{aligned} \tag{2.7}$$

$$\begin{aligned}-\tau\left(\frac{u_i^k}{4h} + \frac{\mu}{h^\alpha}g_2^\alpha\right)u_{i-1}^{k+1} + \left(1 + \frac{\tau\left(u_{i+1}^k - u_{i-1}^k\right)}{4h} - \frac{\mu\tau}{h^\alpha}g_1^\alpha\right)u_i^{k+1}& \\ +\tau\left(\frac{u_i^k}{4h} - \frac{\mu}{h^\alpha}g_0^\alpha\right)u_{i+1}^{k+1} - \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} = u_i^k&\end{aligned} \tag{2.8}$$

When $k = 0$,

$$\begin{aligned}-\tau\left(\frac{u_i^0}{4h} + \frac{\mu}{h^\alpha}g_2^\alpha\right)u_{i-1}^1 + \left(1 + \frac{\tau\left(u_{i+1}^0 - u_{i-1}^0\right)}{4h} - \frac{\mu\tau}{h^\alpha}g_1^\alpha\right)u_i^1& \\ +\tau\left(\frac{u_i^0}{4h} - \frac{\mu}{h^\alpha}g_0^\alpha\right)u_{i+1}^1 - \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^1 = u_i^0&\end{aligned} \tag{2.9}$$

When $k \geq 1$,

$$\begin{aligned}-\tau\left(\frac{u_i^k}{4h} + \frac{\mu}{h^\alpha}g_2^\alpha\right)u_{i-1}^{k+1} + \left(1 + \frac{\tau\left(u_{i+1}^k - u_{i-1}^k\right)}{4h} - \frac{\mu\tau}{h^\alpha}g_1^\alpha\right)u_i^{k+1}& \\ +\tau\left(\frac{u_i^k}{4h} - \frac{\mu}{h^\alpha}g_0^\alpha\right)u_{i+1}^{k+1} - \tau\frac{\mu}{h^\alpha}\sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{k+1} = u_i^k&\end{aligned} \tag{2.10}$$

Rewriting above equation, we get

$$a_i^k u_{i-1}^{k+1} + b_i^k u_i^{k+1} + c_i^k u_{i+1}^{k+1} = u_i^k + d_i^k \tag{2.11}$$

where, $a_i^k = -\tau\left(\frac{u_i^k}{4h} + \frac{\mu}{h^\alpha} g_2^\alpha\right)$, $b_i^k = \left(1 + \frac{\tau\left(u_{i+1}^k - u_{i-1}^k\right)}{4h} - \frac{\mu\tau}{h^\alpha} g_1^\alpha\right)$, $c_i^k = \tau\left(\frac{u_i^k}{4h} - \frac{\mu}{h^\alpha} g_0^\alpha\right)$, $d_i^k = \tau \frac{\mu}{h^\alpha} \sum_{j=3}^{i+1} g_j^\alpha u_{i-j+1}^{k+1}$

The boundary/initial conditions are,

$$u_i^0 = u\left(ih\right), \quad u_0^k = h_1\left(t\right), \quad u_m^k = h_2\left(t\right)$$

where $k = 0, 1, 2, ..., n,$ $i = 0, 1, 2, ..., m.$ The truncation error is $O\left(\tau^2, h^2\right)$.

2.1.1 Stability analysis - implicit FDM

Let us investigate the stability of the numerical implicit scheme (2.8) by using von-Neumann analysis. Let U_j^k is the ES of $u(x,t)$ at the point (x_j, t_k). Define

$$e_j^k = U_j^k - u_j^k \tag{2.12}$$

Then, by substituting Equation (2.12) into Equation (2.11),we have

$$a_i^k e_{i-1}^{k+1} + b_i^k e_i^{k+1} + c_i^k e_{i+1}^{k+1} = e_i^k + d_i^k \tag{2.13}$$

We put $e_i^k = \rho^k\ e^{ipjh}$ $(i = \sqrt{-1})$, in equation (2.6) and p is the wave number.

$$\rho^{k+1}\left[\tau\left(\frac{\rho^k\ e^{ipjh}}{2h}\right) isin\left(ph\right) + \left(1 + \frac{\tau\left(isin\left(ph\right)\rho^k\right)}{2h}\right) - \tau\frac{\mu}{h^\alpha}\sum_{r=0}^{i+1} g_j^\alpha\ e^{ip(1-r)h}\right] = \rho^k \tag{2.14}$$

$$\frac{\rho^{k+1}}{\rho^k} = \frac{1}{\left[\tau\left(\frac{\rho^k\ e^{ipjh}}{2h}\right) isin\left(ph\right) + \left(1 + \frac{\tau\left(isin\left(ph\right)\rho^k\ e^{ipjh}\right)}{2h}\right) - \tau\frac{\mu}{h^\alpha}\sum_{r=0}^{i+1} g_j^\alpha\ e^{ip(1-r)h}\right]} \leq 1 \tag{2.15}$$

It is obvious that the above scheme is unconditionally stable.

2.2 Lax-Friedrichs Scheme

As a result of its application to a nonlinear space fractional problem and the dissipative nature of the solution, the Lax-Friedrichs scheme is considered to be a classic first-order method. The Lax-Friedrichs scheme of the fractional equation (2.1) is approximated by as below:

$$\left(\frac{u_i^{k+1}-\frac{1}{2}\left(u_{i-1}^k+u_{i+1}^k\right)}{\tau}\right)+\frac{u_i^k}{2}\left(\frac{u_{i+1}^{k+1}-u_{i-1}^{k+1}}{2h}\right)+\frac{u_i^{k+1}}{2}\left(\frac{u_{i+1}^k-u_{i-1}^k}{2h}\right)=\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1}g_j^\alpha u_{i-j+1}^{k+1} \tag{2.16}$$

$$-\tau\left(\frac{u^k}{4h}\right)u_{i-1}^{k+1}+\left(1+\frac{\tau(u_{i+1}^k-u_{i-1}^k)}{4h}\right)u_i^{k+1}+\tau\left(\frac{u^k}{4h}\right)u_{i+1}^{k+1}-\tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1}g_j^\alpha u_{i-j+1}^{k+1}=\frac{1}{2}\left(u_{i-1}^k+u_{i+1}^k\right) \tag{2.17}$$

When $k=0$

$$-\tau\left(\frac{u^0}{4h}\right)u_{i-1}^1+\left(1+\frac{\tau(u_{i+1}^0-u_{i-1}^0)}{4h}\right)u_i^1+\tau\left(\frac{u^0}{4h}\right)u_{i+1}^1-\tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1}g_j^\alpha u_{i-j+1}^1=\frac{1}{2}(u_{i-1}^0+u_{i+1}^0) \tag{2.18}$$

When $k\geq 1$

$$-\tau\left(\frac{u^k}{4h}\right)u_{i-1}^{k+1}+\left(1+\frac{\tau(u_{i+1}^k-u_{i-1}^k)}{4h}\right)u_i^{k+1}+\tau\left(\frac{u^k}{4h}\right)u_{i+1}^{k+1}-\tau\frac{\mu}{h^\alpha}\sum_{j=0}^{i+1}g_j^\alpha u_{i-j+1}^{k+1}=\frac{1}{2}(u_{i-1}^k+u_{i+1}^k) \tag{2.19}$$

2.2.1 Stability Analysis - Lax-Friedrichs-implicit FDM

Let us consider the von-Neumann based method in preparation for estimating the stability of Lax-Friedrichs implicit scheme for SFBE. Let U_i^k be the approximate solution of fractional schemes (2.17).

$$e_i^k=U_i^k-u_i^k \tag{2.20}$$

Define, $e_i^k = \rho^k e^{ipjh} (i = \sqrt{(-1)}$ in Eq. (2.17), We get

$$\rho^{k+1} \left[\tau \left(\frac{\rho^k e^{ipjh}}{2h} \right) isin\,(ph) + \left(1 + \frac{\tau\,(\,isin\,(ph)\,e^{ipjh}\;\rho^k)}{2h} \right) - \tau \frac{\mu}{h^\alpha} \sum_{r=0}^{i+1} g_j^\alpha\; e^{ip(1-r)h} \right] = \left(\rho^k\; cos(ph) \right) \tag{2.21}$$

$$\frac{\rho^{k+1}}{\rho^k} = \frac{cos(ph)}{\left[\left(1 + \frac{\tau\,(\,isin\,(ph)\,e^{ipjh}\;\rho^k)}{2h} \right) + \tau \left(\frac{\rho^k e^{ipjh}}{2h} \right) isin\,(ph)\; - \tau \frac{\mu}{h^\alpha} \sum_{r=0}^{i+1} g_j^\alpha\; e^{ip(1-r)h} \right]} \tag{2.22}$$

We know, the value of the $sin\,(ph)\;\; and\; cos\,(ph)\;\; \leq 1$

3 Numerical Results

The verification of NS and accuracy of the schemes (implicit FDM and Lax-Friedrichs-implicit FDM) are illustrated in this section. In addition, the behavior of the solution with respect to change in the parameters are considered. This types of Burger's equation are used in predicting the important real world applications such as fluid flow, contaminant flow, boundary layer flow, aquifer flow, etc.

The accuracy of the FDM based schemes are measured using the L_∞ error norm, which is defined below:

$$L_\infty = \parallel U_k - u_N \parallel_\infty = Max_j |U_k - (u_N)_j| \tag{3.1}$$

where U_k and u_N denotes the ES and NS respectively at the node points x_k, for some fixed time.

3.1 Example 1

Consider the space fractional Burger's equation with source term to find error values as follows:

$$\frac{\partial u(x,t)}{\partial t} + u \frac{\partial u(x,t)}{\partial x} = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t), (x,t) \in [a,b] \times (0, T_{max}] \tag{3.2}$$

with initial and boundary conditions as

$$u\,(x,0) = x; \qquad and \quad u\,(0,t) = 0; \qquad u\,(1,t) = \frac{1}{1+t}$$

The exact solution [27]

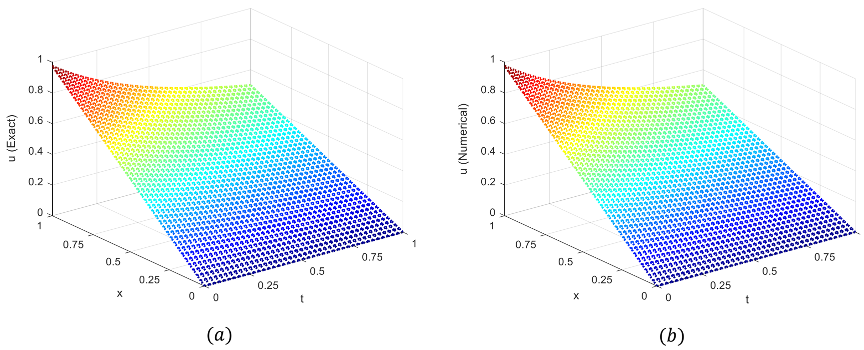

Figure 1: Comparison of (a). ES and (b). NS

$$u\left(x,t\right)=\frac{x}{1+t} \tag{3.3}$$

and the respective source term is,

$$f\left(x,t\right)=-\frac{1}{1+t}.\frac{1}{\Gamma\left(2-\alpha\right)}\mu x^{1-\alpha} \tag{3.4}$$

Table 1: Comparison the Maximum errors (L_∞) between ES and NS

τ	α	implicit FDM	Lax-Friedrichs FDM
1/100	1.9	$6.68689189E-03$	$4.27211449E-03$
1/100	1.7	$1.03293294E-02$	$9.42116044E-03$
1/100	1.5	$1.12734595E-02$	$1.07855788E-02$
1/100	1.3	$1.18441200E-02$	$1.14968475E-02$
1/100	1.1	$1.22581907E-02$	$1.18474280E-02$
1/1000	1.9	$3.30544871E-03$	$1.65415841E-03$
1/1000	1.7	$9.85641548E-03$	$7.98325112E-03$
1/1000	1.5	$9.89541454E-03$	$9.75634282E-03$
1/1000	1.3	$9.90254650E-03$	$9.76487132E-03$
1/1000	1.1	$9.91051481E-03$	$9.80037527E-03$

The verification of NS for the SFBE (2.18) with the ES is illustrated in the Fig. 1. The comparison is done against the time, $t=0$ to 1 and space, $x=0$ to 1. Both the NS and ES are good in comparable. Also, the Table. 3.1 shows the L_2 between the ES and NS. It is found that the Lax-Friedrichs-implicit FDM has lesser L_2 than the implicit FDM for every α.

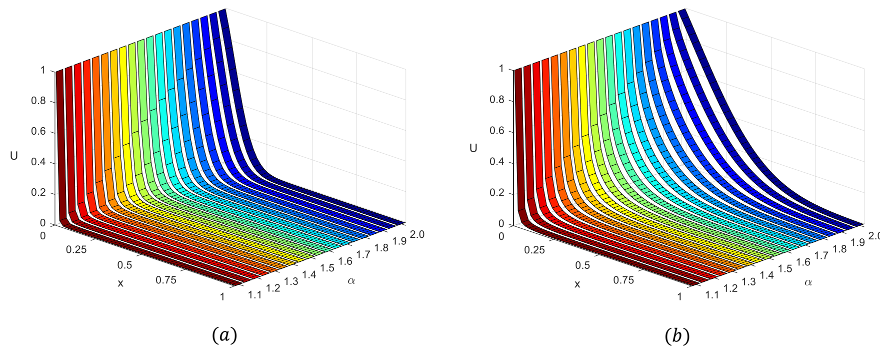

Figure 2: variation of U at (a) $\mu = 0.1$ (b) $\mu = 1.0$

3.2 Example 2

Also, consider the SFBE without source term to find the characteristics of NS as

$$\frac{\partial U(x,t)}{\partial t} + u\frac{\partial U(x,t)}{\partial x} = \mu\frac{\partial^{\alpha} U(x,t)}{\partial x^{\alpha}} \tag{3.5}$$

with initial and boundary conditions as

$$U\left(x,0\right)=0; \qquad U\left(0,t\right)=1; \qquad U\left(1,t\right)=0$$

Figure 2 shows the variation of U with respect to the space fractional parameter (α) and space coordinates (x) at kinematic viscosity $\mu = 0.1$ and 1.0 respectively. It is noted that, by increasing the parameter α, U decreases its intensity and travelling distance along the space.

4 Conclusion

The NS of SFBE has been evaluated by using implicit and Lax-Friedrichs-implicit FDM respectively. It is noted that both the implicit scheme is unconditionally stable and are good in agreement with the ES. It is found that L_2 of the Lax-Friedrichs-implicit is lesser than the implicit FDM. Also, it is found that the variation in space fractional order strongly affects the flow characteristics.

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