Optimal Control of two-strain typhoid transmission using treatment and proper hygiene/sanitation practices

Tsegaye Kebede Irena[1], Sunita Gakkhar

December 26, 2021

Abstract

A mathematical model is developed to predict the optimum level of measures required to control a two-strain typhoid infection. The model considers symptomatic individuals and carriers together with environmental bacteria with different sensitivities to antimicrobials. Treatment for symptomatic individuals in each strain and use of sanitation and proper hygiene practices are considered as control measures. Our simulation results show that combining the three control interventions highly influenced the number of symptomatic individuals and environmental bacteria in both the strains. However, there are still a significant number of asymptomatic carriers in both the strains. This result shows that combating a two-strain typhoid infection requires some control interventions that reduce the number of asymptomatic carriers to near zero, along with optimal treatment combined with proper hygiene/sanitation practices. Further, efficiency analysis is used to investigate the impact of each control strategy on reducing the number of infected individuals and bacteria in both the strains. The study result suggests that implementing the combination of all the three control interventions is the most effective control strategy.

Key words: *Salmonella* Typhi; Two-strain typhoid infection; Asymptomatic carriers; Efficiency analysis

Mathematics Subject Classification(2010): 44A15; 46F12; 54B15; 46F99

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1 Introduction

Typhoid, a disease caused by *Salmonella* Typhi bacteria, is a significant cause of illness and death in low-resource regions worldwide, especially Sub-Saharan

[1]Corresponding author: Department of Mathematics
Indian Institute of Technology Roorkee, Roorkee-247667, India

Africa and South/Southeast Asia [1]. It is a severe febrile illness often accompanied by headache, loss of appetite, malaise, abdominal pain, diarrhea, and (in severe cases) intestinal perforation and neurological complications [2]. It is estimated to cause nearly 12 million cases and over 128 000 deaths globally each year [6]. It is estimated that the case fatality rate for untreated patients ranges between 10 and 20%, but drops to 1–4% with appropriate and timely antimicrobial treatment [4, 3, 5]. The infection is usually spread through contaminated food and water from the environment and direct contact with an infected person [7, 8].

Typhoid fever can be prevented and controlled through public health interventions such as providing safe drinking water, promoting hygiene and sanitation, and ensuring adequate and timely patient care. Antimicrobial treatment is the cornerstone for reducing severe illness and even death. However, misuse of antimicrobials for treatment leads to the emergence of resistant strains of *Salmonella* Typhi, known as treatment-induced acquired resistance [9, 10]. In typhoid endemic areas, clinicians frequently prescribe antimicrobials to patients with suspected typhoid without blood culture confirmation. This practice results in delayed treatment leading to the development of antimicrobial resistance [3, 11, 12, 13]. Treatment-induced acquired resistance has complicated treatment, increasing morbidity and mortality, and is considered one of the most significant challenges in managing the disease [14, 15].

In existing litrature, several typhoid epidemiological models have been developed and analyzed to better understand the transmission dynamics of typhoid [16, 17, 18, 19, 20, 21, 22, 23]. Among them, only a few have explored the effect of control strategies for typhoid with optimal control theory [16, 17]. Optimal control theory is a mathematical optimization that deals with finding a control for a dynamical system over a period of time. Although the importance of optimal control theory in epidemiology is well recognized, its applications in typhoid dynamics are scarce. No attempts have been made to predict the optimal level of control measures required to combat a two-strain typhoid infection. Our aim is to investigate the optimal control strategies in a two-strain dynamic model involving antimicrobial-sensitive and resistant strains of typhoid. A mathematical model for a two-strain typhoid dynamics is explored considering treatment-induced acquired resistance and re-infection [24]. Three time-dependent controls are introduced in this model to explore the optimal control strategy for controlling the disease.

The paper is organized as follows: In Section 2, the model in [24] is modified by adding three time-dependent controls $u_1(t)$, $u_2(t)$ and $u_3(t)$, and three positive parameters ϵ, b_1 and b_2. Also, a description of these parameters is given. In Section 3, a mathematical analysis of the time-dependent model is performed. In Section 4, numerical simulations and discussions of the corresponding results are presented. A short conclusion of the study is made in Section 5.

2 Model with controls

The mathematical model developed by Irena and Gakkhar [24] is considered to investigate the infection dynamics in a two-strain typhoid disease. The state variables I_j, C_j, and $\mathcal{B}_j$ represent the number of symptomatic infectious individuals, asymptomatic carriers, and bacteria for the strain j, respectively, while S represents the susceptible individuals. The model presented in [24] is

$$\begin{cases} \frac{dS}{dt} = \pi - \mu S - (\lambda_1 + \lambda_2)S + (1-p)r_1 I_1 + r_2 I_2 \\ \frac{dI_1}{dt} = (1-\alpha)[\lambda_1 S - \psi\lambda_2 I_1] - (\mu + d_1 + r_1)I_1 + \phi_1 C_1 \\ \frac{dC_1}{dt} = \alpha\lambda_1 S - \psi\lambda_2 C_1 - (\mu + \phi_1)C_1 \\ \frac{d\mathcal{B}_1}{dt} = \delta_1 I_1 + \omega_1 C_1 - \xi_1 \mathcal{B}_1 \\ \frac{dI_2}{dt} = (1-\alpha)\lambda_2[S + \psi(I_1 + C_1)] + pr_1 I_1 - (\mu + d_2 + r_2)I_2 + \phi_2 C_2 \\ \frac{dC_2}{dt} = \alpha\lambda_2(S + \psi C_1) - (\mu + \phi_2)C_2 \\ \frac{d\mathcal{B}_2}{dt} = \delta_2 I_2 + \omega_2 C_2 - \xi_2 \mathcal{B}_2 \end{cases} \tag{2.1}$$

where

$$\lambda_j = \frac{\beta_j(I_j + \theta C_j)}{N} + \eta f(B) g_j(B)$$

and $j = 1, 2$ represent the sensitive and resistant strains, respectively.

On the basis of sensitivity analysis of the model, three time-dependent controls are introduced in the model: (i) treatment of the symptomatic individuals in each strain $(u_1(t), u_2(t))$, which were constant parameters in our previous work [24], and (ii) proper hygiene/sanitation practices in order to prevent contamination of food and water to reduce both direct and environmental transmission $(u_3(t))$. The first two controls, u_1 and u_2, also decrease the bacteria excretion of symptomatic individuals in both strains so that the bacteria shedding rates by symptomatic individuals δ_1 and δ_2 in model (2.1) are replaced by $(1-(1-p)u_1)\delta_1$ and $(1-\epsilon u_2)\delta_2$, respectively. The parameter ϵ represents the efficacy of treatment for symptomatic individuals with resistant strain. Also, the second control u_3 increases the decay rate of bacteria so that the bacteria decay rates ξ_1 and ξ_1 are replaced by $\xi_1 + b_1 u_3$ and $\xi_2 + b_2 u_3$, respectively. The parameters b_1 and b_2 denote the bacteria decay rates (sensitive and AMR strains, respectively) induced by sanitation and proper hygiene practices. The schematic diagram in Figure 1 shows the transmission dynamics of the time-dependent model. Thus, the resulting dynamic model is given by the following

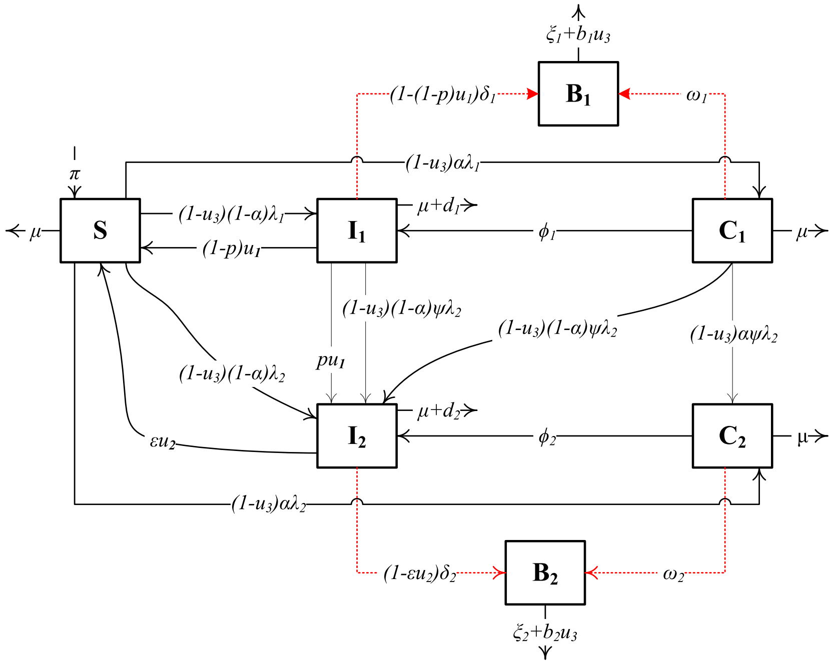

Figure 1: Flow diagram of the model.

system of nonlinear ODEs:

$$\begin{cases} \frac{dS}{dt} = \pi - \mu S - (1-u_3)(\lambda_1+\lambda_2)S + (1-p)u_1 I_1 + \epsilon u_2 I_2 \\ \frac{dI_1}{dt} = (1-u_3)(1-\alpha)[\lambda_1 S - \psi\lambda_2 I_1] - (\mu + d_1 + u_1)I_1 + \phi_1 C_1 \\ \frac{dC_1}{dt} = (1-u_3)[\alpha\lambda_1 S - \psi\lambda_2 C_1] - (\mu+\phi_1)C_1 \\ \frac{d\mathcal{B}_1}{dt} = \delta_1(1-(1-p)u_1)I_1 + \omega_1 C_1 - (\xi_1 + b_1 u_3)\mathcal{B}_1 \\ \frac{dI_2}{dt} = (1-\alpha)(1-u_3)\lambda_2[S + \psi(I_1 + C_1)] + pu_1 I_1 \\ \qquad -(\mu + d_2 + \epsilon u_2)I_2 + \phi_2 C_2 \\ \frac{dC_2}{dt} = (1-u_3)\alpha\lambda_2(S + \psi C_1) - (\mu+\phi_2)C_2 \\ \frac{d\mathcal{B}_2}{dt} = \delta_2(1-\epsilon u_2)I_2 + \omega_2 C_2 - (\xi_2 + b_2 u_3)\mathcal{B}_2 \end{cases} \tag{2.2}$$

The model is associated with the nonnegative initial conditions:

$$S(0), \quad I_j(0), \quad C_j(0), \quad \mathcal{B}_j(0) \quad \text{for} \quad j = 1, 2.$$

The description of the associated model parameters are given in Table 1 and are assumed to be nonnegative.

Table 1: Description the model parameters.

Parameter	Description
α	Fraction of newly infected individuals who becomes asymptomatic carriers
β_1, β_2	Ingestion rate of sensitive and resistant strains of bacteria through human-to-human interaction
δ_1, δ_2	Shedding rate of bacteria by symptomatic cases with sensitive and resistant strains
ϵ	Efficacy of treatment of symptomatic individuals with resistant strain
η	Ingestion rate of bacteria from the contaminated environment
θ	Relative infectiousness of asymptomatic carriers
μ	Natural mortality rate of human population
ξ_1, ξ_2	Decay rate of sensitive and resistant strains of bacteria in the environment
π	Influx rate of individuals into susceptible class
ϕ_1, ϕ_2	Symptoms development rate by asymptomatic carriers with sensitive and resistant strains
ψ	Factor reducing the risk of re-infection with resistant strain due to activates of immune cells to the previous infection with a sensitive strain
ω_1, ω_2	Shedding rate of bacteria by asymptomatic carriers with sensitive and resistant strains
b_1, b_2	Sanitation-induced bacteria decay rates (sensitive and resistant strains)
d_1, d_2	Disease-induced death rate for symptomatic cases with sensitive and resistant strains
p	Fraction of those symptomatic individuals infected with a sensitive strain who acquire treatment-induced resistance

The objective functional to be minimized is

$$J(u_1, u_2, u_3) = \int_0^T \left(\sum_{i=1}^{2} A_i(I_i + C_i) + A_3 \sum_{j=1}^{2} \mathcal{B}_j + \frac{1}{2} \sum_{k=1}^{3} D_k u_k^2 \right) dt \tag{2.3}$$

subject to the state system (2.2), where A_i and D_i $(i = 1, 2, 3)$ are appropriate weight constants. The aim is to minimize the total number of infective individuals as well as bacteria while keeping the implementation cost of the strategies associated to the controls low.
We seek to find an optimal control triplet (u_1^*, u_2^*, u_3^*) such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{\Omega} J(u_1, u_2, u_3)$$

where

$$\Omega = \left\{ (u_1, u_2, u_3) \in L^1(0, T) \mid 0 \leq u_i \leq 1, \; i = 1, 2, 3 \right\}$$

is the control set.

3 Optimal control analysis

The existence of optimal control triplet (u_1^*, u_2^*, u_3^*) is guaranteed due to a priori boundedness of the state solutions, convexity of the integrand of J on Ω, and the *Lipschitz* property of the state system [25].

The necessary conditions that an optimal solution must satisfy come from Pontryagin's Maximum Principle [26]. This principle converts (2.2) and (2.3) into a problem of minimizing pointwise a Hamiltonian $\mathbb{H}$ with respect to u_1, u_2 and u_3:

$$\begin{aligned}
\mathbb{H} = {} & A_1(I_1 + C_1) + A_2(I_2 + C_2) + A_3(\mathcal{B}_1 + \mathcal{B}_2) + \frac{D_1}{2}u_1^2 + \frac{D_2}{2}u_2^2 + \frac{D_3}{2}u_3^2 \\
& +\lambda_1[\pi - \mu S - (1 - u_3)(\lambda_1 + \lambda_2)S + (1 - p)u_1 I_1 + \epsilon u_2 I_2] \\
& +\lambda_2[(1 - u_3)(1 - \alpha)(\lambda_1 S - \psi\lambda_2 I_1) - (\mu + d_1 + u_1)I_1 + \phi_1 C_1] \\
& +\lambda_3[(1 - u_3)(\alpha\lambda_1 S - \psi\lambda_2 C_1) - (\mu + \phi_1)C_1] \\
& +\lambda_4[\delta_1(1 - (1 - p)u_1)I_1 + \omega_1 C_1 - (\xi_1 + b_1 u_3)\mathcal{B}_1] \\
& +\lambda_5[(1 - u_3)(1 - \alpha)\lambda_2(S + \psi(I_1 + C_1)) + p u_1 I_1 \\
& \qquad -(\mu + d_2 + \epsilon u_2)I_2 + \phi_2 C_2] \\
& +\lambda_6[(1 - u_3)\alpha\lambda_2(S + \psi C_1) - (\mu + \phi_2)C_2] \\
& +\lambda_7[\delta_2(1 - \epsilon u_2)I_2 + \omega_2 C_2 - (\xi_2 + b_2 u_3)\mathcal{B}_2]
\end{aligned} \tag{3.1}$$

where λ_i, $i = 1, 2, ..., 7$ are the adjoint functions.

By applying Pontryagin's Maximum Principle [26] and the existence result for the optimal control triplet from [25], the following adjoint system is obtained together with transversality conditions $\lambda_k(T) = 0$:

$$
\begin{aligned}
\frac{d\lambda_1}{dt} =& \frac{1}{\mathcal{B}_1+\mathcal{B}_2}\\
&\times [(1-u_3)\eta f(\mathcal{B})((\lambda_1-(1-\alpha)\lambda_2-\alpha\lambda_3)\mathcal{B}_1+(\lambda_1-(1-\alpha)\lambda_5-\alpha\lambda_6)\mathcal{B}_2)\\
&+\mu(\mathcal{B}_1+\mathcal{B}_2)\lambda_1]\\
&+\frac{(1-u_3)}{N^2}[\mathcal{B}_1(\lambda_1-(1-\alpha)\lambda_2-\alpha\lambda_3)(I_1+\theta C_1)(I_1+C_1+I_2+C_2)]\\
&+\frac{\beta_2(1-u_3)(I_2+\theta C_2)}{N^2}[(\lambda_1-(1-\alpha)\lambda_5-\alpha\lambda_6+(\lambda_5-\lambda_2)(1-\alpha)\psi)I_1\\
&+(\lambda_1-(\lambda_5(1-\alpha)+\alpha\lambda-6)(1-\psi)-\psi\lambda_3)C_1\\
&+(\lambda_1-(1-\alpha)\lambda_5-\alpha\lambda_6)I_2+(\lambda_1-(1-\alpha)\lambda_5-\alpha\lambda_6)C_2],\\
\frac{d\lambda_2}{dt} =& -A_1-[(1-p)u_1-S(1-u_3)\frac{\beta_1(S+(1-\theta)C_1+I_2+C_2)-\beta_2(I_2+\theta C_2)}{N^2}]\lambda_1\\
&+(\mu+d_1+u_1)\lambda_2\\
&-(1-u_3)(1-\alpha)[\frac{\beta_1 S(S+(1-\theta)C_1+I_2+C_2)}{N^2}-\frac{\beta_2\psi(S+C_1+I_2+C_2)(I_2+\theta C_2)}{N^2}\\
&-\frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1+\mathcal{B}_2}]\lambda_2-(1-u_3)[\frac{\alpha\beta_1 S(S+(1-\theta)C_1+I_2+C_2)}{N^2}+\frac{\beta_2\psi C_1(I_2+\theta C_2)}{N^2}]\lambda_3\\
&-(1-u_1)\delta_1\lambda_4-pu_1\lambda_5\\
&-(1-u_3)(1-\alpha)[\frac{\beta_2(I_2+\theta C_2)(S(\psi-1)+\psi(I_2+C_2))}{N^2}+\frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1+\mathcal{B}_2}]\lambda_5\\
&+\frac{(1-u_3)\alpha\beta_2(S+\psi C_1)(I_2+\theta C_2)}{N^2}\lambda_6,\\
\frac{d\lambda_3}{dt} =& -A_1+S(1-u_3)\frac{\beta_1(-I_1+\theta(S+I_1+I_2+C_2))-\beta_2(I_2+\theta C_2)}{N^2}\lambda_1+(\mu+\phi_1)\lambda_3\\
&+[(1-u_3)(1-\alpha)\frac{\beta_1 S(-I_1+\theta(S+I_1+I_2+C_2))+\beta_2\psi I_1(I_2+\theta C_2)}{N^2}+\phi_1]\lambda_2\\
&-(1-u_3)\frac{\alpha\beta_1 S(-I_1+\theta(S+I_1+I_2+C_2))-\beta_2\psi(I_2+\theta C_2)(S+I_1+I_2+C_2)}{N^2}\lambda_3\\
&+(1-u_3)\frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1+\mathcal{B}_2}\lambda_3-\omega_1\lambda_4\\
&-(1-u_3)(1-\alpha)[\frac{\beta_2(I_2+\theta C_2)(S(\psi-1)+\psi(I_2+C_2))}{N^2}+\frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1+\mathcal{B}_2}]\lambda_5\\
&-(1-u_3)\alpha[\frac{\beta_2(I_2+\theta C_2)(S(\psi-1)+\psi(I_1+I_2+C_2))}{N^2}+\frac{\psi\eta f(\mathcal{B})\mathcal{B}_2}{\mathcal{B}_1+\mathcal{B}_2}]\lambda_6,
\end{aligned}
$$

7

$$
\begin{aligned}
\frac{d\lambda_4}{dt} =& - A_3 + \frac{(1-u_3)\eta S}{(1+\mathcal{B}_1)^2(1+\mathcal{B}_2)}\lambda_1 + (\xi_1 + b_1 u_3)\lambda_4 \\
&- \frac{(1-u_3)\eta f(\mathcal{B})\mathcal{B}_2}{(\mathcal{B}_1+\mathcal{B}_2)^2}[(1-\alpha)((S+\psi I_1)\lambda_2 - (S+\psi(I_1+C_1))\lambda_5) \\
&+ (\alpha S + \psi C_1)\lambda_3 - \alpha(S+\psi C_1)\lambda_6] - (1-u_3)\eta[\frac{\mathcal{B}_1 S((1-\alpha)\lambda_2 + \alpha\lambda_3)}{(1+\mathcal{B}_1)^2(1+\mathcal{B}_2)(\mathcal{B}_1+\mathcal{B}_2)} \\
&+ \frac{\mathcal{B}_2\{-\psi C_1\lambda_3 + \alpha\lambda_6(S+\psi C_1) + \psi(-1+\alpha)\lambda_2 I_1 + (1-\alpha)\lambda_5\ (S+\psi(C_1+I_1))\}}{(1+\mathcal{B}_1)^2(1+\mathcal{B}_2)(\mathcal{B}_1+\mathcal{B}_2)}], \\
\frac{d\lambda_5}{dt} =& - A_2 - [u_2 + S(1-u_3)\frac{\beta_1(I_1+\theta C_1) - \beta_2(S+I_1+C_1+(1-\theta)C_2)}{N^2}]\lambda_1 \\
&+ (1-u3)(1-\alpha)\frac{\beta_1 S(I_1+\theta C_1) + \beta_2\psi I_1(S+I_1+C_1+(1-\theta)C_2)}{N^2}\lambda_2 \\
&+ (1-u3)\frac{\beta_1\alpha S(I_1+\theta C_1) + \beta_2\psi C_1(S+I_1+C_1+(1-\theta)C_2)}{N^2}\lambda_3 \\
&+ [(\mu+d_2+u_2) - \frac{(1-u_3)(1-\alpha)(S+I_1+C_1+(1-\theta)C_2)(S+\psi(I_1+C_1))\beta_2}{N^2}]\lambda_5 \\
&- \frac{(1-u_3)\alpha(S+\psi C_1)(S+I-1+C_1+(1-\theta)C_2)\beta_2}{N^2}\lambda_6 - (1-\epsilon u_2)\delta_2\lambda_7, \\
\frac{d\lambda_6}{dt} =& - A_2 + S(1-u_3)\frac{\beta_2(-I_2+\theta(S+I_1+C_1+I_2)) - \beta_1(I_1+\theta C_1)}{N^2}\lambda_1 \\
&+ (1-u_3)(1-\alpha)\frac{\beta_1 S(I_1+\theta C_1) + \beta_2\psi I_1(-I_2+\theta(C_1+S+I_1+I_2))}{N^2}\lambda_2 \\
&+ (1-u_3)\frac{\alpha\beta_1 S(I_1+\theta C_1) + \beta_2\psi C_1(-I_2+\theta(S+I_1+C_1+I_2))}{N^2}\lambda_3 \\
&- [\frac{(1-u_3)(1-\alpha)(S+\psi(I_1+C_1))(-I_2+\theta(S+I_1+C_1+I_2))\beta_2}{N^2} + \phi_2]\lambda_5 + (\mu+\phi_2)\lambda_6 \\
&- \frac{(1-u_3)\alpha(S+\psi C_1)(-I_2+\theta(S+I_1+C_1+I_2))\beta_2}{N^2}\lambda_6 - \omega_2\lambda_7, \\
\frac{d\lambda_7}{dt} =& - A_3 + \frac{(1-u_3)\eta S}{(1+\mathcal{B}_1)(1+\mathcal{B}_2)^2}\lambda_1 + (\xi_1 + b_1 u_3)\lambda_4 \\
&- \frac{(1-u_3)\eta f(\mathcal{B})\mathcal{B}_1}{(\mathcal{B}_1+\mathcal{B}_2)^2}[(1-\alpha)((S+\psi I_1)\lambda_2 - (S+\psi(I_1+C_1))\lambda_5) \\
&+ (\alpha S + \psi C_1)\lambda_3 - \alpha(S+\psi C_1)\lambda_6] - (1-u_3)\eta[\frac{\mathcal{B}_1 S((1-\alpha)\lambda_2 + \alpha\lambda_3)}{(1+\mathcal{B}_1)(1+\mathcal{B}_2)^2(\mathcal{B}_1+\mathcal{B}_2)} \\
&+ \frac{\mathcal{B}_2\{-\psi C_1\lambda_3 + \alpha\lambda_6(S+\psi C_1) + \psi(-1+\alpha)\lambda_2 I_1 + (1-\alpha)\lambda_5(S+\psi(C_1+I_1))\}}{(1+\mathcal{B}_1)(1+\mathcal{B}_2)^2(\mathcal{B}_1+\mathcal{B}_2)}]
\end{aligned}
\tag{3.2}
$$

Furthermore, the optimal control characterization is

$$\begin{aligned}
u_1^* &= max\left\{0, min\left(\frac{(\lambda_2+(1-p)(\delta_1\lambda_4-\lambda_1)-p\lambda_5)I_1^*}{D_1}, 1\right)\right\} \\
u_2^* &= max\left\{0, min\left(\frac{\epsilon(\lambda_5+\delta_2\lambda_7-\lambda_1)I_2^*}{D_2}, 1\right)\right\} \\
u_3^* &= max\left\{0, min\left(\tilde{u}_3, 1\right)\right\}
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\tilde{u}_3 = \frac{\eta f(\mathcal{B})}{D_3}&[-\lambda_1 S^* + \frac{S^*\mathcal{B}_1^*(1-\alpha)\lambda_2+\alpha\lambda_3}{\mathcal{B}_1^*+\mathcal{B}_2^*} \\
&+\frac{\mathcal{B}_2^*\left(\psi(1-\alpha)(\lambda_5-\lambda_2)I_1^*+((1-\alpha)\lambda_5+\alpha\lambda_6)(S^*+\psi C_1^*)-\psi\lambda_3 C_1^*\right)}{\mathcal{B}_1^*+\mathcal{B}_2^*}] \\
&+\frac{b_1\lambda_4\mathcal{B}_1^*+b_2\lambda_7\mathcal{B}_2^*}{D_3} - \frac{\beta_1}{D_3N^*}(I_1^*+\theta C_1^*)(\lambda_1-\lambda_2+\alpha(\lambda_2-\lambda_3))S^* \\
&-\frac{\beta_2}{D_3N^*}(I_2^*+\theta C_2^*)[\lambda_1 S^*+\psi\lambda_3 C_1^*+(S^*+\psi C_1^*)(-\lambda_5+\alpha(\lambda_5-\lambda_6)) \\
&+\psi(1-\alpha)(\lambda_2-\lambda_5)I_1^*].
\end{aligned}$$

4 Numerical results

This section presents the numerical simulation results by solving the optimality system, which comprises the state system (2.2), adjoint system (3.2), control characterization (3.3), and corresponding initial and final conditions, using the forward-backward sweep method [27, 28].

For numerical simulations, we consider the model parameter values presented in Table 2.

Table 2: Model parameter values used in numerical simulations [24], the unit is per week if appropriate.

$\alpha = 0.3$	$\beta_1 = 0.006$	$\beta_2 = 0.0052$	$\delta_1 = 1.0$
$\delta_2 = 1.05$	$\eta = 1.379 \times 10^{-10}$	$\theta = 0.35$	$\mu = 0.0005$
$\xi_1 = 0.2415$	$\xi_2 = 0.2415$	$\pi = 10^5/52$	$\phi_1 = 0.00096$
$\phi_2 = 0.0017$	$\psi = 0.95$	$\omega_1 = 0.05$	$\omega_2 = 0.06$
$d_1 = 0.00125$	$d_2 = 0.002$	$p = 0.1$	

Additionally, the following parameter values are chosen:

$$A_1 = A_2 = 10,\ A_3 = 25,\ D_1 = 5,\ D_2 = 8,\ D_3 = 10,\ b_1 = 0.2,\ b_2 = 0.1,$$
$$\epsilon = 0.75,\ T = 100 \text{ weeks}.$$

The following control strategies are explored in order to determine the optimum strategy that significantly reduces typhoid transmission:

A: Treatment of the symptomatic individuals in each strain (u_1, u_2) only;

B: Employing sanitation and proper hygiene (u_3) only;

C: Employing all the three control interventions $(u_1,\ u_2,\ u_3)$.

The control profile for each control strategy is shown in Fig. 2, and the effect of each control strategy on the reduction of infection is depicted in Fig. 3.

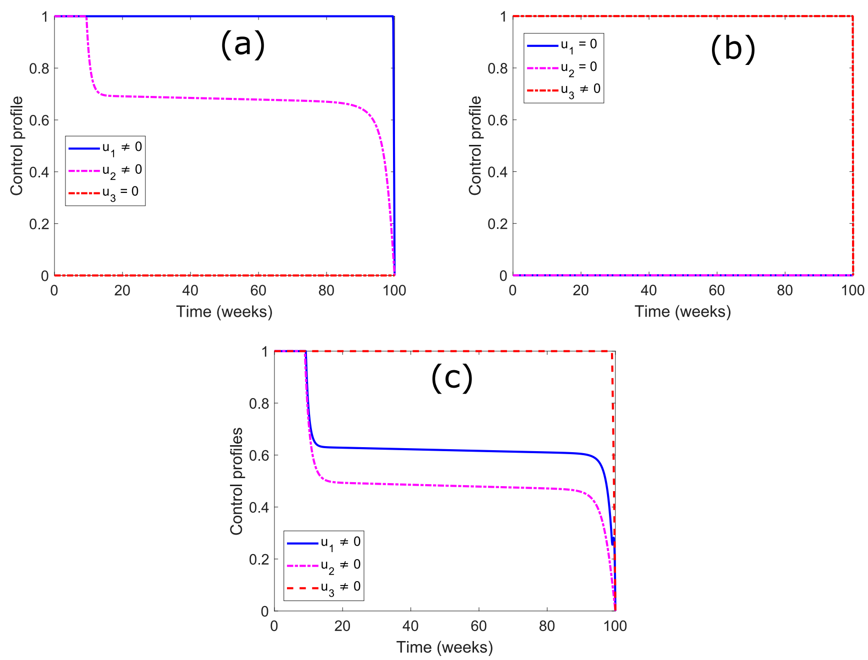

Figure 2: Control profile for (a) optimal treatment only, (b) optimal sanitation and proper hygiene only, and (c) optimal treatment combined with sanitation and proper hygiene

Our simulation results reveal that the combination of all control interventions highly influenced the symptomatic individuals and environmental bacteria in both the strains. However, there are still a significant number of asymptomatic carriers in both the strains, which play an important role in the evolution and transmission of typhoid infections. This reflects that asymptomatic carriers may have long-term impacts on the spread of typhoid infection even in the presence of the two control interventions.

4.1 Efficiency analysis

Here an efficiency analysis is performed to determine the best control strategy without considering costs associated with each control strategy [29, 30]. So, we

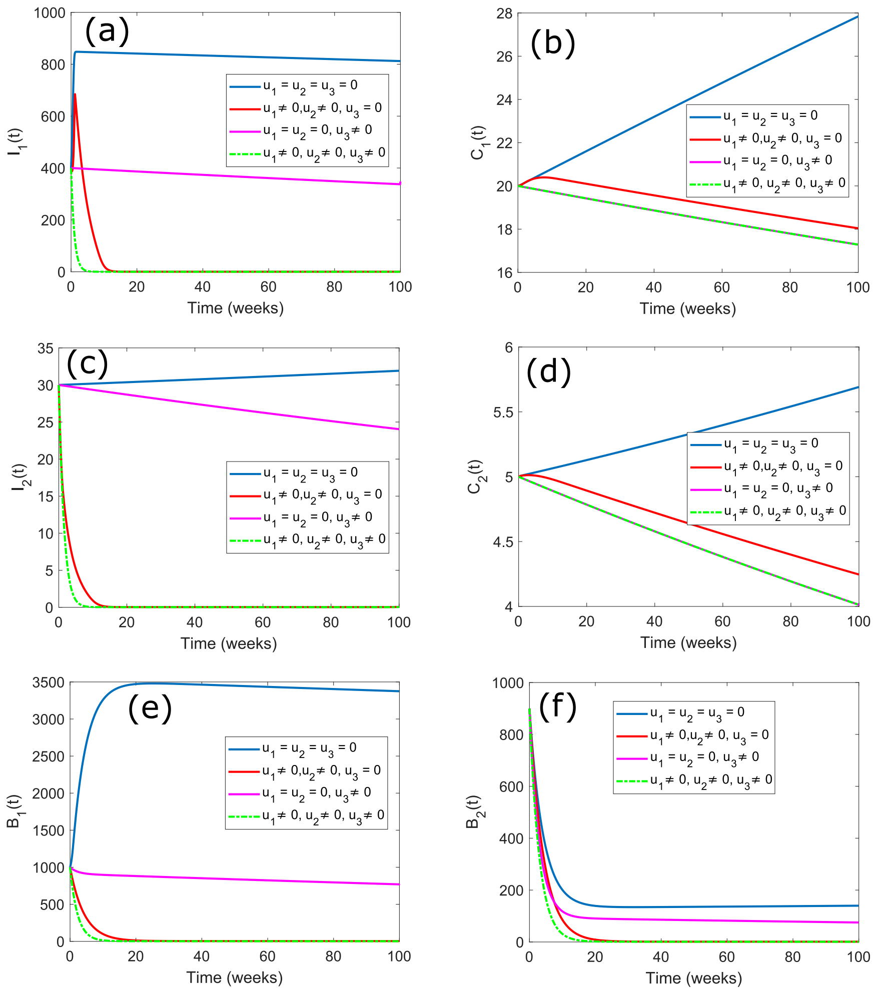

Figure 3: Effect of each control strategy on reducing the number of infectious humans and bacteria: (a) Symptomatic individuals with sensitive strain, (b) Asymptomatic carriers with sensitive strain, (c) Symptomatic individuals with resistant strain, (d) Asymptomatic carriers with resistant strain, (e) Sensitive strain of bacteria in the environment, (f) Resistant strain of bacteria in the environment

investigate the impact of each control strategies on the reduction of infectious humans and bacteria by introducing the efficiency index, $\mathbb{F}$. The efficiency index for human and bacteria population in the strain j are, respectively, computed as:

$$\mathbb{F}^{I_j+C_j} = \left(1 - \frac{A_c^{I_j+C_j}}{A_o^{I_j+C_j}}\right) \times 100 \quad \text{and} \quad \mathbb{F}^{\mathcal{B}_j} = \left(1 - \frac{A_c^{\mathcal{B}_j}}{A_o^{\mathcal{B}_j}}\right) \times 100$$

where

$$A^{I_j+C_j} = \int_0^T \Big(I_j(t) + C_j(t)\Big)dt \quad \text{and} \quad A^{\mathcal{B}_j} = \int_0^T \mathcal{B}_j(t)dt$$

represent the cumulative number of infectious humans and bacteria with strain j, respectively, during the time interval $[0, T]$. The efficiency index is calculated for human and bacteria population in both the strains and presented in Table 3. Note that the control strategy with the highest efficiency index will be the best. From Table 3, it follows that strategy C is the most effective for reducing

Table 3: Efficiency index

Strategy	$A_c^{I_1+C_1}$	$A_c^{\mathcal{B}_1}$	$\mathbb{F}^{I_1+C_1}$	$\mathbb{F}^{\mathcal{B}_1}$	$A_c^{I_2+C_2}$	$A_c^{\mathcal{B}_2}$	$\mathbb{F}^{I_2+C_2}$	$\mathbb{F}^{\mathcal{B}_2}$
No control	87084	340679	0.0	0.0	3617	16767	0.0	0.0
A	1958	4567	97.75	98.66	508	3933	85.96	76.54
B	6541	12887	92.45	96.22	3143	10817	13.11	35.49
C	1918	2525	97.80	99.23	493	2792	86.37	83.35

the disease burden, followed by strategy A and strategy B.

5 Conclusions

The novelty of this study is its ability to predict the optimal level of control interventions that include treatment and proper hygiene/ sanitation practices. On the basis of sensitivity analysis of a two-strain typhoid model incorporating symptomatic infection, asymptomatic carriers, and environmental bacteria, some control measures were suggested in [24]. Accordingly, the time-dependent functions representing the treatment of sensitive and resistant strains are considered as control measures. Proper hygiene and sanitation are also considered as another control measure to prevent contamination of food and water. The necessary and sufficient conditions for the existence of optimal controls are established and the optimality system is developed. The characterization of the optimal control is determined by the Pontryagin's maximum principle. The numerical simulations are performed for every single control and combination of the two controls. The simulation results reveal that with the combination of the two control interventions, the number of symptomatic individuals and doses of *S.* Typhi bacteria in both the strains reduced to near zero. However, there is still a significant number of asymptomatic carriers in both strains, which play an essential role in the evolution and transmission of typhoid infections. So, additional preventive measures need to be implemented in order to further reduce the population of asymptomatic carriers. The effects of each control strategy on the reduction of infection in both the strains is investigated through efficiency analysis. From the study results, we conclude that the fight against a two-strain typhoid infection requires some control interventions that reduce the number of asymptomatic carriers to near zero, along with optimal treatment combined with sanitation and proper hygiene.

Acknowledgement

The first author is financially supported by Indian Council for Cultural Relations (ICCR) during his Ph.D. study. The authors are thankful to the anonymous reviewers for their valuable comments and suggestions, which helped us to improve the quality of our original manuscript.

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Bicomplex Laplace Transform of Fractional Order, Properties and applications

Urvashi Purohit Sharma and Ritu Agarwal*

December 25, 2021

Abstract

The aim of this research article is to define bicomplex Laplace transform of fractional order or fractional Laplace transform by the application of the Mittag-Leffler function. Various properties of bicomplex fractional Laplace transform along with the convolution theorem have also been given. Inverse bicomplex fractional Laplace transform has also been defined. Application of bicomplex fractional Laplace transform in the solution of diffusion equation has been given.

Key words: Bicomplex numbers, Fractional derivative, fractional Laplace transform, Mittag-Leffler function.
Mathematics Subject Classification(2010): 30G35, 44A10, 33E12.

1 Introduction

In recent years, mathematicians and physicists have focused their efforts on bicomplex algebra. In 1882, Segre [25] introduced bicomplex numbers. Detailed study of bicomplex numbers are presented by Riley [20], Price [18], Rönn [24]. A bicomplex number is defined as an ordered pair of complex numbers, similar like how a complex number is defined as an ordered pair of real numbers.

In recent years, the fractional order differential equations with boundary conditions have gained more attention in a variety of scientific and engineering domains. The Mittag-Leffler function (see, e.g. [7, 10]) has an important contribution in the study of fractional calculus, it has been used to solve fractional order differential equations. The Mittag-Leffler function has caught the interest of a number of authors working in the field of fractional calculus (FC) and its applications such as, usage of a fractional operator involving Mittag–Leffler function for the generalized Casson fluid flow [29], to established the fractional calculus operators with Appell function kernels and Caputo-type fractional differential operators [16], Epidemiological analysis of fractional order COVID-19 model with Mittag-Leffler kernel [6]. In recent developments authors have worked on

*Corresponding author: Department of Mathematics, Malaviya National Institute of Technology, Jaipur-302017, INDIA, Email: ragarwal.maths@mnit.ac.in

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the area of fractional calculus such as, to study a guava fruit model associated with a non-local additionally non-singular fractional derivative [27], the approximate solution of nonlinear Caudrey-Dodd-Gibbon equation of fractional order [28], analysis of fractional blood alcohol model [26].

Many authors have studied the applications of the fractional integral transform [9, 13, 14, 17, 23]. Efforts have been made by authors to introduce the Mittag-Leffler function (ML function) in bicomplex space along with applications to fractional calculus and integral transform [4, 5]. In 2011 bicomplex Laplace transform is introduced by Kumar et al. [15] and its convolution theorem and applications in bicomplex space are discussed by Agarwal et al. [1], bicomplex double Laplace transform is derived by Goswami et al. [8].

Following the path, efforts are made to extend the fractional Laplace transform in bicomplex space. Fractional Laplace transformation method is a effective and strong tool for finding a solution of the fractional differential equation. In this article bicomplex fractional Laplace transform and its properties in bicomplex space are introduced.

2 Preliminaries

2.1 Bicomplex Numbers

Definition 2.1 (Bicomplex Number)**.** A bicomplex number $\xi \in \mathbb{T}$ can be written as [25]

$$\xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3, \text{ where } \; x_0,\; x_1,\; x_2,\; x_3 \; \in \mathbb{R}. \tag{2.1}$$

Here $\mathbb{T}$, $\mathbb{R}$ represents the set of bicomplex numbers and real numbers respectively.

We shall use the notations, $x_0 = \mathrm{Re}(\xi),\; x_1 = \mathrm{Im}_{i_1}(\xi),\; x_2 = \mathrm{Im}_{i_2}(\xi),\; x_3 = \mathrm{Im}_j(\xi)$.

Idempotent representation is particularly important since it allows for term-by-term addition, multiplication, and division.

Definition 2.2 (Idempotent Representation)**.** Every bicomplex number has following idempotent representation [18]

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2. \tag{2.2}$$

Hence if $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$ then

$$\xi = \xi_1 e_1 + \xi_2 e_2, \tag{2.3}$$

where $e_1,\; e_2$ are idempotent elements in $\mathbb{T}$ such that $e_1 = \dfrac{1 + i_1 i_2}{2} = \dfrac{1 + j}{2}$, $e_2 = \dfrac{1 - i_1 i_2}{2} = \dfrac{1 - j}{2}$ and $e_1 + e_2 = 1,\; e_1.e_2 = 0$.

Projection Mappings

$P_1 : \mathbb{T} \to T_1 \subseteq \mathbb{C}$, $P_2 : \mathbb{T} \to T_2 \subseteq \mathbb{C}$ for a bicomplex number $\xi = z_1 + i_2 z_2$ are

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given by (see, e.g. [2, 22]):

$$P_1(\xi) = P_1(z_1 + i_2 z_2) = (z_1 - i_1 z_2) \in T_1, \tag{2.4}$$

and

$$P_2(\xi) = P_2(z_1 + i_2 z_2)(z_1 + i_1 z_2) \in T_2, \tag{2.5}$$

where

$$T_1 = \{\xi_1 = z_1 - i_1 z_2 \, | z_1, z_2 \in \mathbb{C}\} \text{ and } T_2 = \{\xi_2 = z_1 + i_1 z_2 \, | z_1, z_2 \in \mathbb{C}\}. \tag{2.6}$$

2.2 Bicomplex One-Parameter Mittag-Leffler Function

The bicomplex one parameter ML function defined Agarwal et al. [5] is given by

$$\mathbb{E}_\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(\alpha n + 1)}, \tag{2.7}$$

where $\xi, \alpha \in \mathbb{T},\ \xi = z_1 + i_2 z_2$ and $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$.

2.3 Modified Riemann- Liouville Derivative

Definition 2.3 (Modified Riemann-Liouville Derivative, [12])**.** Let $g : \mathbb{R} \to \mathbb{R},\ y \to g(y)$ represents a continuous function (not necessarily differentiable) function

1. If $g(y)$ is a constant M then its fractional derivative of order μ is given by

$$^J D_y^\mu M = \begin{cases} \frac{M}{\Gamma(1-\mu) y^\mu} & if\ \mu \leq 0, \\ 0 & if\ \mu > 0. \end{cases}$$

2. If $g(y)$ is not a constant then its fractional derivative of order μ is given by

$$^J D_y^\mu (g(y) - g(0)) = \frac{1}{\Gamma(-\mu)} \int_0^y \frac{g(\zeta) d\zeta}{(y-\zeta)^{\mu+1}},\ \mu < 0, \tag{2.8}$$

$$^J D_y^\mu (g(y) - g(0)) = {}^J D_y^\mu g(y) = ^J D_y(g^{\mu-1}(y)),\ \mu > 0, \tag{2.9}$$

$$(g^\mu(y)) = \left(g^{\mu-n}(y)\right)^{(n)},\ n \leq \mu \leq n+1,\ n \geq 1. \tag{2.10}$$

2.4 Laplace Transform of Fractional Order

Let $g(x)$ denotes the function which vanishes for negative values of the variable x. Its Laplace transform (LT) of order α is defined by the expression (see, e.g. [13, 14, 19]), when it is finite,

$$L_\alpha(g(x)) = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) g(x) (dx)^\alpha,\ 0 < \alpha < 1, \tag{2.11}$$

where $s \in \mathbb{C}$.
Sufficient condition for this integral to be finite is that (see, e.g.[13])

$$\int_0^\infty |g(x)|(dx)^\alpha < M < \infty. \tag{2.12}$$

If $g(u)$ is a continuous function, the integral with respect to $(du)^\alpha$ is defined as (see, e.g. [14]) the fractional differential equation's solution $y(u)$

$$dy = g(t)(du)^\alpha,\ x \geq 0,\ y(0) = 0, \tag{2.13}$$

where

$$y = \int_0^u g(v)(dv)^\alpha = \alpha \int_0^u \frac{g(v)}{(u-v)^{1-\alpha}} dv,\ 0 < \alpha < 1. \tag{2.14}$$

Jumarie [11] gave the the proof of the above result as follows:

$$x^{(\alpha)}(u) = g(u),\ 0 < \alpha \leq 1. \tag{2.15}$$

Its solution is obtained by fractional derivative as

$$x(u) = D^{-\alpha} g(u) = \frac{1}{\Gamma\alpha} \int_0^u (u-t)^{\alpha-1} g(t) dt. \tag{2.16}$$

Again

$$d^\alpha x = g(u)(du)^\alpha, \tag{2.17}$$

or

$$\Gamma(\alpha+1) dx = g(u)(du)^\alpha. \tag{2.18}$$

On integrating

$$x(u) = \frac{1}{\Gamma(\alpha+1)} \int_0^u g(t)(dt)^\alpha. \tag{2.19}$$

From equations (2.16) and (2.19), equation (2.14) can be obtained.

3 Bicomplex Laplace transform of Fractional order

In this section we introduce the bicomplex fractional Laplace transform with convergence conditions using the bicomplex ML function.

Definition 3.1 (Class $\mathcal{C}$)**.** Let $\mathcal{C}$ be the class of bicomplex-valued functions defined with the following properties, for any $f \in \mathcal{C}$

1. $f(x)$ vanishes for negative values of the variable x.
2. f is piecewise continuous in the interval $(0, a]$ for any $a \in (0, +\infty)$.

3. $\int_0^\infty |f(x)|_j (dx)^\alpha < M < \infty$.

Now we introduce the bicomplex Laplace transform of fractional order α as follows:
Let Laplace transform of order α of $f(t) \in \mathcal{C}$ for $t \geq 0$ can be written as

$$L_\alpha(f(t))_{s_1} = F_\alpha(s_1) = \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha) f(t)(dt)^\alpha, \ 0 < \alpha < 1, \tag{3.1}$$

where $s_1 \in \mathbb{C}$ and take another LT of order α of $f(t) \in \mathcal{C}$ for $s_2 \in \mathbb{C}$

$$L_\alpha(f(t))_{s_2} = F_\alpha(s_2) = \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha) f(t)(dt)^\alpha, \ 0 < \alpha < 1. \tag{3.2}$$

Now we take linear combination of $F_\alpha(s_1)$ and $F_\alpha(s_2)$ with e_1 and e_2 such as

$$\begin{aligned}
&L_\alpha(f(t))_{s_1} e_1 + L_\alpha(f(t))_{s_2} e_2 \\
&\quad = F_\alpha(s_1)e_1 + F_\alpha(s_2)e_2 \\
&\quad = \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha) f(t)(dt)^\alpha e_1 + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha) f(t)(dt)^\alpha e_2 \\
&\quad = \int_0^\infty \left(E_\alpha(-s_1^\alpha t^\alpha)e_1 + E_\alpha(-s_2^\alpha t^\alpha)e_2\right) f(t)(dt)^\alpha \\
&\quad = \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) f(t)(dt)^\alpha \\
&\quad = F_\alpha(\xi) \\
&\quad = L_\alpha(f(t))_\xi,
\end{aligned} \tag{3.3}$$

where $\xi = s_1 e_1 + s_2 e_2 \in \mathbb{T}$.
Since $F_\alpha(s_1)$ and $F_\alpha(s_2)$ are complex valued functions which are convergent and analytic for respectively, so by application of decomposition theorem of Ringleb [21], (see, e.g. [20]) bicomplex valued function $F_\alpha(\xi) = F_\alpha(s_1)e_1 + F_\alpha(s_2)e_2$ will be convergent and analytic.

Definition 3.2 (Bicomplex Laplace Transform of Fractional Order). Let $g(t) \in \mathcal{C}$ be a bicomplex valued function. Then bicomplex Laplace transform of fractional order α of $g(t)$ for $t \geq 0$ can be defined as

$$L_\alpha(g(t))_\xi = G_\alpha(\xi) = \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) g(t)(dt)^\alpha = \lim_{\mathcal{M}\to\infty} \int_0^{\mathcal{M}} E_\alpha(-\xi^\alpha t^\alpha) g(t)(dt)^\alpha, \tag{3.4}$$

where $0 < \alpha < 1, \ \xi = s_1 e_1 + s_2 e_2 \in \mathbb{T}, \ s_1, \ s_2 \in \mathbb{C}$.

3.1 Some Basic Properties of Bicomplex Fractional Laplace Transform

Theorem 3.3 (Linearity Property). *Let $F_\alpha(\xi)$ and $G_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of class $\mathcal{C}$ functions $f(t)$ and $g(t)$ respectively, then*

$$L_\alpha\left(f(t)+g(t)\right)=F_\alpha(\xi)+G_\alpha(\xi). \tag{3.5}$$

Proof. Let $\xi=s_1e_1+s_2e_2\in\mathbb{T},\ s_1,\ s_2\in\mathbb{C}$ and $0<\alpha<1$ then

$$\begin{aligned}
L_\alpha\left(f(t)+g(t)\right)&=\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)\left(f(t)+g(t)\right)(dt)^\alpha\\
&=\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)\left(f_1(t)+g_1(t)\right)(dt)^\alpha e_1\\
&\quad+\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)\left(f_2(t)+g_2(t)\right)(dt)^\alpha e_2\\
&=\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1+\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)g_1(t)(dt)^\alpha e_1\\
&\quad+\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2+\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)g_2(t)(dt)^\alpha e_2\\
&=\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)f(t)(dt)^\alpha+\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)g(t)(dt)^\alpha\\
&=L_\alpha\left(f(t)\right)+L_\alpha\left(g(t)\right)\\
&=F_\alpha(\xi)+G_\alpha(\xi).
\end{aligned} \tag{3.6}$$

□

Theorem 3.4. *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of function $f(t)\in\mathcal{C}$ and $\xi=s_1e_1+s_2e_2\in\mathbb{T},\ s_1,\ s_2\in\mathbb{C},\ 0<\alpha<1$ and k is a constant then*

$$L_\alpha\left(kf(t)\right)=kF_\alpha(\xi). \tag{3.7}$$

Proof. Let $\xi=s_1e_1+s_2e_2\in\mathbb{T},\ s_1,\ s_2\in\mathbb{C}$ and $0<\alpha<1$, then

$$\begin{aligned}
L_\alpha\left(kf(t)\right)&=\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)\left(kf(t)\right)(dt)^\alpha\\
&=\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)\left(kf_1(t)\right)(dt)^\alpha e_1+\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)\left(kf_2(t)\right)(dt)^\alpha e_2\\
&=k\int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1+k\int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2\\
&=k\int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)f(t)(dt)^\alpha\\
&=kL_\alpha\left(f(t)\right)\\
&=kF_\alpha(\xi).
\end{aligned} \tag{3.8}$$

□

Theorem 3.5 (Bicomplex Fractional Laplace Transform of Derivatives)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional LT of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$L_\alpha\left({}^JD^\alpha f(t)\right) = \xi^\alpha F_\alpha(\xi) - f(0), \tag{3.9}$$

where ${}^JD^\alpha$ is defined in the definition (2.3).

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T}$ and $0 < \alpha < 1$ then

$$\begin{aligned} L_\alpha\left({}^JD^\alpha f(t)\right) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)\left(D^\alpha f(t)\right)(dt)^\alpha \\ &= \left[f(t)E_\alpha(-\xi^\alpha t^\alpha)\right]_0^\infty - \int_0^\infty f(t)\left(-\xi^\alpha E_\alpha(-\xi^\alpha t^\alpha)\right)(dt)^\alpha \\ &= -f(0) + \xi^\alpha \int_0^\infty f(t)E_\alpha(-\xi^\alpha t^\alpha)(dt)^\alpha \\ &= \xi^\alpha L_\alpha\left(f(t)\right) - f(0) \\ &= \xi^\alpha F_\alpha(\xi) - f(0). \end{aligned} \tag{3.10}$$

□

Corollary 3.6. *Let $F_\alpha(\xi)$ be the bicomplex fractional LT of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$L_\alpha\left({}^JD^{2\alpha} f(t)\right) = \xi^{2\alpha} F_\alpha(\xi) - \xi^\alpha f(0) - f^\alpha(0), \tag{3.11}$$

where ${}^JD^{2\alpha}$ is defined in the definition (2.3).

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ and $D^\alpha f(t) = F(t)$ then

$$\begin{aligned} L_\alpha\left({}^JD^{2\alpha} f(t)\right) &= L_\alpha\left({}^JD^\alpha F(t)\right) \\ &= \xi^\alpha L_\alpha\left(F(t)\right) - F(0) \\ &= \xi^\alpha L_\alpha\left({}^JD^\alpha f(t))\right) - f^\alpha(0) \\ &= \xi^\alpha\left(\xi^\alpha L_\alpha\left(f(t)\right) - f(0))\right) - f^\alpha(0) \\ &= \xi^{2\alpha} L_\alpha\left(f(t)\right) - \xi^\alpha f(0) - f^\alpha(0) \\ &= \xi^{2\alpha} F_\alpha(\xi) - \xi^\alpha f(0) - f^\alpha(0). \end{aligned} \tag{3.12}$$

□

Proceeding in similar manner, we obtain the result contained in the following corollary:

Corollary 3.7. *Let $F_\alpha(\xi)$ be the bicomplex fractional LT of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$\begin{aligned} &L_\alpha\left({}^JD^{n\alpha} f(t)\right) \\ &= \xi^{n\alpha} F_\alpha(\xi) - \left(\xi^{n\alpha-\alpha} f(0) + \xi^{n\alpha-2\alpha} f^\alpha(0) + \xi^{n\alpha-3\alpha} f^{2\alpha}(0) + \cdots + f^{n\alpha-\alpha}(0)\right), \end{aligned} \tag{3.13}$$

where ${}^JD^{n\alpha}$ is defined in the definition (2.3).

Theorem 3.8 (Change of Scale Property)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C},\ a > 0$ and $0 < \alpha < 1$ then*

$$L_\alpha(f(at)) = (1/a)^\alpha F_\alpha\left(\frac{\xi}{a}\right). \tag{3.14}$$

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equations (3.4) and (2.14) we have

$$\begin{aligned} L_\alpha(f(at)) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) f(at)(dt)^\alpha \\ &= \lim_{\mathcal{M}\to\infty} \int_0^{\mathcal{M}} E_\alpha(-\xi^\alpha t^\alpha) f(at)(dt)^\alpha \\ &= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-\xi^\alpha t^\alpha) f(at)(dt) \\ &= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_1^\alpha t^\alpha) f_1(at)(dt)e_1 \\ &\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_2^\alpha t^\alpha) f_2(at)(dt)e_2, \end{aligned} \tag{3.15}$$

putting $at = x \Longrightarrow dt = \frac{dx}{a},\ a > 0$,

$$\begin{aligned} L_\alpha(f(at)) &= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \left(\mathcal{M} - \frac{x}{a}\right)^{\alpha-1} E_\alpha\left(-s_1^\alpha \frac{x^\alpha}{a^\alpha}\right) f_1(x)\frac{dx}{a}e_1 \\ &\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \left(\mathcal{M} - \frac{x}{a}\right)^{\alpha-1} E_\alpha\left(-s_2^\alpha \frac{x^\alpha}{a^\alpha}\right) f_2(x)\frac{dx}{a}e_2 \\ &= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \frac{(a\mathcal{M}-x)^{\alpha-1}}{a^{\alpha-1}} E_\alpha\left(-s_1^\alpha \frac{x^\alpha}{a^\alpha}\right) f_1(x)\frac{dx}{a}e_1 \\ &\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{a\mathcal{M}} \frac{(a\mathcal{M}-x)^{\alpha-1}}{a^{\alpha-1}} E_\alpha\left(-s_2^\alpha \frac{x^\alpha}{a^\alpha}\right) f_2(x)\frac{dx}{a}e_1 \\ &= (1/a)^\alpha F_\alpha\left(\frac{\xi_1}{a}\right)e_1 + (1/a)^\alpha F_\alpha\left(\frac{\xi_2}{a}\right)e_2 \\ &= (1/a)^\alpha F_\alpha\left(\frac{\xi}{a}\right). \end{aligned} \tag{3.16}$$

□

Theorem 3.9 (Shifting Property)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C},\ c > 0$ and $0 < \alpha < 1$ then*

$$L_\alpha(f(t-c)) = E_\alpha(\xi^\alpha c^\alpha) F_\alpha(\xi). \tag{3.17}$$

Proof. Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equations (3.4) and (2.14) we have

$$\begin{aligned}
L_\alpha(f(t-c)) &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha) f(t-c)(dt)^\alpha \\
&= \lim_{\mathcal{M}\to\infty} \int_0^{\mathcal{M}} E_\alpha(-\xi^\alpha t^\alpha) f(t-c)(dt)^\alpha \\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-\xi^\alpha t^\alpha) f(t-c)(dt) \qquad (3.18) \\
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_1^\alpha t^\alpha) f_1(t-c)(dt)e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}} (\mathcal{M}-t)^{\alpha-1} E_\alpha(-s_2^\alpha t^\alpha) f_2(t-c)(dt)e_2.
\end{aligned}$$

Putting $t - c = x \Longrightarrow dt = dx$

$$\begin{aligned}
&= \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_1^\alpha (x+c)^\alpha) f_1(x)(dx)e_1 \\
&\quad + \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_2^\alpha (x+c)^\alpha) f_2(x)(dx)e_2 \\
&= E_\alpha(-s_1^\alpha c^\alpha) \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_1^\alpha x^\alpha) f_1(x)(dx)e_1 \qquad (3.19) \\
&\quad + E_\alpha(-s_2^\alpha c^\alpha) \lim_{\mathcal{M}\to\infty} \alpha \int_0^{\mathcal{M}-c} (\mathcal{M}-x-c)^{\alpha-1} E_\alpha(-s_2^\alpha x^\alpha) f_2(x)(dx)e_2 \\
&= E_\alpha(\xi^\alpha c^\alpha) L_\alpha(f(t)) \\
&= E_\alpha(\xi^\alpha c^\alpha) F_\alpha(\xi).
\end{aligned}$$

□

Theorem 3.10 (Bicomplex Fractional Laplace Transform of Integrals)**.** *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ 0 < \alpha < 1$ then*

$$L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right) = \frac{1}{\xi^\alpha \Gamma(1+\alpha)} L_\alpha(f(t)). \qquad (3.20)$$

Proof. Since

$$ {}^J D_t^\alpha \int_0^t f(v)(dv)^\alpha = \alpha!\, f(t), \qquad (3.21)$$

by using equation (3.9)

$$\begin{aligned}
L_\alpha\left({}^J D_t^\alpha \int_0^t f(v)(dv)^\alpha\right) &= \xi^\alpha L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right), \\
L_\alpha(\alpha!\, f(t)) &= \xi^\alpha L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right).
\end{aligned} \qquad (3.22)$$

Hence,

$$L_\alpha\left(\int_0^t f(v)(dv)^\alpha\right) = \Gamma(\alpha+1)\xi^{-\alpha}L_\alpha\left(f(t)\right). \tag{3.23}$$

□

Theorem 3.11. *Let* $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ *and* $0 < \alpha < 1$ *then*

(i) $L_\alpha(t^\alpha f(t)) = -{}^J D_\xi^\alpha L_\alpha(f(t))$,

(ii) $L_\alpha(E_\alpha(-c^\alpha t^\alpha)f(t))_\xi = F_\alpha(\xi + c)$,

(iii) $L_\alpha(-t^\alpha f(t)) = {}^J D_\xi^\alpha L_\alpha(f(t))$.

Proof. (i) Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equation (3.4) we have

$$\begin{aligned}
&L_\alpha(t^\alpha f(t))\\
&\quad= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)t^\alpha f(t)(dt)^\alpha\\
&\quad= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)t^\alpha f_1(t)(dt)^\alpha e_1 + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)t^\alpha f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_{s_1}^\alpha \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1 -\ {}^J D_{s_2}^\alpha \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_\xi^\alpha L_\alpha(f(t)).
\end{aligned} \tag{3.24}$$

(ii) Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equation (3.4) we have

$$\begin{aligned}
L_\alpha(E_\alpha(-c^\alpha t^\alpha)f(t))_\xi &= \int_0^\infty E_\alpha(-\xi^\alpha t^\alpha)E_\alpha(-c^\alpha t^\alpha)f(t)(dt)^\alpha\\
&= \int_0^\infty E_\alpha(-(\xi + c)^\alpha t^\alpha)f(t)(dt)^\alpha\\
&= F_\alpha(\xi + c).
\end{aligned} \tag{3.25}$$

(iii) Let $\xi = s_1e_1 + s_2e_2 \in \mathbb{T},\ s_1,\ s_2 \in \mathbb{C}$ and $0 < \alpha < 1$ then from equation (3.4) we have

$$\begin{aligned}
&L_\alpha(-t^\alpha f(t))\\
&\quad= \int_0^\infty E_\alpha(-\xi^\alpha(-t)^\alpha)t^\alpha f(t)(dt)^\alpha\\
&\quad= \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)t^\alpha f_1(t)(dt)^\alpha e_1 + \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)t^\alpha f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_{s_1}^\alpha \int_0^\infty E_\alpha(-s_1^\alpha t^\alpha)f_1(t)(dt)^\alpha e_1 -\ {}^J D_{s_2}^\alpha \int_0^\infty E_\alpha(-s_2^\alpha t^\alpha)f_2(t)(dt)^\alpha e_2\\
&\quad= -\ {}^J D_\alpha^\xi L_\alpha(f(t)).
\end{aligned} \tag{3.26}$$

□

3.2 Convolution Theorem

Convolution is a mathematical operation on two functions $f,\ g$. which is useful in signal theory, image processing. Convolution of order μ of the functions $f(t),\ g(t)$ defined by Jumarie [14] given by

$$(f * g)(t) = \int_0^t f(t-v)g(v)(dv)^{\mu}. \tag{3.27}$$

Theorem 3.12. *Let $f,\ g \in \mathcal{C}$ and Let $F_\alpha(\xi)$ and $G_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of functions $f(t)$ and $g(t)$ respectively, then*

$$L_\alpha (f * g)(t) = F_\alpha(\xi)G_\alpha(\xi) = L_\alpha(f(t))\, L_\alpha(g(t)). \tag{3.28}$$

Proof.

$$\begin{aligned}
L_\alpha (f(t) * g(t)) &= \int_0^\infty (dt)^\alpha E_\alpha(-\xi^\alpha t^\alpha) \int_0^t f(t-v)g(v)(dv)^\alpha \\
&= \int_0^\infty (dt)^\alpha E_\alpha(-(s_1e_1+s_2e_2)^\alpha t^\alpha) \int_0^t f(t-v)g(v)(dv)^\alpha \\
&= \left(\int_0^\infty (dt)^\alpha E_\alpha(-s_1^\alpha t^\alpha) \int_0^t f(t-v)g(v)(dv)^\alpha\right) e_1 \\
&\quad + \left(\int_0^\infty (dt)^\alpha E_\alpha(-s_2^\alpha t^\alpha) \int_0^t f(t-v)g(v)(dv)^\alpha\right) e_2 \\
&= \left(\int_0^\infty (dt)^\alpha E_\alpha(-s_1^\alpha (t-v)^\alpha) E_\alpha(-s_1^\alpha v^\alpha) \int_0^t f(t-v)g(v)(dv)^\alpha\right) e_1 \\
&\quad + \left(\int_0^\infty (dt)^\alpha E_\alpha(-s_2^\alpha (t-v)^\alpha) E_\alpha(-s_2^\alpha v^\alpha) \int_0^t f(t-v)g(v)(dv)^\alpha\right) e_2.
\end{aligned} \tag{3.29}$$

Put $p = t - v,\ q = v$, to obtain

$$\begin{aligned}
L_\alpha (f(t) * g(t)) &= \left(\int_0^\infty \int_0^\infty (dp)^\alpha E_\alpha(-s_1^\alpha p^\alpha) E_\alpha(-s_1^\alpha q^\alpha) f(p)g(q)(dq)^\alpha\right) e_1 \\
&\quad + \left(\int_0^\infty \int_0^\infty (dp)^\alpha E_\alpha(-s_2^\alpha p^\alpha) E_\alpha(-s_2^\alpha q^\alpha) f(p)g(q)(dq)^\alpha\right) e_2 \\
&= \left(\int_0^\infty \int_0^\infty E_\alpha(-s_1^\alpha p^\alpha) E_\alpha(-s_1^\alpha q^\alpha) f(p)g(q)(dp)^\alpha(dq)^\alpha\right) e_1 \\
&\quad + \left(\int_0^\infty \int_0^\infty E_\alpha(-s_2^\alpha p^\alpha) E_\alpha(-s_2^\alpha q^\alpha) f(p)g(q)(dp)^\alpha(dq)^\alpha\right) e_2
\end{aligned} \tag{3.30}$$

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Hence,

$$\begin{aligned} L_\alpha\left(f(t)*g(t)\right) &= \left(F_\alpha(s_1)G_\alpha(s_1)\right)e_1 + \left(F_\alpha(s_2)G_\alpha(s_2)\right)e_2 \\ &= F_\alpha(\xi)G_\alpha(\xi) \\ &= L_\alpha(f(t))L_\alpha(g(t)). \end{aligned} \tag{3.31}$$

□

4 Bicomplex Fractional Inverse Laplace Transform

Definition 4.1. Generalized Dirac's function $\delta_\alpha(x)$ of fractional order α, $0 < \alpha < 1$ is given by (see, e.g. [14])

$$\int_{\mathbb{R}} f(x)\delta_\alpha(x)(dx)^\alpha = \alpha f(0). \tag{4.1}$$

The relation between Dirac's function and ML function is given by (see, e.g. [14]) the following result

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-i_1\infty}^{+i_1\infty} E_\alpha(i_1(-\omega x)^\alpha)(d\omega)^\alpha = \delta_\alpha(x), \tag{4.2}$$

where M_α is the period of the complex-valued ML function defined by the relation $E_\alpha(i_1(M_\alpha)^\alpha) = 1$.

Theorem 4.2. *Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of order α of function $f(t) \in \mathcal{C}$ and $\xi = s_1e_1 + s_2e_2 \in \mathbb{T}$ and $0 < \alpha < 1$ then*

$$f(t) = \frac{1}{(M_\alpha)^\alpha}\int_H E_\alpha(\xi^\alpha x^\alpha)F_\alpha(\xi)(d\xi)^\alpha, \tag{4.3}$$

where H is closed contour in $\mathbb{T}$.

Proof. Let $F_\alpha(\xi)$ be the bicomplex fractional Laplace transform of bicomplex-valued function $f(t)$. Then $F_\alpha(\xi) = F_\alpha(s_1)e_1 + F_\alpha(s_2)e_2$. The inverse formula for complex fractional Laplace transform (see, e.g. [14]) are

$$\begin{aligned} f_1(t) &= \frac{1}{(M_\alpha)^\alpha}\int_{-i_1\infty}^{+i_1\infty} E_\alpha(s_1^\alpha x^\alpha)F_\alpha(s_1)(ds_1)^\alpha \\ &= \frac{1}{(M_\alpha)^\alpha}\int_{\gamma_1} E_\alpha(s_1^\alpha x^\alpha)F_\alpha(s_1)(ds_1)^\alpha, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} f_2(t) &= \frac{1}{(M_\alpha)^\alpha}\int_{-i_1\infty}^{+i_1\infty} E_\alpha(s_2^\alpha x^\alpha)F_\alpha(s_2)(ds_2)^\alpha \\ &= \frac{1}{(M_\alpha)^\alpha}\int_{\gamma_2} E_\alpha(s_2^\alpha x^\alpha)F_\alpha(s_2)(ds_2)^\alpha, \end{aligned} \tag{4.5}$$

where M_α is the period of the complex-valued ML function defined by the relation $E_\alpha(i_1(M_\alpha)^\alpha) = 1$ and γ_1 and γ_2 be closed contours taken along the the vertical lines as follows $\gamma_1 = -i_1\infty$ to $i_1\infty$,$\gamma_2 = -i_1\infty$ to $i_1\infty$.

Now, using complex inversions (4.4) and (4.5), we get

$$\begin{aligned}
&f(t) = f_1(t)e_1 + f_2(t)e_2\\
&= \frac{1}{(M_\alpha)^\alpha}\left(\int_{\gamma_1} E_\alpha(s_1^\alpha x^\alpha)F_\alpha(s_1)(ds_1)^\alpha\ e_1 + \int_{\gamma_2} E_\alpha(s_2^\alpha x^\alpha)F_\alpha(s_2)(ds_2)^\alpha\ e_2\right)\\
&= \frac{1}{(M_\alpha)^\alpha}\int_{(\gamma_1,\gamma_2)} E_\alpha((s_1e_1+s_2e_2)^\alpha x^\alpha)F_\alpha(s_1e_1+s_2e_2)\left((ds_1)^\alpha e_1 + (ds_2)^\alpha e_2\right)\\
&= \frac{1}{(M_\alpha)^\alpha}\int_H E_\alpha(\xi^\alpha x^\alpha)F_\alpha(\xi)(d\xi)^\alpha,
\end{aligned} \tag{4.6}$$

where $H = (\gamma_1, \gamma_2)$ and

$$\xi = s_1e_1 + s_2e_2 \Rightarrow d\xi = ds_1e_1 + ds_2e_2 \Rightarrow (d\xi)^\alpha = (ds_1)^\alpha e_1 + (ds_2)^\alpha e_2. \tag{4.7}$$

□

5 Application of Bicomplex Fractional Laplace Transform

Agarwal et al.[3] discussed fractional differential equations in bicomplex space. Bicomplex fractional Laplace transform has great advantage in finding the solution of fractional order differential equations. We have solved the following homogeneous fractional order differential equations using bicomplex fractional Laplace transform.

$$(D^{2\alpha} + 2D^\alpha + 2)y(t) = 0, \tag{5.1}$$

where $y(0) = 1$ and $y^\alpha(0) = -1$.
By taking bicomplex fractional LT on both sides of order α, we get

$$L_\alpha\left(y^{2\alpha} + 2y^\alpha + 2y\right) = 0, \tag{5.2}$$

$$s^{2\alpha}L_\alpha(y(t)) - s^\alpha y(0) - y^\alpha(0) + 2\left(s^\alpha L_\alpha y(t) - y(0)\right) + 2L_\alpha y(t) = 0, \tag{5.3}$$

$$\left(s^{2\alpha} + 2s^\alpha + 2\right)L_\alpha y(t) = s^\alpha + 1, \tag{5.4}$$

$$\Rightarrow \quad L_\alpha y(t) = \frac{s^\alpha + 1}{(s^\alpha+1)^2 + 1}. \tag{5.5}$$

Hence,

$$L_\alpha y(t) = L_\alpha\left(\mathbb{E}_\alpha(-t^\alpha)\cos_\alpha(t^\alpha)\right). \tag{5.6}$$

Therefore

$$y(t) = \mathbb{E}_\alpha(-t^\alpha)\cos_\alpha(t^\alpha), \tag{5.7}$$

where $\cos_\alpha(t^\alpha)$ is fractional order cosine function (see, e.g. [13, 19]).

5.1 Application to Diffusion equation

Consider the following partial fractional differential equation

$$D_t^{\alpha} u(x,t) = cD_x^{\beta} u(x,t),\ 0 < \alpha, \beta < 1, \tag{5.8}$$

with initial condition $u(x,t) = f(x)$. It is very simple case of diffusion equation (see, e. g. [13]).
By taking bicomplex fractional LT of the equation (5.8) with respect to t,

$$s^{\alpha}\bar{u}(x,s) - f(x) = cD_x^{\beta}\bar{u}(x,s). \tag{5.9}$$

Taking fractional Fourier transform of equation (5.9) defined by Jumarie [13] with respect to x,

$$s^{\alpha}\hat{\bar{u}}(\zeta,s) - \hat{f}(\zeta) = c(-i_1\zeta^{\beta})\hat{\bar{u}}(\zeta,s), \tag{5.10}$$

or

$$(s^{\alpha} + i_1 c\zeta^{\beta})\hat{\bar{u}}(\zeta,s) = \hat{f}(\zeta), \tag{5.11}$$

$$\hat{\bar{u}}(\zeta,s) = \frac{\hat{f}(\zeta)}{(s^{\alpha} + i_1 c\zeta^{\beta})}. \tag{5.12}$$

By taking inverse Bicomplex fractional Laplace transform

$$\hat{u}(\zeta,t) = \hat{f}(\zeta)E_{\alpha}(-i_1 c\zeta^{\beta}t^{\alpha}), \qquad \text{(From [19, Property 3.4])}. \tag{5.13}$$

Finally by taking Inverse Fractional Fourier transform defined by Jumarie [13] of the equation (5.13)

$$u(x,t) = \frac{1}{(M_{\beta})^{\beta}}\int_{-\infty}^{+\infty} E_{\beta}(i_1\zeta^{\beta}x^{\beta})\ E_{\alpha}(-i_1 c\zeta^{\beta}t^{\alpha})\ \hat{f}(\zeta)\ (d\zeta)^{\alpha}. \tag{5.14}$$

6 Conclusion

In this paper, the Laplace transform of fractional order or fractional Laplace transform in bicomplex space, the extension of complex Laplace transform of fractional order has been derived. Various properties along with the convolution theorem have also been derived. Bicomplex fractional Laplace transform may be used in finding the solution of bicomplex fractional Schrödinger equation.

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