J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 30, NO.2, 2022, COPYRIGHT 2022 EUDOXUS PRESS, LLC

Solving System of Boundary Value Problems using Non polynomial Spline Methods Based on Off-step Mesh

Sucheta Nayak*, Arshad Khan and R. K. Mohanty

January 1, 2022

Abstract

We present two non polynomial spline methods based on quasi-variable mesh using off-step points to solve the system of boundary value problems which are nonlinear. We also discuss how the methods handle the presence of singularity. The proposed methods has been shown second and third-order convergent for a model linear problem. The methods are implemented on existing problems which are linear, non linear as well as singular. The obtained numerical results approximate the exact solutions very well and validate the theoretical findings.

Key words: Off-step, non polynomial, quasi-variable mesh, singular, nonlinear, system.
Mathematics Subject Classification(2010): 65L10.

1 Introduction

In this paper, we seek solution for the following system of M boundary value problems(BVPs) which are non linear as well as singular.

$$\frac{d^2y^i}{dx^2} = f^i(x, y^1, ..., y^i, ..., y^M, \frac{dy^1}{dx}, ..., \frac{dy^i}{dx}, ..., \frac{dy^M}{dx}), \tag{1.1}$$

$$y^i(0) = a_i, y^i(1) = b_i, \quad \text{where} \quad a_i, b_i \in R, \ i = 1(1)M. \tag{1.2}$$

We consider $-\infty < y^i, \frac{dy^i}{dx} < \infty$ and the conditions such that f^i is continuous and its partial derivatives w.r.t. y^j and $\frac{dy^i}{dx}$ exist, continuous and are positive. Also partial derivative w.r.t. $\frac{dy^i}{dx}$ is bounded by some $K > 0, \ j, i = 1(1)M$, to ensure the existence [13] of a unique solution $(1.1) - (1.2)$.

*Corresponding author: Department of Mathematics, Lady Shri Ram College for Women, University of Delhi, New Delhi-24, India suchetanayak@ lsr.edu.in

1

Such systems like the fourth order Euler differential equations [5], coupled Navier stokes in fluid dynamics and Maxwell's equations of electromagnetism[9], system of differential equations[28], fourth order non linear differential equations [27] simulates many real world problems. A few more examples are as follows:
(i)In Plate deflection theory

$$(-1)^n y^{(2n)}(x) = f(x, y(x)), n \in N$$
$$y^{(2i)}(a) = A_{2i}, y^{(2i)}(b) = B_{2i}, i \in [0, k-1]$$

(ii)Three box cars on a level track connected by springs is modelled as follows:

$$mx_1'' = -sx_1 + sx_2,$$
$$mx_2'' = sx_1 - 2sx_2 + sx_3,$$
$$mx_3'' = -sx_3,$$

where m, x_1, x_2, x_3 and s are masses,positions of the boxcars and Hooke's constant.

(iii)A horizontal earthquake wave F affects every floor of a building. If there are three floors, then equations for the floor is modelled as follows:

$$M_1x_1'' = -(r_1 + r_2)x_1 + r_2x_2,$$
$$M_2x_2'' = r_2x_1 - (r_2 + r_3)x_2 + r_3x_3,$$
$$M_3x_3'' = r_3x_2 - (r_3 + r_4)x_3,$$

where M_i, x_1, x_2, x_3 and r_i are point masses of each floor, location of masses and Hooke's constant.

Such systems of BVPs comprising first or second order BVPs not only models many real life problems but are also instrumental in solving many higher order problems by decomposing them. Authors like Aftabizadeh[1], Agarwal[2], Regan[24] have developed theories related to existence and uniqueness of solutions for these BVPs. But, for our work, we focus on system of second order BVPs which are non linear as well singular in nature. These problems have extensive application and has been the cause of interest for many authors. Many efficient numerical methods have been developed to solve second order BVPs and 'Splines' have been very instrumental for solving such problems. Mohanty et.al.([15],[16], [18], [21]) developed AGE iterative methods. In these methods, using Taylor's theorem derivatives are approximated and accordingly a finite difference scheme was developed. Then the resultant system solved by splitting the coefficient matrix into sum of three matrices. Also Mohanty et.al.([17],[19],[20],[22]) derived polynomial and non polynomial spline methods based on uniform and variable mesh to solve class of problems ranging from linear, nonlinear, singular and singularly perturbed BVP. A third order cubic spline method based on non uniform mesh was developed by Kadalbajoo et. al[12] to solve singularly perturbed BVPs. BVPs of eighth order were solved by Akram and Rehman[4] using kernel space method. Eighth and sixth order BVPs were solved by Siddiqi and Akram ([30], [31]) using non-polynomial and

2

septic spline. Jha and Bieniasz[11] developed a scheme based on geometric mesh to solve sixth order differential equation by converting it into system of second order differential equations. Infact, very recently, apart from the schemes based on classical finite differences some other kinds of methods were also developed. Bhrawy et. al.[6] developed collocation method based on Jacobi polynomials and solved nonlinear second-order initial value problems. Dwivedi and Singh [7] developed collocation method based on Fibbonacci polynomial to solve sub diffusion equations. Singh et.al.[32] developed finite difference scheme based on homotopy analysis transform technique to solve fractional non-linear coupled problem. Such considerable amount of work has motivated us to develop a numerical method to solve the higher order problem as well as system of linear and non linear singular BVPs.

In this paper, generalized non polynomial spline schemes have been developed which are based on off-step points using quasi-variable mesh. We use a second order BVP to derive the methods. As per the methods developed, we decompose the higher order BVP into system of second order BVPs (1) alongwith modifying the boundary conditions. Also, we have solved singular BVPs. The off-step points used in the method allows us to overcome the singularity. Moreover, since we use the quasi-variable mesh the error gets uniformly distributed throughout the solution domain. Finally, as we use the boundary conditions in the scheme, we get a tri-diagonal matrix with block elements representing the system of equations to be solved.

We have solved seven problems and demonstrated the accuracy of the proposed methods. The BVPs considered in this paper have been solved by other methods as well. Twizell [29] used modified extrapolation method to solve fourth order linear BVPs, Akram and Siddiqi[3] used non polynomial spline method which is second order convergent to solve linear sixth order BVPs. Khan and Khandelwal[14]and Sakai and Usmani[25] used splines to solve nonlinear fourth and sixth order BVPs.

2 Method Formulation

We use a non linear BVP of second order and derive the method in scalar form:

$$y'' = f(x, y, y'), \text{ subject to } y(0) = a, y(1) = b. \tag{2.1}$$

Now, we divide the solution region [0,1] into $N + 1$ points such as $x_j, j = 0(1)N$ with mesh size h_j such that $x_j = x_{j-1} + h_j$, $\frac{h_{j+1}}{h_j} = \sigma_j, j = 1(1)N - 1$ where the σ_j is the mesh ratio. When mesh ratio is one, the quasi-variable mesh converts to a uniform mesh with width, say h. Now, we choose $\sigma_j = \sigma$ a constant $\forall j$ without loss of generality. Also, let the exact solution of (2.1) be $y(x_j)$ or y_j at the grid points x_j. Now, we define the the following non polynomial spline function:

$$S_j(x) = d_j \sin(kx - kx_j) + c_j \cos(kx - kx_j) + b_j(x - x_j) + a_j,\ x_{j-1} \le x \le x_j. \tag{2.2}$$

Here, $S_j(x)$ has continuous second derivative in $[0,1]$ and $S_j(x), S_j'(x)$ interpolates at the mesh points x_j. Using the definition of the spline, we determine values for the unknowns a_j, b_j, c_j and d_j as:

$$a_j = y_j + \frac{f_j}{k^2}, \tag{2.3}$$

$$b_j = \frac{f_j - f_{j-\frac{1}{2}}}{k^2 h_j} - \frac{y_{j-1} - y_j}{h_j},$$

$$c_j = -\frac{f_j}{k^2},$$

$$\text{and} \quad d_j = -\frac{f_k \cos\theta_j - f_{j-\frac{1}{2}}}{k^2 \sin kh_j}. \tag{2.4}$$

Using the spline's first derivative continuity conditions, we get the non polynomial spline method based on off-step points as:

$$\sigma y_{j-1} - y_j(1+\sigma) + y_{j+1} = h_j^2(P\sigma f_{j-\frac{1}{2}} + Q\sigma f_j + R\sigma f_{j+\frac{1}{2}}) + T_j^3, \tag{2.5}$$

where

$$R = \frac{2kh_{j+1} - \sin kh_{j+1}}{2k^2 h_{j+1} \sin kh_{j+1}}, P = \frac{kh_j - \sin kh_j \cos kh_j}{kh_j \sin kh_j}, \tag{2.6}$$

$$Q = \frac{2(\sigma_j+1)(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j)) - 2\sigma kh_j \sin(kh_j\sigma + kh_j)}{(kh_j)^2(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j))} \tag{2.7}$$

Now, we also derive the consistency condition using (2.5) – (2.7) i.e.,

$$\tan(\frac{kh_j}{2}) + \tan(\frac{kh_{j+1}}{2}) = \frac{kh_j}{2} + \frac{kh_{j+1}}{2}. \tag{2.8}$$

We solve equation (2.8) for kh_j and consider the non-zero smallest positive root $kh_j = 8.98681891$. But, with this, the order of error term T_j^3 in (2.5) remains four. Now, we derive another off-step method using Taylor's expansion for $j = 1(1)N-1$ as

$$y_{j-1} - y_j(1+\sigma) + \sigma y_{j+1} = h_j^2\sigma(Af_{j-\frac{1}{2}} + Bf_{j+\frac{1}{2}}) + T_j^2, \tag{2.9}$$

$$\text{for } A = \frac{(2+\sigma)}{6}, \ B = \frac{(2\sigma+1)}{6}. \tag{2.10}$$

Now, the following approximations are defined at $x_j, j = 1(1)N-1$,

$$
\begin{aligned}
S_j &= (\sigma+1)\sigma, && (2.11)\\
\bar{y}_{j+\frac{1}{2}} &= \frac{y_j + y_{j+1}}{2}, && (2.12)\\
\bar{y}_{j-\frac{1}{2}} &= \frac{y_j + y_{j-1}}{2}, && (2.13)\\
\bar{y'}_{j+\frac{1}{2}} &= \frac{y_{j+1} - y_j}{h_j\sigma}, && (2.14)\\
\bar{y'}_{j-\frac{1}{2}} &= \frac{y_j - y_{j-1}}{h_j}, && (2.15)\\
\bar{y'}_j &= \frac{y_{j+1} - y_j(1-\sigma^2) - \sigma^2 y_{j-1}}{S_j h_j}, && (2.16)\\
\bar{f}_j &= f(x_j, y_j, \bar{y'}_j), && (2.17)\\
\bar{f}_{j-\frac{1}{2}} &= f(x_{j-\frac{1}{2}}, \bar{y}_{j-\frac{1}{2}}, \bar{y'}_{j-\frac{1}{2}}), && (2.18)\\
\bar{f}_{j+\frac{1}{2}} &= f(x_{j+\frac{1}{2}}, \bar{y}_{j+\frac{1}{2}}, \bar{y'}_{j+\frac{1}{2}}). && (2.19)
\end{aligned}
$$

Next, we define higher order approximation of y_j and y'_j to raise order of the error term T_j^3 in equation (2.5) :

$$
\begin{aligned}
\hat{y}_j &= y_j + h_j^2\delta(\bar{f}_{j-\frac{1}{2}} + \bar{f}_{j+\frac{1}{2}}), && (2.20)\\
\hat{y'}_j &= \bar{y'}_j - h_j\gamma(\bar{f}_{j-\frac{1}{2}} - \bar{f}_{j+\frac{1}{2}}), && (2.21)
\end{aligned}
$$

where γ, δ are unknowns. This gives us the modified $\bar{f}_j$ i.e.,

$$\hat{f}_j = f(x_j, \hat{y}_j, \hat{y'}_j). \qquad (2.22)$$

Now, expanding the approximations (2.12) – (2.22) we get the following:

$$
\begin{aligned}
P &= \frac{\sigma}{3} + O(k{h_j}^2), && (2.23)\\
R &= \frac{1}{3} + O(k{h_j}^2), && (2.24)\\
Q &= \frac{(\sigma+1)}{6} + O(k{h_j}^2), && (2.25)\\
\bar{y}_{j+\frac{1}{2}} &= y_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2}{8}y_j'' + O(h_j^3), && (2.26)\\
\bar{y}_{j-\frac{1}{2}} &= y_{j-\frac{1}{2}} + \frac{(h_j)^2}{8}y_j'' + O(h_j^3), && (2.27)\\
\bar{y'}_{j+\frac{1}{2}} &= {y'}_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2}{24}y_j''' + O(h_j^3), && (2.28)\\
\bar{y'}_{j-\frac{1}{2}} &= {y'}_{j-\frac{1}{2}} + \frac{(h_j)^2}{24}y_j''' + O(h_j^3), && (2.29)\\
\hat{y}_j &= y_j + \delta h_j^2(2y_j'') + O(h_j^3), \sigma \neq 1, && (2.30)\\
\hat{y'}_j &= {y'}_j + \frac{h_j^2 y_j'''}{6}((1+3\gamma)\sigma + 3\gamma) + O(h_j^3), && (2.31)\\
\bar{f}_{j+\frac{1}{2}} &= f_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2 y_j''}{8}\frac{\partial f}{\partial y_j} + \frac{(h_j\sigma)^2 y_j'''}{24}\frac{\partial f}{\partial y_j'} + O(h_j^3), && (2.32)\\
\bar{f}_{j-\frac{1}{2}} &= f_{j-\frac{1}{2}} + \frac{h_j^2 y_j''}{8}\frac{\partial f}{\partial y_j} + \frac{h_j^2 y_j'''}{24}\frac{\partial f}{\partial y_j'} + O(h_j^3), && (2.33)\\
\hat{f}_j &= f_j + 2h_j^2\delta y_j''\frac{\partial f}{\partial y_j} + \frac{h_j^2}{6}(\sigma + 3\gamma(1+\sigma))y_j'''\frac{\partial f}{\partial y_j'} + O(h_j^3). && (2.34)
\end{aligned}
$$

Thus, we develop the first method by discretizing the proposed BVP (2.1) based on the method (2.9) as:

$$
\sigma y_{j-1} - (1+\sigma)y_j + y_{j+1} = h_j^2(A\sigma\bar{f}_{j-\frac{1}{2}} + B\sigma\bar{f}_{j+\frac{1}{2}}) + T_j^2. \quad (2.35)
$$

In this method, we can show that for $\sigma \neq 1$, the order of truncation error T_j^2 is $O(h_j^4)$ using the approximations $(2.32) - (2.33)$. Also, if we use the off-step non-polynomial scheme(2.5) along with the approximation $(2.26) - (2.34)$, we get the second method as:

$$
\begin{aligned}
y_{j+1} - (1+\sigma)y_j + \sigma y_{j-1} &= h_j^2\sigma(P\bar{f}_{j-\frac{1}{2}} + Q\hat{f}_j + R\bar{f}_{j+\frac{1}{2}})\\
&- h_j^4[(\frac{R\sigma^2 + Q4((1+3\gamma)\sigma+3\gamma) + P}{24})y_j'''\frac{\partial f}{\partial y_j'}\\
&+ (\frac{R\sigma^2}{8} + 2Q\delta + \frac{P}{8})y_j''\frac{\partial f}{\partial y_j}] + T_j^3. \quad (2.36)
\end{aligned}
$$

The coefficients of order four of h_j is equated to zero to get the value of δ, γ so as to raise the order of local truncation error T_j^3. Thus, we get

6

$\gamma = -\frac{(R\sigma^2+P+4Q\sigma)}{12Q(1+\sigma)}, \delta = -\frac{(R+P\sigma^2)}{16Q}$. In case of uniform mesh, the local truncation error becomes of order six. We also ensure the necessary condition for convergence of the methods provided by Jain [10], that the coefficients A, B in method (2.35)and in method (2.36) P, Q and R are positive for $\sigma > 0$. Hence, both the proposed off-step three point discretization using the approximate solutions Y_j at x_j are as follows:

$$Y_{j+1} - (\sigma+1)Y_j + \sigma Y_{j-1} \quad = \quad h_j^2\sigma(A\bar{F}_{j-\frac{1}{2}} + B\bar{F}_{j+\frac{1}{2}}), \tag{2.37}$$

and

$$Y_{j+1} - (\sigma+1)Y_j + \sigma Y_{j-1} = h_j^2\sigma(R\bar{F}_{j+\frac{1}{2}} + Q\hat{F}_j + P\bar{F}_{j-\frac{1}{2}}). \tag{2.38}$$

3 Generalised Methods

We develop the generalized methods by using the following approximations and scalar methods developed in the last section, thus, solving $(1.1) - (1.2)$ we get,

$$S_j = (\sigma+1)\sigma, \tag{3.1}$$

$$\bar{Y}^i_{j+\frac{1}{2}} = \frac{Y^i_j + Y^i_{j+1}}{2}, \tag{3.2}$$

$$\bar{Y}^i_{j-\frac{1}{2}} = \frac{Y^i_j + Y^i_{j-1}}{2}, \tag{3.3}$$

$$\bar{Y'}^i_{j+\frac{1}{2}} = \frac{Y^i_j - Y^i_{j+1}}{h_j\sigma}, \tag{3.4}$$

$$\bar{Y'}^i_{j-\frac{1}{2}} = \frac{Y^i_j - Y^i_{j-1}}{h_j}, \tag{3.5}$$

$$\bar{Y'}^i_j = \frac{Y^i_{j+1} - (1-\sigma^2)Y^i_j - \sigma^2 Y^i_{j-1}}{S_j h_j}, \tag{3.6}$$

$$\bar{f}^i_j = f^i(x_j, Y_j, Y^{(1)}_j, Y^{(2)}_j, ..., Y^i_j, ..., Y^{(M)}_j, \bar{Y'}^{(1)}_j, \bar{Y'}^{(2)}_j, ..., \bar{Y'}^i_j, ..., \bar{Y'}^{(M)}_j), \quad (3.7)$$

$$\bar{f}^i_{j-\frac{1}{2}} = f^i(x_{j-\frac{1}{2}}, \bar{Y}_{j-\frac{1}{2}}, \bar{Y}^{(1)}_{j-\frac{1}{2}}, \bar{Y}^{(2)}_{j-\frac{1}{2}}, ..., \bar{Y}^i_{j-\frac{1}{2}}, ..., \bar{Y}^{(M)}_{j-\frac{1}{2}}, \bar{Y'}^{(1)}_{j-\frac{1}{2}}, \bar{Y'}^{(2)}_{j-\frac{1}{2}}, ..., \bar{Y'}^i_{j-\frac{1}{2}}, ..., \bar{Y'}^{(M)}_{j-\frac{1}{2}}), \quad (3.8)$$

$$\bar{f}^i_{j+\frac{1}{2}} = f^i(x_{j+\frac{1}{2}}, \bar{Y}_{j+\frac{1}{2}}, \bar{Y}^{(1)}_{j+\frac{1}{2}}, \bar{Y}^{(2)}_{j+\frac{1}{2}}, ..., \bar{Y}^i_{j+\frac{1}{2}}, ..., \bar{Y}^{(M)}_{j+\frac{1}{2}}, \bar{Y'}^{(1)}_{j+\frac{1}{2}}, \bar{Y'}^{(2)}_{j+\frac{1}{2}}, ..., \bar{Y'}^i_{j+\frac{1}{2}}, ..., \bar{Y'}^{(M)}_{j+\frac{1}{2}}), \quad (3.9)$$

$$\hat{Y}^i_j = Y^i_j + h^2_j \delta_i (\bar{f}^i_{j+\frac{1}{2}} + \bar{f}^i_{j-\frac{1}{2}}), \quad (3.10)$$

$$\hat{Y'}^i_j = \bar{Y'}^i_j + h_j \gamma_i (\bar{f}^i_{j+\frac{1}{2}} - \bar{f}^i_{j-\frac{1}{2}}), \quad (3.11)$$

$$\hat{f}^i_j = f^i(x_j, \hat{Y}_j, \hat{Y}^{(1)}_j, \hat{Y}^{(2)}_j, ..., \hat{Y}^i_j, ..., \hat{Y}^{(M)}_j, \hat{Y'}^{(1)}_j, \hat{Y'}^{(2)}_j, ..., \hat{Y'}^i_j, ..., \hat{Y'}^{(M)}_j), \quad (3.12)$$

$$Y^i_{j+1} - (1+\sigma) Y^i_j + \sigma Y^i_{j-1} = h^2_j \sigma (A \bar{f}^i_{j-\frac{1}{2}} + B \bar{f}^i_{j+\frac{1}{2}}), \quad (3.13)$$

$$Y^i_{j+1} - (1+\sigma) Y^i_j + \sigma Y^i_{j-1} = h^2_j \sigma (R \bar{f}^i_{j+\frac{1}{2}} + Q \hat{f}^i_j + P \bar{f}^i_{j-\frac{1}{2}}), \quad (3.14)$$

where

$$A = \frac{(2+\sigma)}{6}, \ B = \frac{(2\sigma+1)}{6}, \quad (3.15)$$

$$P = \frac{kh_j - \sin kh_j \cos kh_j}{k^2 kh_j \sin kh_j}, \ R = \frac{2kh_{j+1} - \sin kh_{j+1}}{2k^2 kh_{j+1} \sin kh_{j+1}}, \quad (3.16)$$

$$Q = \frac{2(\sigma+1)(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j)) - 2\sigma kh_j \sin(kh_j\sigma + kh_j)}{(kh_j)^2(\cos(kh_j\sigma - kh_j) - \cos(kh_j\sigma + kh_j))} \quad (3.17)$$

4 Illustration of the Method

Consider a linear singular BVP of fourth order as follows:

$$\frac{d^4 y(x)}{dx^4} = a(x)y(x) + d(x), x \neq 0, \quad (4.1)$$

$$y(0) = c_1, y(1) = d_1, \frac{d^2 y}{dx^2}(0) = c_2, \frac{d^2 y}{dx^2}(1) = d_2. \quad (4.2)$$

where $a(x)$ is singular and c_1, c_2, d_1, d_2 are real constants. Using (1.1), we write the problem (4.1) – (4.2) as follows:

$$\frac{d^2 y}{dx^2}(x) = z(x), \quad (4.3)$$

$$\frac{d^2 z}{dx^2}(x) = a(x)y(x) + d(x), \quad (4.4)$$

$$y(0) = c_1, y(1) = d_1, \quad (4.5)$$

$$z(0) = c_2, z(1) = d_2. \quad (4.6)$$

We use the method(3.14) to the BVP $(4.3)-(4.6)$. The method is given as follows:

$$\begin{aligned}
\sigma Y_{j-1}-Y_j(1+\sigma)+Y_{j+1} &= h_j^2\sigma(R\bar{Z}_{j+\frac{1}{2}}+Q\hat{Z}_j+P\bar{Z}_{j-\frac{1}{2}}), && (4.7)\\
\sigma Z_{j-1}-Z_j(1+\sigma)+Z_{j+1} &= h_j^2\sigma(R(a_{j+\frac{1}{2}}\bar{Y}_{j+\frac{1}{2}}+d_{j+\frac{1}{2}})\\
&\quad +Q(a_j\hat{Y}_j+d_j)+P(a_{j-\frac{1}{2}}\bar{Y}_{j-\frac{1}{2}}+d_{j-\frac{1}{2}})). && (4.8)
\end{aligned}$$

Then, we approximate $a_{j\pm\frac{1}{2}}$ for the BVP $(4.7)-(4.8)$ as

$$a_{j-\frac{1}{2}}=a_j-\frac{h_ja_j'}{2}+\frac{h_j^2a_j''}{8}+O(h_j^3), \tag{4.9}$$

$$a_{j+\frac{1}{2}}=a_j+\frac{\sigma h_ja_j'}{2}+\frac{(h_j\sigma)^2a_j''}{8}+O(h_j^3). \tag{4.10}$$

Similarily, we approximate $d_{j\pm\frac{1}{2}}$. Using the relations $(4.9)-(4.10)$ in $(4.7)-(4.8)$ we get,

$$\begin{aligned}
\sigma Y_{j-1}-Y_j(1+\sigma)+Y_{j+1} &= h_j^2\sigma(R\bar{Z}_{j+\frac{1}{2}}+Q\hat{Z}_j+P\bar{Z}_{j-\frac{1}{2}}), && (4.11)\\
\sigma Z_{j-1}-Z_j(1+\sigma)+Z_{j+1} &= h_j^2\sigma(R(a_{j+\frac{1}{2}}\bar{Y}_{j+\frac{1}{2}}+d_{j+\frac{1}{2}})\\
&\quad +Q(a_j\hat{Y}_j+d_j)+P(a_{j-\frac{1}{2}}\bar{Y}_{j-\frac{1}{2}}+d_{j-\frac{1}{2}})). && (4.12)
\end{aligned}$$

Finally, substituting $(3.1)-(3.12)$ in $(4.11)-(4.12)$ we get the difference equation of BVP $(4.3)-(4.6)$as follows:

$$\begin{bmatrix} b_j^{11} & b_j^{12} \\ b_j^{21} & b_j^{22} \end{bmatrix}\begin{bmatrix} Y_{j-1} \\ Z_{j-1} \end{bmatrix}+\begin{bmatrix} d_j^{11} & d_j^{12} \\ d_j^{21} & d_j^{22} \end{bmatrix}\begin{bmatrix} Y_j \\ Z_j \end{bmatrix}+\begin{bmatrix} p_j^{11} & p_j^{12} \\ p_j^{21} & p_j^{22} \end{bmatrix}\begin{bmatrix} Y_{j+1} \\ Z_{j+1} \end{bmatrix}=\begin{bmatrix} \psi_j^1 \\ \psi_j^2 \end{bmatrix}, \tag{4.13}$$

where

$$\begin{aligned}
b_j^{11} &= -\sigma + \frac{h_j^4}{2}\sigma^2 Q\delta a_j, \quad b_j^{12} = \frac{h_j^2\sigma^2}{2},\\
b_j^{21} &= \frac{h_j^2}{2}\sigma^2 R a_{j-\frac{1}{2}}, \qquad b_j^{22} = -\sigma + \frac{h_j^4}{2}\sigma^2 Q\delta a_j,\\
d_j^{11} &= (1+\sigma) + Q\delta a_j h_j^4\sigma^2, \qquad d_j^{12} = \frac{h_j^2\sigma^2(2Q+R+P)}{2},\\
d_j^{21} &= \frac{\sigma^2}{2}[h_j^2 a_j(2Q+R+P) + h_j^3(-P+\sigma R)\frac{a_j'}{2} + R\frac{h_j^4\sigma^2 a_j''}{8}],\\
&\qquad\qquad diag_j^{22} = (1+\sigma) + Q\delta a_j h_j^4\sigma^2,\\
p_j^{11} &= -1 + \frac{a_j h_j^4\sigma^2 Q\delta}{2}, \quad p_j^{12} = \frac{Rh_j^2\sigma^2}{2},\\
p_j^{21} &= h_j^2\sigma^2\frac{R}{2}a_{j+\frac{1}{2}}, \qquad p_j^{22} = -1 + \frac{a_j h_j^4\sigma^2 Q\delta}{2},\\
\psi_j^1 &= -h_j^4 2b_j Q\sigma^2,\\
&\psi_j^2 = -\sigma^2[h_j^2 d_j(2Q+P+R) + \frac{d_j' h_j^3}{2}(-P+R) + \frac{d_j'' h_j^4}{8}(R+P)].
\end{aligned}$$

5 Convergence Analysis

We provide the convergence of method (3.14) for the coupled second order BVP (4.3) − (4.6). The convergence of scalar singular BVP has been already provided by Mohanty [23]. Now, once the condition (4.5) − (4.6) is substituted in the difference equation (4.13), it is written in matrix form as follows:

$$H\hat{Y} + \hat{\psi} = \begin{bmatrix} b_j & d_j & p_j \end{bmatrix}\begin{bmatrix} \hat{Y_{j-1}} \\ \hat{Y_j} \\ \hat{Y_{j+1}} \end{bmatrix} + \hat{\psi}_j = \hat{0}, \tag{5.1}$$

where b_j, p_j, d_j are block elements of order 2 in tridiagonal block matrix H.

$$\begin{aligned}
\hat{Y} &= [\hat{Y}_1, \hat{Y}_2, ..., \hat{Y}_j, ...\hat{Y}_{N-1}]^T, where \qquad \hat{Y}_j = [Y_j, Z_j]^T,\\
\hat{\psi} &= [\hat{\psi}_1 + b_1[c_1, c_2]^T, \hat{\psi}_2, ..., \hat{\psi}_j, ...\hat{\psi}_{N-1} + p_{N-1}[d_1, d_2]^T]^T, \ where \ \hat{\psi}_j = [\psi_j^1, \psi_j^2]^T,\\
&\quad \hat{0} \ \text{ is a zero vector with } N-1 \text{ components.}
\end{aligned}$$

Let $[[y_1, z_1]^T, [y_2, z_2]^T ......, [y_j, z_j]^T, ...[y_{N-1}, z_{N-1}]^T]^T \cong \hat{y}$ be the exact solution satisfying

$$H\hat{y} + \hat{\psi} + \hat{T}_j^3 = 0, \tag{5.2}$$

where $\hat{T}_j^3$ is the truncation error, then the error vector E is given by $\hat{y} - \hat{Y}$. We get the error equation from(5.1)and(5.2), *i.e.*, $HE = \hat{T}_j^3$. (5.3)

For some $k_1, k_2 > 0$, let $|a_j| \le k_1$ and $|a_j'| \le k_2$. Using (4.13) and neglecting the higher order terms of h_j we get,

$$\|p_j\|_\infty \le \max_{1\le j\le N-2}\begin{cases} 1 + \frac{h_j^2\sigma^2 P}{2}, \\ 1 + \frac{h_j^2\sigma^2 P}{2}(k_1 + \frac{h_j\sigma}{2}k_2), \end{cases} \tag{5.4}$$

10

$$\|b_j\|_\infty \leq \max_{2\leq j\leq N-1} \begin{cases} \sigma + \frac{h_j^2\sigma^2 R}{2}, \\ \sigma + \frac{h_j^2\sigma^2 R}{2}(k_1 + \frac{h_j}{2}k_2). \end{cases} \tag{5.5}$$

We prove the irreducibility of H for sufficiently small h_j as well as $\|b_j\|_\infty \leq \sigma$ and $\|p_j\|_\infty \leq 1$ from $(5.4) - (5.5)$.
Let the sum of elements of j_{th} row of H be sum_j,

$$sum_j = \begin{cases} \sigma + \frac{h_j^2\sigma^2}{12}(P + 2(R+Q)), j = 1, \\ \sigma + \frac{h_j^2\sigma^2 a_j'}{24}(R + 2(P+Q))a_j + \frac{\sigma^2}{2}(h_j^2 R a_j + h_j^3(-P + 2R\sigma)), \ j = 2, \end{cases} \tag{5.6}$$

$$sum_j = \begin{cases} \frac{h_j^2\sigma^2}{2}(R+Q+P), \ j = 3(2)N-4, \\ \frac{h_j^2\sigma^2 a_j}{2}(R+Q+P) + \frac{h_j^3\sigma^2 a_j'}{4}(-2P + \sigma R), \ j = 4(2)N-3, \end{cases} \tag{5.7}$$

$$sum_j = \begin{cases} 1 + \frac{h_j^2\sigma^2}{12}(R + 2(Q+P)), \ j = N-2, \\ 1 + \frac{h_j^2\sigma^2 a_j}{12}(R + 2(Q+P)) + \frac{h_j^3\sigma^2 a_j'}{4}(-2R + P\sigma), \ j = N-1. \end{cases} \tag{5.8}$$

We can easily prove that H is Monotone using $0 < L \leq min(L_1, L_2)$ in $(5.6) - (5.8)$ and for sufficiently small h_j. Therefore, $H^{-1} \geq 0$ and exist. Hence by (5.3) we have,

$$||E|| = ||H^{-1}||||\hat{T}_j^3||. \tag{5.9}$$

Now for sufficiently small h_j, by $(2.23) - (2.25)$ and $(5.6) - (5.8)$ we can say that:

$$sum_j > \begin{cases} \frac{h_j^2\sigma(2+3\sigma)}{12}, \quad j = 1, \\ \frac{h_j^2\sigma(2+3\sigma)L}{12}, j = 2, \end{cases} \tag{5.10}$$

$$sum_j \geq \begin{cases} \frac{h_j^2(\sigma+1)}{2}, \quad j = 3(2)N-4, \\ \frac{h_j^2(\sigma+1)L}{2}, \quad j = 4(2)N-3, \end{cases} \tag{5.11}$$

$$sum_j > \begin{cases} \frac{h_j^2\sigma(2\sigma+3)}{12}, \quad j = N-2, \\ \frac{h_j^2\sigma(2\sigma+3)L}{12}, j = N-1. \end{cases} \tag{5.12}$$

We can also say for $\sigma \neq 0$:

$$\begin{aligned} sum_j &> \max[\frac{h_j^2\sigma(2+3\sigma)}{12}, \frac{h_j^2\sigma(2+3\sigma)L}{12}] \\ &= \frac{h_j^2\sigma(2+3\sigma)L}{12}, \text{for} \quad j = 1, 2, \end{aligned} \tag{5.13}$$

$$\begin{aligned} sum_j &\geq \max[\frac{h_j^2(1+\sigma)}{2}, \frac{h_j^2(1+\sigma)L}{2}] \\ &= \frac{h_j^2(1+\sigma)L}{2}, \quad \text{for } j = 3(1)N-3, \end{aligned} \tag{5.14}$$

$$\begin{aligned} sum_j &> \max[\frac{h_j^2\sigma(2\sigma+3)}{12}, \frac{h_j^2\sigma(2\sigma+3)L}{12}] \\ &= \frac{h_j^2\sigma(2\sigma+3)L}{12}, \text{for} \quad j = N-2, N-1. \end{aligned} \tag{5.15}$$

11

Then, we use a result proved by Varga [33] for $i = 1(1)N-1$,

$$H_{i,j}{}^{-1} \leq \frac{1}{sum_j}, \quad \text{where } H_{i,j}{}^{-1} \text{ is the } (i,j)^{th}\text{element of } H^{-1} \tag{5.16}$$

By using $(5.13)-(5.15)$, we have

$$\frac{1}{sum_j} \leq \begin{cases} \frac{12}{h_j^2(3\sigma+2)\sigma L}, & j = 1,2, \\ \frac{2}{h_j^2(\sigma+1)L}, & j = 3(1)N-3, \\ \frac{12}{h_j^2(2\sigma+3)\sigma L}, & j = N-2, N-1. \end{cases} \tag{5.17}$$

Now, we show that the error defined in equation (5.9) is bounded and is of order $O(h_j^3)$. For this, we define norm of H^{-1} and $\hat{T}_j^3$ such that,

$$\parallel H_{j,i}{}^{-1} \parallel = \max_{j\in[1,N-1]} \sum_{i=1}^{N-1} \mid H_{j,i}{}^{-1} \mid, also \parallel T \parallel = \max_{j\in[1,N-1]} \mid \hat{T}_j^3 \mid . \tag{5.18}$$

Thus, using (5.3) and $(5.16)-(5.18)$ we get the bound for the error term as follows:

$$\parallel E \parallel \leq O(h_j^5)\frac{12}{h_j^2 L\sigma}\frac{(6\sigma^3+18\sigma^2+16\sigma+5)}{(6\sigma^3+19\sigma^2+19\sigma+6)} = O(h_j^3). \tag{5.19}$$

This proves the method (3.14) has third order convergence for BVPs $(4.1)-(4.2)$. Therefore, we can say that method (3.14) has third order convergence for BVP $(1.1)-(1.2)$. Similarly, method (3.13) has second order convergence.

6 Numerical Illustrations

We have solved seven problems. For quasi-variable mesh and uniform mesh, we have tabulated root mean square errors and maximum absolute errors respectively in Tables 1-7. We have chosen $h_1 = \frac{(1-\sigma)}{(1-\sigma^N)}, \sigma \neq 1$. The remaining h_j 's are calculated by the relation $h_j = \sigma h_{j-1}, j = 2(1)N-1$. Figures 1-7 presents the graphs of numerical solution and the exact solution in case of fourth order method based on uniform mesh. Related numerical results are provided in Table 1-7.

Gauss Elimination and Newton's method for block elements has been used for solving system of linear and nonlinear BVPs respectively with initial approximation $y_0 = 0$. The order of convergence (OC) for fourth order method based on uniform mesh is also provided. Matlab 07 has been used for doing all calculations.

Problem 6.1 (Nonlinear boundary value problem)

$$\frac{d^4y(x)}{dx^4} = 6e^{-4\ y} - \frac{12}{(1+x)^4},$$
$$y(0) = 0, y(1) = .6931, \frac{d^2y}{dx^2}(0) = -1, \frac{d^2y}{dx^2}(1) = -.25.$$

12

Table 1: Problem 6.1

	Off-step mesh			
N	$Method1$	$Method2$	$Uniform\ mesh\ method$	[29]
8	4.4398e-003	4.8610e-006	7.2499e-007	0.37e-005
16	2.0758e-003	1.2961e-006	4.6937e-008	0.29e-006
32	1.3702e-003	6.7628e-007	2.9600e-009	0.19e-007

The exact solution is given by $y(x) = \log(1 + x)$. In Table 1, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

Problem 6.2 (Sixth order linear boundary value problem):

$$\begin{aligned}
&\frac{d^6y(x)}{dx^6} + y(x) = 6(5\sin(x) + 2x\cos(x)), x \in [0, 1]\\
&y(0) = 0, \frac{d^2y}{dx^2}(0) = 0, \frac{d^4y}{dx^4}(0) = 0,\\
&y(1) = 0, \frac{d^2y}{dx^2}(1) = 3.84416, \frac{d^4y}{dx^4}(1) = -14.42007.
\end{aligned}$$

The exact solution is $y(x) = (x^2 - 1)\sin(x)$. In Table 2, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

Problem 6.3 (Fourth order non linear boundary value problem)

$$\begin{aligned}
&\frac{d^4y(x)}{dx^4} = 3(\frac{dy}{dx})^2 + 4.5y^3, x \in [0, 1]\\
&y(0) = 4, \frac{d^2y}{dx^2}(0) = 24, y(1) = 1, \frac{d^2y}{dx^2}(1) = 1.5e.
\end{aligned}$$

The exact solution is $y(x) = \frac{4}{(1+2x+x^2)}$. In Table 3, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

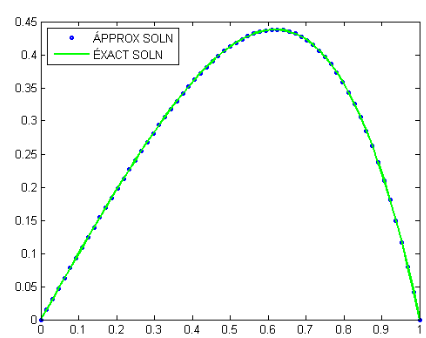

Figure 1: Exact solution vs Numerical solution in uniform mesh method

13

Table 2: *Problem 6.2*

	Off-step mesh				
N	$Method1$	$Method2$	$Uniform\ mesh\ method$	[3]	[26]
8	6.4952e-004	4.2946e-006	6.5901e-007	1.5379 e-006	8.1514e-005
16	5.9397e-004	9.9183e-007	4.1831e-008	1.9790 e-007	2.1052 e-005
32	5.1433e-004	4.7874e-007	2.6133e-009	4.0596 e-008	5.3084 e-006

Table 3: *Problem 6.3*

	Off-step mesh			
N	$Method1$	$Method2$	$Uniform\ mesh\ method$	[25]
8	1.2451e-003	8.4710e-005	2.2780e-005	1.44 e-003
16	2.7555e-004	2.1499e-005	1.5362e-006	9.33 e-004
32	4.0919e-004	9.7519e-006	9.8628e-008	5.90 e-005
64	3.1887e-004	6.4390e-006	6.2300e-009	3.69 e-006

Problem 6.4 (Sixth order non linear boundary value problem)

$$\begin{aligned}
&\frac{d^6y(x)}{dx^6} = y^2e^{-x},\ x \in [0,1]\\
&y(0) = 1, y(1) = e,\\
&\frac{d^2y}{dx^2}(0) = 1, \frac{d^2y}{dx^2}(1) = e,\\
&\frac{d^4y}{dx^4}(0) = 1, \frac{d^4y}{dx^4}(1) = e.
\end{aligned}$$

The exact solution is $y(x) = e^x$. In Table 4, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

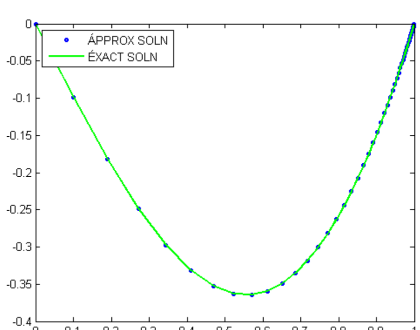

Figure 2: Exact solution vs Numerical solution in uniform mesh method

14

Table 4: *Problem 6.4*

	Off-step mesh			
N	*Method*1	*Method*2	*Uniform mesh method*	[14]
8	2.64952e-004	2.0457e-007	5.1651e-008	7.02e-006
16	5.9397e-004	4.9805e-008	3.2495e-009	4.35e-006
32	5.1433e-004	2.4007e-008	2.0334e-010	7.87e-007

Problem 6.5 (Fourth order non-linear singular boundary value problem)

$$x\frac{d^4y(x)}{dx^4} + \frac{4d^3y(x)}{dx^3} = xy^2 - 4\cos(x) - xsin(x), x \neq 0.$$

The exact solution is $y(x) = \sin(x)$. In Table 5, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

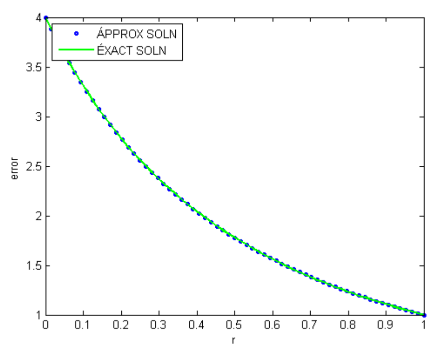

Figure 3: Exact solution vs Numerical solution in uniform mesh method

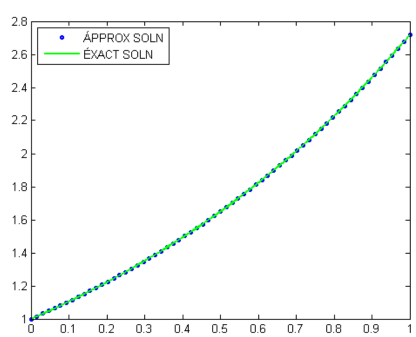

Figure 4: Exact solution vs Numerical solution in uniform mesh method

Table 5: *Problem 6.5*

	Off-step mesh			
N	$Method1$	$Method2$	$Uniform\ mesh\ method$	OC
8	1.4374e-004	4.2067e-006	1.6791e-006	-
16	8.8426e-005	1.2629e-006	1.5413e-007	3.4455
32	5.7765e-005	6.0728e-007	1.2889e-008	3.5810
64	4.0494e-005	4.0146e-007	7.7938e-010	4.1476

Problem 6.6 (Sixth order non-linear singular boundary value problem)

$$x\frac{d^6y(x)}{dx^6} + 6\frac{d^5y(x)}{dx^5} + 2xy(x) = xe^y, x \neq 0.$$

The exact solution is $y(x) = e^x$. In Table 6, results for quasi-variable mesh taking $\sigma = 0.9$ and for uniform mesh is tabulated.

Table 6: *Problem 6.6*

	Off-step mesh			
N	$Method1$	$Method2$	$Uniform\ mesh\ method$	OC
8	6.1864e-004	5.6862e-007	4.5498e-007	-
16	3.1623e-004	1.4817e-007	3.7876e-008	3.5865
32	2.2083e-004	7.6631e-008	2.9520e-009	3.6815
64	2.0493e-004	6.7278e-008	2.2125e-010	3.7379

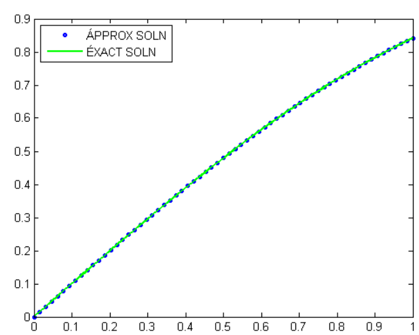

Figure 5: Exact solution vs Numerical solution in uniform mesh method

16

Problem 6.7 (System of second order boundary value problem)

$$\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx} + xy(x) + \frac{dz(x)}{dx} + 2xz(x) = g_1(x),$$
$$\frac{d^2z(x)}{dx^2} + z(x) + 2\frac{dy(x)}{dx} + x^2y(x) = g_2(x),$$
$$y(0) = 0, z(0) = 1, y(1) = 0, z(1) = 1,$$

where $g_1(x) = -2\cos(x)(1+x) + \pi cos(x\pi) + 2xsin(x\pi) + 2\sin(x)(2x-2-x^2)$, $g_2(x) = -4\cos(x)(x-1) + 2\sin(x)(2-x^2+x^3) + (1-\pi^2)\sin(x\pi)$ and $x \in [0,1]$. The exact solution is $y(x) = 2(1-x)\sin(x), z(x) = \sin(x\pi)$.

Table 7: *Problem 6.7*

	z		y	
N	[8]	*Uniform mesh method*	[8]	*Uniform mesh method*
.08	7.5e-004	3.5686e-007	2.2e-004	1.8284e-006
.24	8.2e-004	1.4754e-006	2.3e-004	2.0723e-006
.40	6.5e-004	2.5123e-006	2.3e-004	6.2430e-007
.56	2.8e-004	3.1366e-006	2.2e-004	3.9577e-006
.72	2.6e-004	2.9899e-006	2.6e-004	5.6498e-006
.88	8.0e-004	1.2382e-005	5.5e-004	3.9716e-006
.96	4.8e-004	1.4964e-006	3.1e-004	1.5857e-006

7 Final Remarks

In this paper, two methods of second and third order respectively have been developed to solve singular BVPs both linear as well as nonlinear. For numerical illustration, we have considered seven problems consisting of fourth and sixth order linear and nonlinear BVPs. Table $1-4, 7$ proves improvement in results when compared with problems

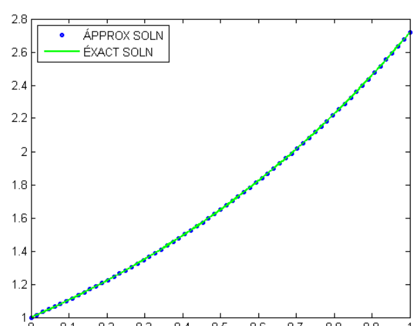

Figure 6: Exact solution vs Numerical solution in uniform mesh method

solved by methods using extrapolation, polynomial and non polynomial splines and also by using reproducing kernel space method.

In our methods minimal grid points i.e., three grid points at a time has been used as compared to existing methods. Due to the use of three grid points, the numerical scheme is converted to a tri-diagonal representation of system of difference equations which can be easily solved by any standard method available in the literature. Also, due to the use of off-step mesh, singularity has been controlled in singular BVPs. We have also solved nonlinear singular BVP and so far such kind of BVP has not been solved. Therefore, for such problems we have presented the numerical order of convergence(OC) based on uniform mesh.

The methods developed are effective and straight forward and can be extended to solve boundary value problems with cartesian as well as polar coordinates. Due to the ability to operate with polar coordinate, many problems on fluid flow with polar symmetry can be attended. Moreover, we can also use the methods to solve wide variety of higher order singularly perturbed BVPs.

References

[1] Aftabizadeh, A.R.(1986), Existence and uniqueness theorems for fourth-order boundary value problems, Journal of Mathematical Analysis and Applications, 116(2), 415-426.

[2] Agarwal, R.P.(1981), Boundary value problems for higher order differential equations, Bulletin of the Institute of Mathematics, Academia Sinica, 9(1), 47-61.

[3] Akram, G. and Siddiqi, S.S.(2006), Solution of sixth order boundary value problems using non-polynomial spline technique, Applied Mathematics and Computation, 181(1), 708-720.

[4] Akram, G. and Rehman, Hamood Ur(2013), Numerical solution of eighth order boundary value problems in reproducing kernel space, Numerical Algorithm , 62, 527-540.

[5] Bernis, F.(1982), Compactness of the support in convex and non-convex fourth order elasticity problem, Nonlinear Analysis, 6, 1221-1243.

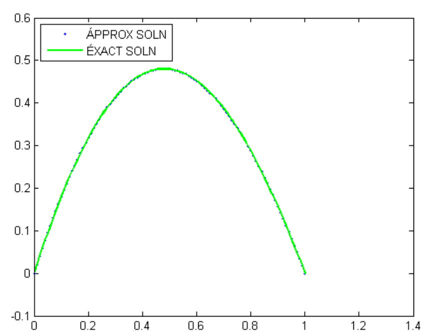

Figure 7: Exact solution vs Numerical solution in uniform mesh method

18

[6] Bhrawy, A.H., Alofi, A.S., Van Gorder, R.A.(2014), An Efficient Collocation Method for a Class of Boundary Value Problems Arising in Mathematical Physics and Geometry, Abstract and Applied Analysis, Article ID 425648, 9 pages.

[7] Dwivedi, K. and Singh, J.(2021), Numerical solution of two-dimensional fractional-order reaction advection sub-diffusion equation with finite-difference Fibonacci collocation method, Mathematics and Computers in Simulation, 181, 38-50.

[8] Geng,F. and Cui,M.(2007), Solving a nonlinear system of second order boundary value problems, Journal of Mathematical Analysis and Application , 327, 1167-1181.

[9] Glatzmaier, G.A.(2014), Numerical simulations of stellar convection dynamics at the base of the convection zone, Geophysical Fluid Dynamics, 31(1985), 137-150.

[10] Jain, M.K.(2014), Numerical Solution of Differential Equations(third edition), New Age International(P) Ltd., New Delhi.

[11] Jha, N., Bieniasz, L.K.(2015), A Fifth (Six) Order Accurate, Three-Point Compact Finite Difference Scheme for the Numerical Solution of Sixth Order Boundary Value Problems on Geometric Meshes, Journal of Scientific Compution, 64, 898–913.

[12] Kadalbajoo, M.K. and Bawa, R.K.(1993), Third Order Variable-Mesh Cubic Spline Methods for Singularly Perturbed Boundary Value Problems, Applied Mathematics and Computation, 59, 117-129.

[13] Keller, H.B.(1968), Numerical Methods for Two point Boundary value problems, Blaisdell Publications Co., New York.

[14] Khan, A. and Khandelwal, P.(2011), Solution of Non-linear Sixth-order Two Point Boundary-value Problems Using Parametric Septic Splines, International Journal of Nonlinear Science ,12(2), 184-195.

[15] Mohanty,R.K. and Evans, D.J.(2003), A Fourth Order Accurate Cubic Spline Alternating Group Explicit Method for Non-linear Singular Two Point Boundary Value Problems, International Journal of Computer Mathematics, 80, 479-492.

[16] Mohanty, R.K.,Sachdev, P.L. and Jha, N.(2004), An $O(h^4)$ Accurate Cubic Spline TAGE Method for Non-linear Singular Two Point Boundary Value Problems, Applied Mathematics and Computations, 158, 853-868.

[17] Mohanty,R.K., Jha, N. and Evans, D.J.(2004), Spline in Compression Method for the Numerical Solution of Singularly Perturbed Two Point Singular Boundary Value Problems , International Journal of Computer Mathematics, 81, 615-627 .

[18] Mohanty, R.K., Evans, D.J. and Khosla, N.(2005), An $O(h^3)$ Non-uniform Mesh Cubic Spline TAGE Method for Non-linear Singular Two-point Boundary Value Problems , International Journal of Computer Mathematics, 82, 1125-1139.

[19] Mohanty, R.K., Evans, D.J. and Arora,U.(2005) ,Convergent Spline in Tension Methods for Singularly Perturbed Two Point Singular Boundary Value Problems , International Journal of Computer Mathematics , 82, 55-66.

[20] Mohanty, R.K. and Jha, N. (2005), A Class of Variable Mesh Spline in Compression Methods for Singularly Perturbed Two Point Singular Boundary Value Problems, Applied Mathematics and Computations , 168, 704-716.

[21] Mohanty, R.K. and Khosla, N. (2005), A Third Order Accurate Variable Mesh TAGE Iterative Method for the Numerical Solution of Two Point Non-linear Singular Boundary Value Problems, International Journal of Computer Mathematics, 82, 1261-1273.

[22] Mohanty, R.K. and Arora, U.(2006), A Family of Non-uniform Mesh Tension Spline Methods for Singularly Perturbed Two Point Singular Boundary Value Problems with Significant First Derivatives, Applied Mathematics and Computations , 172, 531-544.

[23] Mohanty, R.K.(2006), A class of non-uniform mesh three point arithmetic average discretization for $y'' = f(x, y, y')$ and the estimates of y' , Applied Mathematics and Computation, 183, 477-485.

[24] O'Regan, D (1991), Solvability of some fourth (and higher) order singular boundary value problems, Journal of Mathematical Analysis and Applications, 161(1), 78-116.

[25] Sakai, M. and Usmani,R.(1983), Spline Solutions for Nonlinear Fourth-Order Two-Point Boundary Value Problems , Publication of Research Institute for Mathematical Sciences, Kyoto University , 19, 135-144.

[26] Siddiqi, S. S. and Twizell, E.H.(1996), Spline solutions of linear sixth- order boundary value problems, International Journal of Computer Mathematics, 60, 295-304.

[27] Terril, R. M.(1964), Laminar flow in a uniformly porous channel, Aeronaut Quarterly, 15(3), 299-310.

[28] Toomre, J., Zahn, J.R., Latour, J. and Spiegel, E.A.(1976), Stellar convection theory II:Single-mode study of the second convection zone in A-type stars, The Astophysical Journal, 207, 545-563.

[29] Twizell, E.H.(1986), A fourth-order extrapolation method for special nonlinear fourth-order boundary value problems, Communications in Applied Numerical Methods,John Wiley and Sons, 2, 593-602.

[30] Siddiqi, S. S. and Akram ,G.(2007), Solution of eighth-order boundary value problems using the non-polynomial spline technique, International Journal of Computer Mathematics, 84(3), 347-368.

[31] Siddiqi, S. S. and Akram ,G.(2008), Septic spline solutions of sixth-order boundary value problems, Journal of Computational and Applied Mathematics, 215, 288-301.

[32] Singh, J., Kilicman, A., Kumar D., Swaroop, R. and Md. Ali F.(2019), Numerical Study For Fractional Model Of Non-Linear Predator-Prey Biological Population Dynamical System, Thermal Science, 23 (6) , S2017-S2025.

[33] Varga, R.S. (1962), Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ.