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A Note On Nielsen-Type Integrals, Logarithmic Integrals And Higher Harmonic Sums

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Abstract

Due to the great success of hypergeometric functions, we provide the analytical solutions of certain definite logarithmic integrals and Nielsen-type integrals in terms of multi-variable Kampé de Fériet functions with suitable convergence conditions and higher harmonic sums by using series rearrangement technique and incomplete Gamma function.
Further we also obtain the solution of other related logarithmic integrals in terms of generalized hypergeometric functions and Kummer's confluent hypergeometric functions by using series rearrangement technique.
The results presented in the paper and comparable outcomes are hoped to be supplied by the use of computer-aid programs, for example, Mathematica.

Key Words and Phrases. Polylogarithm functions; Harmonic sums; Finite Mellin transforms; Nielsen-type integrals.
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1. Introduction, definitions and known results

Here and elsewhere, we use the following standard notations, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} \quad ,$$
$$\mathbb{Z}_0^- = \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- = \{-1, -2, -3, \dots\},$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

The incomplete gamma function is denoted by $\gamma(z, \alpha)$ ([23, p.127, Question (2)], see also [13, p. 15, Question. (10)]) and is defined by :

$$\gamma(z, \alpha) := \int_0^\alpha e^{-t} t^{z-1} \mathrm{d}t \qquad ; (\Re(z) > 0, \ |arg(\alpha)| < \pi),$$
$$= \frac{\alpha^z}{z} {}_1F_1 \left[\begin{array}{c} z; \\ z+1; \end{array} \ -\alpha \right]. \tag{1.1}$$

The polylogarithm function (also known as Jonquière's function) ([27, pp.197–198], see also [14] and [15]) $\mathrm{Li}_s(z)$, is defined for any complex s and z;

$$\mathrm{Li}_s(z) = F(z,s) = \mathrm{PolyLog}[s,z] := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^s}; \qquad s \neq 1,$$

$$\Big(|z|<1, s\in\mathbb{C}\backslash\{1\}; |z|=1, z\neq 1, \Re(s)>0; z=1, \Re(s)>1\Big).$$

The polylogarithm integrals ([3, p.79, Equation (14)], see also [11, p.1232, Equation (1.1)]) are given by:

$$\mathrm{Li}_n(x) := S_{n-1,1}(x) = \frac{(-1)^{n-1}}{(n-2)!} \int_0^1 \frac{1}{z}[\ell n(z)]^{n-2} \ell n(1-zx)dz,$$

$$\Big(n\in\mathbb{N}\backslash\{1\}\ ;\ x\in\mathbb{C}\Big).$$

The Nielsen-integrals ([3, p. 77, Equation (4)], see also [10, p. 647, Equation (1)], [11, p.1232, Equation (1.3)] and [18]) are given by:

$$S_{n,p}(x) := \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{1}{z}[\ell n(z)]^{n-1}[\ell n(1-zx)]^p dz,$$

$$\Big(n,p\in\mathbb{N}\ ;\ x\in\mathbb{C}\Big).$$

Generalized Nielsen-integrals ([3, p. 80, Equation (18)], see also [12], [24]) are given by:

$$S_{n,p,q}(x) := \frac{(-1)^{n+p+q-1}}{(n-1)!p!q!} \int_0^1 \frac{1}{z}[\ell n(z)]^{n-1}[\ell n(1-zx)]^p[\ell n(1+zx)]^q dz, \tag{1.2}$$

$$\Big(\ p,n\in\mathbb{N}\ ;\ x\in\mathbb{C}\ ;\ q\in\mathbb{N}_0\Big).$$

2. Development of finite Mellin transform and harmonic sums

The Development of finite Mellin transform and harmonic sums is in the continuation of ([20, p.1, Equations (1), (3) and (3') and pp.4-5, Equations (6) and (7)], see also [7], [21]).

One-sided or unilateral Laplace transform is defined by:

$$L[f(t);z] = \int_0^{\infty} e^{-tz} f(t)dt = \phi(z), \tag{2.1}$$

then by the substitution $t = -\ell n(x)$, the one-sided Laplace transform (2.1) is converted into a finite Mellin transforms, given by (2.2):

$$M[f(-\ell n(x));z] = \phi(z) = \int_0^1 x^{z-1} f(-\ell n(x))dx, \tag{2.2}$$

provided that the integrals (2.1) and (2.2) exist subject to suitable convergence condition on real part of complex parameter z.
The infinite Mellin transform is defined by:

$$M[g(x);z] = \int_0^{\infty} x^{z-1} g(x)dx = \psi(z), \tag{2.3}$$

provided that above integral exist.

This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory, mathematical statistics and the theory of asymptotic expansions, it is closely related to the Laplace transform and the Fourier transform, and the theory of the Gamma function and allied special functions. Also the Mellin transform is extremely useful for certain applications including solving Laplace equation in polar coordinates, as well as for estimating integrals.
The substitution $x = e^{-t}$ transforms (2.3) into two-sided Laplace transforms (2.4) or into the sum of two, one-sided Laplace transforms (2.5), therefore

$$\psi(z) = \int_{-\infty}^{+\infty} e^{-tz} g(e^{-t}) dt, \tag{2.4}$$

$$= \int_0^\infty e^{-tz} g(e^{-t}) dt + \int_0^\infty e^{-t(-z)} g(e^t) dt. \tag{2.5}$$

In the literature one often defines the transform shifted over "one" as in [34, p.2042, Equation (30)]. One may consider the Mellin-transformation ([4, p.1, Equation (2)], see also [16, p. 159]) for the function $f(z)$, in the form:

$$M\{f(z); N\} = \int_0^1 z^{N-1} f(z) dz,$$

provided that the above integral exists. Here N denotes the integer moment-index, (which is even or odd positive integers depending on the quantity being studied).
The Mellin transform of just a power of $\ell n(1-z)$ can be replaced immediately using the formula, ([3, p.89, Equation (84)], [34, pp.2042–2043, Equations (35) and (36)]):

$$\int_0^1 z^m \ell n^p (1-z) dz = \frac{(-1)^p p!}{(m+1)} S_{\underbrace{1, \ldots, 1}_{p}}(m+1), \tag{2.6}$$

in which the S-function has p indices that are all 1.
[7, p137, Equation (1) and p. 312, Equation (1)] see also [20], [21]

$$\int_0^\infty e^{-\alpha x} x^{s-1} \mathrm{d}x = \frac{\Gamma(s)}{\alpha^s} \qquad ; (\Re(\alpha) > 0, \Re(s) > 0). \tag{2.7}$$

Using suitable substitution in equation (2.7) and further adjustment of parameters, we can derive the following integral:

$$\int_0^1 z^m \ell n^p (z) dz = \frac{(-1)^p \Gamma(p+1)}{(m+1)^{p+1}}, \tag{2.8}$$

$$(\Re(m) > -1, \Re(p) > -1).$$

The functions emerging in perturbation calculations in massless Quantum Field Theories belong to the class discussed by Nielsen [18] and their Mellin-convolutions. By explicit calculation we will show that the Mellin-transforms of such functions can be represented by linear combinations of the finite harmonic sums (see [3, p.77, Equation (3)], [4, p.1, Equation (4)]).

$$S_{k_1,\ldots,k_m}(N) = \sum_{n_1=1}^{N} \frac{(sign(k_1))^{n_1}}{n_1^{|k_1|}} \sum_{n_2=1}^{n_1} \frac{(sign(k_2))^{n_2}}{n_2^{|k_2|}} \ldots \sum_{n_m=1}^{n_{m-1}} \frac{(sign(k_m))^{n_m}}{n_m^{|k_m|}};\ N \in \mathbb{N}, \forall \ell, k_\ell \neq 0.$$

The notation that is used for the various functions and series in this paper is closely related to how useful it can be for a computer program. This notation stays as closely as possible to existing ones. The harmonic series [34, p.2037, Equations (1) and (2)] is defined by:

$$S_m(n) = \sum_{i=1}^{n} \frac{1}{i^m},$$

$$S_{-m}(n)=\sum_{i=1}^{n}\frac{(-1)^i}{i^m},$$

in which $m>0$. The general single harmonic sums $S_{\pm k}(N),\ k>0$ [4, p. 3, Equations (14), (15), (16) and (17)] are obtained by :

$$S_k(N)=\frac{(-1)^{k-1}}{(k-1)!}\int_0^1[\ell n(x)]^{k-1}\left(\frac{x^N-1}{x-1}\right)dx,$$

$$S_{-k}(N)=\frac{(-1)^{k-1}}{(k-1)!}\int_0^1[\ell n(x)]^{k-1}\left(\frac{(-x)^N-1}{x+1}\right)dx,$$

$$\sum_{k=1}^{N}\frac{x^k}{k^\ell}=\frac{(-1)^{\ell-1}}{(\ell-1)!}\int_0^x[\ell n(z)]^{\ell-1}\left(\frac{z^N-1}{z-1}\right)dz,$$

$$\sum_{k=1}^{N}\frac{(-x)^k}{k^\ell}=\frac{(-1)^{\ell-1}}{(\ell-1)!}\int_0^x[\ell n(z)]^{\ell-1}\left(\frac{(-z)^N-1}{z+1}\right)dz.$$

One can define higher harmonic series [34, pp.2037–2038, Equations (3), (4) and (5)] given by:

$$S_{m,j_1,\ldots,j_p}(n)=\sum_{i=1}^{n}\frac{1}{i^m}S_{j_1,\ldots,j_p}(i),$$

$$S_{-m,j_1,\ldots,j_p}(n)=\sum_{i=1}^{n}\frac{(-1)^i}{i^m}S_{j_1,\ldots,j_p}(i),$$

with the same conditions on m. The m and the j_i, $(1\le i\le p)$ are referred to as the indices of the harmonic series. Hence

$$S_{1,-5,3}(n)=\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i}\frac{(-1)^j}{j^5}\sum_{k=1}^{j}\frac{1}{k^3}.$$

For numerical computations one may use the recursion relations [4, p.22, Equations (163) and (164)] for complex values of N, in terms of products of single harmonic sums only

$$S_{\underbrace{-1,\ldots,-1}_{k}}(N)=\frac{1}{k}\sum_{\ell=1}^{k}S_{(-1)^\ell|\ell|}(N)S_{\underbrace{-1,\ldots,-1}_{k-\ell}}(N),$$

$$S_{\underbrace{1,\ldots,1}_{k}}(N)=\frac{1}{k}\sum_{\ell=1}^{k}S_\ell(N)S_{\underbrace{1,\ldots,1}_{k-\ell}}(N).$$

The finite harmonic sums are connected by various algebraic relations. We will only consider the multiple harmonic sums into a single sum: [4, p.19, Equation (126)] see also [34, p. 2056, Equation (92)]

$$S_{1,1}(N)=\frac{1}{2}\left[S_1^2(N)+S_2(N)\right].$$

[4, p.20, Equation (144)] see also [34, p. 2056, Equation (93)]

$$S_{1,1,1}(N)=\frac{1}{6}S_1^3(N)+\frac{1}{2}S_1(N)S_2(N)+\frac{1}{3}S_3(N).$$

[4, p.21, Equation (156)] see also [34, p. 2056, Equation (94)]

$$S_{1,1,1,1}(N)=\frac{1}{4}S_4(N)+\frac{1}{8}S_2^2(N)+\frac{1}{3}S_3(N)S_1(N)+\frac{1}{4}S_2(N)S_1^2(N)+\frac{1}{24}S_1^4(N).$$

The multi-variable extension of Kampé de Fériet double hypergeometric function [28, p. 454] see also [9], [31, pp.65-66], [32, p. 1127, Eq. (4.1)] is given in the form:

$$F^{p:q_1;q_2;\ldots;q_n}_{\ell:m_1;m_2;\ldots;m_n}\left[\begin{array}{ll}(a_p):(b^{(1)}_{q_1});\ldots;(b^{(n)}_{q_n}); & \\ & x_1,\ldots,x_n\\ (\alpha_\ell):(\beta^{(1)}_{m_1});\ldots;(\beta^{(n)}_{m_n}); & \end{array}\right]$$

$$= \sum_{s_1,\ldots,s_n=0}^{\infty}\Lambda(s_1,\ldots,s_n)\frac{{x_1}^{s_1}}{s_1!}\cdots\frac{{x_n}^{s_n}}{s_n!}\ , \tag{2.9}$$

where

$$\Lambda(s_1,\ldots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\ldots+s_n}\prod_{j=1}^{q_1}(b^{(1)}_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b^{(n)}_j)_{s_n}}{\prod_{j=1}^{\ell}(\alpha_j)_{s_1+\ldots+s_n}\prod_{j=1}^{m_1}(\beta^{(1)}_j)_{s_1}\cdots\prod_{j=1}^{m_n}(\beta^{(n)}_j)_{s_n}}\ ,$$

and, for convergence of the multiple hypergeometric series in (2.9),

$$\text{When}\qquad 1+\ell+m_k-p-q_k>0,\qquad k=1,\ldots,n$$

then $|x_1|<\infty,...,|x_n|<\infty$.

$$\text{When}\qquad 1+\ell+m_k-p-q_k=0,\qquad k=1,\ldots,n;\quad p>\ell$$

then $\qquad |x_1|^{\frac{1}{p-\ell}}+\ldots+|x_n|^{\frac{1}{p-\ell}}<1.$

$$\text{When}\qquad 1+\ell+m_k-p-q_k=0,\qquad k=1,\ldots,n;\quad p\leqq\ell$$

then $\qquad max\{|x_1|,\ldots,|x_n|\}<1.$
For absolutely and conditionally convergence of above multiple series (2.9), the readers and researchers can refer a beautiful paper of Hai *et al.* [8, pp.113-114, Theorums 4, 5 and 6], when $x_1,x_2,...,x_n\in\{-1,1\}$. Niukkanen [19] discovers several possible applications of such multiple hypergeometric functions (2.9).

For positive integers $m_1,m_2,m_3,\ldots,m_r$ $(r\geq 1)$, the following multiple series identity [31, p.102, Equation (16)], holds true:

$$\sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\Phi(k_1,k_2,\ldots,k_r;n)\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k_1,k_2,\ldots,k_r=0}^{k_1m_1+k_2m_2+\ldots+k_rm_r\leq n}\Phi(k_1,k_2,\ldots,k_r;n-m_1k_1-m_2k_2-\ldots-m_rk_r)\right), \tag{2.10}$$

provided that the above multiple series are absolutely convergent.

The paper considers Kampé de Fériet and related (generalized) hypergeometric functions at special (constants) arguments implied by Mellin transforms of special ordinary harmonic Polylogarithm. Some of the integrals are related to Mellin transforms of Nielsen integrals.

The present article is motivated by the work of the researchers: Blümlein *et.al* [4], [5], Kölbig *et.al* [10], [11], [12], Nielsen [18], Qureshi-Baboo [22], Remiddi *et.al* [24] and Vermaseren [34], see also sharma *et.al* [25], [26] and Tyagi *et.al* [33].

- In sections 3 and 4, we provide the analytical solution of the logarithmic integral: $\int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])^\ell \mathrm{d}z$ in terms of multi-variable Kampé de Fériet function and higher harmonic sums.

- In section 5, we also yield the solution of Nielsen-type integrals and related integrals: $\int_0^1 \frac{1}{z}[\ell n(z)]^{n-1}[\ell n(1-zx)]^p[\ell n(1+zx)]^q dz$ in terms of multi-variable Kampé de Fériet function with suitable convergence conditions.
- In section 6, we evaluate special integrals: $\int_{-1}^0 \frac{(\ell n[1+z])^m}{z^n}\mathrm{d}z$, $\int_0^1 \frac{(\ell n[1+z])^p m}{z^n}\mathrm{d}z$ in terms of generalized hypergeometric functions using series rearrangement technique.
- In section 7, we obtain the solution of the general integral: $\int_a^b \frac{(\ell n[1+z])^c}{z^d}\mathrm{d}z$ in terms of Kummer's confluent hypergeometric function using incomplete Gamma function

3. Evaluation of $\int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])^\ell \mathrm{d}z$ in terms of multi-variable Kampé de Fériet function ; where $k,\ell \in \mathbb{N}$ and $m \in \mathbb{C}$

Theorem 3.1. *The following result holds true:*

$$L_1 = \int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])^\ell dz = \frac{(-1)^k}{(1+m+k+\ell)}\times$$

$$\times F^{1:2;\ldots;2;2;\ldots;2}_{1:1;\ldots;1;1;\ldots;1}\left[\begin{array}{ll} 1+m+k+\ell : \overbrace{1,1;\ldots;1,1}^{k};\overbrace{1,1;\ldots;1,1}^{\ell}; & \\ & \underbrace{1,\ldots,1}_{k},\underbrace{-1,\ldots,-1}_{\ell} \\ 2+m+k+\ell : \underbrace{2\ ;\ \ldots\ ;\ 2}_{k};\underbrace{2\ ;\ \ldots\ ;\ 2}_{\ell}; & \end{array}\right], \tag{3.1}$$

where $\Re(m+k+\ell) \neq -1,-2,-3,.....$.

Remark: In view of the theorem of Hải *et al.*[8, pp 113–114, Theorem 4, Equations (3.1), (3.2) and (3.3)], the right hand side (i.e. multiple hypergeometric series) of equation (3.1) is absolutely convergent since arguments $\in \{-1,1\}$.
Proof: Since

$$\ell n(1+z) = -\sum_{q=1}^{\infty}\frac{(-1)^q z^q}{q}\ ; \qquad -1 < z \leq 1,$$

$$\ell n(1-z) = -\sum_{p=1}^{\infty}\frac{z^p}{p}\ ; \qquad -1 \leq z < 1.$$

Therefore

$$L_1 = \int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])^\ell \mathrm{d}z$$

$$= \int_0^1 z^m\left(-\sum_{p_1=1}^{\infty}\frac{z^{p_1}}{p_1}\right)\left(-\sum_{p_2=1}^{\infty}\frac{z^{p_2}}{p_2}\right)\cdots\left(-\sum_{p_k=1}^{\infty}\frac{z^{p_k}}{p_k}\right)\left(-\sum_{q_1=1}^{\infty}\frac{(-1)^{q_1}z^{q_1}}{q_1}\right)\left(-\sum_{q_2=1}^{\infty}\frac{(-1)^{q_2}z^{q_2}}{q_2}\right)\cdots\left(-\sum_{q_\ell=1}^{\infty}\frac{(-1)^{q_\ell}z^{q_\ell}}{q_\ell}\right)\mathrm{d}z$$

$$= (-1)^{k+\ell}\sum_{p_1=1}^{\infty}\sum_{p_2=1}^{\infty}\cdots\sum_{p_k=1}^{\infty}\frac{1}{p_1p_2\ldots p_k}\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\cdots\sum_{q_\ell=1}^{\infty}\frac{(-1)^{q_1+q_2+\ldots+q_\ell}}{q_1q_2\ldots q_\ell}\int_0^1 z^{m+p_1+p_2\ldots+p_k+q_1+q_2+\ldots+q_\ell}\mathrm{d}z$$

$$= (-1)^{k+\ell}\sum_{p_1=1}^{\infty}\sum_{p_2=1}^{\infty}\cdots\sum_{p_k=1}^{\infty}\frac{1}{p_1p_2\ldots p_k}\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\cdots\sum_{q_\ell=1}^{\infty}\frac{(-1)^{q_1+q_2+\ldots+q_\ell}}{q_1q_2\ldots q_\ell}\times$$

$$\times\frac{1}{(1+m+p_1+p_2...+p_k+q_1+q_2+...+q_\ell)}$$

$$=(-1)^k\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}...\sum_{p_k=0}^{\infty}\frac{1}{(1+p_1)(1+p_2)...(1+p_k)}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}...\sum_{q_\ell=0}^{\infty}\frac{(-1)^{q_1+q_2+...+q_\ell}}{(1+q_1)(1+q_2)...(1+q_\ell)}\times$$

$$\times\frac{1}{\{(1+m+k+\ell)+(p_1+p_2+...+p_k+q_1+q_2+...+q_\ell)\}}$$

$$=\frac{(-1)^k}{(1+m+k+\ell)}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}...\sum_{p_k=0}^{\infty}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}...\sum_{q_\ell=0}^{\infty}\frac{(1)_{p_1}(1)_{p_2}...(1)_{p_k}}{(2)_{p_1}(2)_{p_2}....(2)_{p_k}}\frac{(1)_{q_1}(1)_{q_2}...(1)_{q_\ell}}{(2)_{q_1}(2)_{q_2}....(2)_{q_\ell}}\times$$

$$\times\frac{(1+m+k+\ell)_{p_1+p_2+...+p_k+q_1+q_2+...+q_\ell}}{(2+m+k+\ell)_{p_1+p_2+...+p_k+q_1+q_2+...+q_\ell}}(1)^{p_1+p_2+...+p_k}(-1)^{q_1+q_2+...+q_\ell}$$

$$=\frac{(-1)^k}{(1+m+k+\ell)}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}...\sum_{p_k=0}^{\infty}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}...\sum_{q_\ell=0}^{\infty}\frac{(1)_{p_1}(1)_{p_2}...(1)_{p_k}}{(2)_{p_1}(2)_{p_2}....(2)_{p_k}}\frac{(1)_{q_1}(1)_{q_2}...(1)_{q_\ell}}{(2)_{q_1}(2)_{q_2}....(2)_{q_\ell}}\times$$

$$\times\frac{(1)_{p_1}(1)_{p_2}...(1)_{p_k}}{p_1!p_2!....p_k!}\frac{(1)_{q_1}(1)_{q_2}...(1)_{q_\ell}}{q_1!q_2!....q_\ell!}\times$$

$$\times\frac{(1+m+k+\ell)_{p_1+p_2+...+p_k+q_1+q_2+...+q_\ell}}{(2+m+k+\ell)_{p_1+p_2+...+p_k+q_1+q_2+...+q_\ell}}(1)^{p_1+p_2+...+p_k}(-1)^{q_1+q_2+...+q_\ell}.$$

Now applying the definition (2.9) of multi-variable extension of Kampé de Fériet function, we obtain the right hand side of the integral L_1.

4. Evaluation of $\int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])^\ell \mathbf{d}z$ in terms of harmonic sums; where k,ℓ and $m\in\mathbb{N}$

Theorem 4.1. *The following results hold true:*
Case I: When $\ell\geq 2$, *then*

$$L_2=\int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])^\ell dz$$

$$=(-1)^k k!\sum_{q_1=0}^{\infty}\left(\sum_{q_2,q_3,...q_\ell=0}^{q_2+q_3+...+q_\ell\leq q_1}(-1)^{q_1}\frac{S_{\underbrace{1,...,1}_{k}}(1+m+\ell+q_1)}{(1+q_1-q_2-q_3-...-q_\ell)(1+q_2)...(1+q_\ell)(1+m+\ell+q_1)}\right). \tag{4.1}$$

Case II: When $\ell=1$, *then*

$$L_3=\int_0^1 z^m(\ell n[1-z])^k(\ell n[1+z])dz$$

$$=(-1)^k k!\sum_{q=0}^{\infty}\left(\frac{(-1)^q}{(1+q)(2+m+q)}S_{\underbrace{1,...,1}_{k}}(2+m+q)\right).$$

Proof of case I: Since $\ell n(1+z) = -\sum_{q=1}^{\infty} \frac{(-1)^q \ z^q}{q}; \qquad -1 < z \leq 1$, therefore

$$L_2 = \int_0^1 z^m (\ell n[1-z])^k (\ell n[1+z])^{\ell} \mathrm{d}z$$

$$= (-1)^{\ell} \sum_{q_1=1}^{\infty} \sum_{q_2=1}^{\infty} ... \sum_{q_\ell=1}^{\infty} \frac{(-1)^{q_1+q_2+...+q_\ell}}{q_1 q_2 ... q_\ell} \int_0^1 z^{m+q_1+q_2+...+q_\ell} (\ell n[1-z])^k \mathrm{d}z.$$

Now using the integral (2.6), we have

$$L_2 = (-1)^{\ell} \sum_{q_1=1}^{\infty} \sum_{q_2=1}^{\infty} ... \sum_{q_\ell=1}^{\infty} \frac{(-1)^{q_1+q_2+...+q_\ell}}{q_1 q_2 ... q_\ell} \times$$

$$\times \frac{(-1)^k \ k!}{(1+m+q_1+q_2+...+q_\ell)} S_{\underbrace{1,\,...,\,1}_{k}} (1+m+q_1+q_2+...+q_\ell) \, .$$

Now replacing q_1 by $1+q_1$, q_2 by $1+q_2$, q_3 by $1+q_3$, ... and q_ℓ by $1+q_\ell$, we obtain

$$L_2 = (-1)^{\ell+k} k! \sum_{q_1=0}^{\infty} \left(\sum_{q_2=0}^{\infty} ... \sum_{q_\ell=0}^{\infty} \frac{(-1)^{\ell+q_1+q_2+...+q_\ell}}{(1+q_1)(1+q_2)...(1+q_\ell)(1+m+\ell+q_1+q_2+...+q_\ell)} \times \right.$$

$$\left. \times S_{\underbrace{1,\,...,\,1}_{k}} (1+m+\ell+q_1+q_2+...+q_\ell) \right) .$$

Now replacing q_1 by $q_1 - q_2 - q_3 - ... - q_\ell$ and applying multiple series identity (2.10), we get the right hand side of assertion (4.1). Similarly we can derive case second when $\ell = 1$.

Some deductions of case I:

(i): When $k = 1$ and $\ell \geq 2$, then

$$\int_0^1 z^m (\ell n[1-z])(\ell n[1+z])^{\ell} \mathrm{d}z$$

$$= \sum_{q_1=0}^{\infty} \left(\sum_{q_2,q_3,...q_\ell=0}^{q_2+q_3+...+q_\ell \leq q_1} (-1)^{1+q_1} \frac{S_1 (1+m+\ell+q_1)}{(1+q_1-q_2-q_3-...-q_\ell)(1+q_2)...(1+q_\ell)(1+m+\ell+q_1)} \right) .$$

(ii): When $k = 2$ and $\ell \geq 2$ and applying the harmonic series relation (2.6), we have

$$\int_0^1 z^m (\ell n[1-z])^2 (\ell n[1+z])^{\ell} \mathrm{d}z$$

$$= \sum_{q_1=0}^{\infty} \left(\sum_{q_2,q_3,...q_\ell=0}^{q_2+q_3+...+q_\ell \leq q_1} (-1)^{q_1} \frac{\{S_1^2 (1+m+\ell+q_1) + S_2 (1+m+\ell+q_1)\}}{(1+q_1-q_2-q_3-...-q_\ell)(1+q_2)...(1+q_\ell)(1+m+\ell+q_1)} \right) .$$

(iii): When $k = 3$ and $\ell \geq 2$ and applying the harmonic series relation (2.7), we get

$$\int_0^1 z^m (\ell n[1-z])^3 (\ell n[1+z])^{\ell} \mathrm{d}z$$

$$= \sum_{q_1=0}^{\infty} \left(\sum_{q_2,q_3,...q_\ell=0}^{q_2+q_3+...+q_\ell \leq q_1} (-1)^{1+q_1} \frac{\{S_1^3 (\beta) + 3 S_1 (\beta) S_2 (\beta) + 2 S_3 (\beta)\}}{(1+q_1-q_2-q_3-...-q_\ell)(1+q_2)...(1+q_\ell)(1+m+\ell+q_1)} \right) ,$$

where $\beta = 1 + m + \ell + q_1$.
(iv): When $k = 4$ and $\ell \geq 2$ and applying the harmonic series relation (2.8), we obtain

$$\int_0^1 z^m (\ell n[1-z])^4 (\ell n[1+z])^\ell \mathrm{d}z$$

$$= \sum_{q_1=0}^{\infty} \left(\sum_{q_2,q_3,\ldots q_\ell=0}^{q_2+q_3+\ldots+q_\ell \leq q_1} \frac{(-1)^{q_1}\{6S_4(\beta) + 3S_2^2(\beta) + 8S_3(\beta)S_1(\beta) + 6S_2(\beta)S_1^2(\beta) + S_1^4(\beta)\}}{(1+q_1-q_2-q_3-\ldots-q_\ell)(1+q_2)\ldots(1+q_\ell)(1+m+\ell+q_1)} \right),$$

where $\beta = 1 + m + \ell + q_1$.

Some deductions of case II:
(a): When $k = 1$ and $\ell = 1$, then

$$\int_0^1 z^m (\ell n[1-z])(\ell n[1+z])\mathrm{d}z = \sum_{q=0}^{\infty} \left(\frac{(-1)^{1+q}}{(1+q)(2+m+q)} S_1(2+m+q) \right).$$

(b): When $k = 2$ and $\ell = 1$, then

$$\int_0^1 z^m (\ell n[1-z])^2 (\ell n[1+z])\mathrm{d}z = \sum_{q=0}^{\infty} \left(\frac{(-1)^q \{S_1^2(2+m+q) + S_2(2+m+q)\}}{(1+q)(2+m+q)} \right).$$

(c): When $k = 3$ and $\ell = 1$, then

$$\int_0^1 z^m (\ell n[1-z])^3 (\ell n[1+z])\mathrm{d}z$$

$$= \sum_{q=0}^{\infty} \left(\frac{(-1)^{1+q}}{(1+q)(2+m+q)} \{S_1^3(\lambda) + 3S_1(\lambda)S_2(\lambda) + 2S_3(\lambda)\} \right),$$

where $\lambda = 2 + m + q$.
(d): When $k = 4$ and $\ell = 1$, then

$$\int_0^1 z^m (\ell n[1-z])^4 (\ell n[1+z])\mathrm{d}z$$

$$= \sum_{q=0}^{\infty} \left(\frac{(-1)^q}{(1+q)(2+m+q)} \{6S_4(\lambda) + 3S_2^2(\lambda) + 8S_3(\lambda)S_1(\lambda) + 6S_2(\lambda)S_1^2(\lambda) + S_1^4(\lambda)\} \right),$$

where $\lambda = 2 + m + q$.

5. Evaluation of Nielsen-type integrals and related integrals in terms of multi-variable Kampé de Fériet function; where $n, p \in \mathbb{N}$ and $q \in \mathbb{N}_0$

Theorem 5.1. *The following result for Nielsen-type integrals holds true:*

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!p!q!} \int_0^1 \frac{1}{z} [\ell n(z)]^{n-1} [\ell n(1-zx)]^p [\ell n(1+zx)]^q dz$$

$$= \frac{(-1)^q x^{p+q}}{(p+q)^n p! q!} \times$$

$$\times F^{n:2;\ldots;2;2;\ldots;2}_{n:1;\ldots;1;1;\ldots;1}\left[\begin{array}{lllll}\overbrace{p+q,\ldots,p+q}^{n} & : & \overbrace{1,1;\ldots;1,1}^{p};\overbrace{1,1;\ldots;1,1}^{q}; & & \\ & & & \underbrace{x,\ldots,x}_{p},\underbrace{-x,\ldots,-x}_{q} \\ \underbrace{1+p+q,\ldots,1+p+q}_{n} & : & \underbrace{2\;;\ldots;\;2}_{p};\underbrace{2\;;\ldots;\;2}_{q}; & & \end{array}\right], \quad (5.1)$$

$$\Big(|x|\leq 1\;;\quad n,p\in\mathbb{N}\;;\quad q\in\mathbb{N}_0\Big).$$

Note: $S_{n,p,0}(x)\equiv S_{n,p}(x)\quad;\quad S_{n-1,1}(x)\equiv \mathrm{Li}_n(x)$
and Kölbig integrals[10, p.647, Equation (2)]:

$$\int_0^1 z^{-1}[\ell n(z)]^{n-1}[\ell n(1-z)]^p dz=(-1)^{n+p-1}(n-1)!p!S_{n,p}(1).$$

Proof
Since

$$\ell n(1-z)=-\sum_{r=1}^{\infty}\frac{z^r}{r}\;;\qquad -1\leq z<1,$$

$$\ell n(1+z)=-\sum_{s=1}^{\infty}\frac{(-1)^s z^s}{s}\;;\qquad -1<z\leq 1.$$

Therefore

$$S_{n,p,q}(x)=\frac{(-1)^{n+p+q-1}}{(n-1)!p!q!}\int_0^1\frac{1}{z}[\ell n(z)]^{n-1}[\ell n(1-zx)]^p[\ell n(1+zx)]^q dz$$

$$=\frac{(-1)^{n+p+q-1}}{(n-1)!p!q!}\int_0^1\frac{1}{z}[\ell n(z)]^{n-1}\left(-\sum_{r_1=1}^{\infty}\frac{z^{r_1}x^{r_1}}{r_1}\right)\cdots\left(-\sum_{r_p=1}^{\infty}\frac{z^{r_p}x^{r_p}}{r_p}\right)\left(-\sum_{s_1=1}^{\infty}\frac{(-1)^{s_1}z^{s_1}x^{s_1}}{s_1}\right)\cdots\left(-\sum_{s_q=1}^{\infty}\frac{(-1)^{s_q}z^{s_q}x^{s_q}}{s_q}\right)dz \quad (5.2)$$

$$=\frac{(-1)^{n-1}}{(n-1)!p!q!}\sum_{r_1=1}^{\infty}\cdots\sum_{r_p=1}^{\infty}\frac{x^{r_1+\ldots+r_p}}{(r_1)\ldots(r_p)}\sum_{s_1=1}^{\infty}\cdots\sum_{s_q=1}^{\infty}\frac{(-1)^{s_1+\ldots+s_q}x^{s_1+\ldots+s_q}}{(s_1)\ldots(s_q)}\times$$
$$\times\int_0^1 z^{r_1+\ldots+r_p+s_1+\ldots+s_q-1}[\ell n(z)]^{n-1}dz$$

Now using the result (2.8), we get

$$S_{n,p,q}(x)=\frac{1}{p!q!}\sum_{r_1=1}^{\infty}\cdots\sum_{r_p=1}^{\infty}\sum_{s_1=1}^{\infty}\cdots\sum_{s_q=1}^{\infty}\frac{(-1)^{s_1+\ldots+s_q}x^{r_1+\ldots+r_p+s_1+\ldots+s_q}}{(r_1+\ldots+r_p+s_1+\ldots+s_q)^n(s_1)\ldots(s_q)(r_1)\ldots(r_p)}.$$

Now replacing r_1 by $1+r_1$, r_2 by $1+r_2$, ..., r_p by $1+r_p$, and s_1 by $1+s_1, s_2$ by $1+s_2$, ..., s_q by $1+s_q$, we obtain

$$S_{n,p,q}(x)=\frac{1}{p!q!}\sum_{r_1=0}^{\infty}\cdots\sum_{r_p=0}^{\infty}\sum_{s_1=0}^{\infty}\cdots\sum_{s_q=0}^{\infty}\frac{(-1)^{q+s_1+\ldots+s_q}x^{p+q+r_1+\ldots+r_p+s_1+\ldots+s_q}}{\{(p+q)+(r_1+\ldots+r_p+s_1+\ldots+s_q)\}^n}\times$$
$$\times\frac{(1)_{r_1}(1)_{r_2}\ldots(1)_{r_p}(1)_{s_1}(1)_{s_2}\ldots(1)_{s_q}}{(2)_{r_1}(2)_{r_2}\ldots(2)_{r_p}(2)_{s_1}(2)_{s_2}\ldots(2)_{s_q}}.$$

Now applying the definition (2.9) of multi-variable extension of Kampé de Fériet function, we get the required result (5.1).

6. Evaluation of $\int_{-1}^{0}\frac{(\ell n[1+z])^m}{z^n}\mathbf{d}z$, $\int_{0}^{1}\frac{(\ell n[1+z])^m}{z^n}\mathbf{d}z$ in terms of generalized hypergeometric functions; where $m,n\in\mathbb{N}$

Theorem 6.1. *The following results hold true:*
When $m \geq n$, then

$$L_4=\int_{-1}^{0}\frac{(\ell n[1+z])^m}{z^n}dz=(-1)^{m-n}m!\ {}_{m+2}F_{m+1}\left[\begin{array}{c}\overbrace{1,1,\ldots,1}^{m+1},n;\\ \underbrace{2,2,\ldots,2}_{m+1};\end{array}1\right], \tag{6.1}$$

When $m \geq 2$, then

$$L_5=\int_{0}^{1}(\ell n[1+z])^m dz=(-1)^{m+1}m!+2(\ell n[2])^m\ {}_2F_0\left[\begin{array}{c}-m,1;\\ -;\end{array}(\ell n[2])^{-1}\right], \tag{6.2}$$

When $m \geq n \geq 2$, then

$$L_6=\int_{0}^{1}\frac{(\ell n[1+z])^m}{z^n}dz=\frac{(m)!}{(n-1)^{m+1}}\ {}_{m+1}F_m\left[\begin{array}{c}\overbrace{n-1,n-1,\ldots,n-1}^{m+1};\\ \underbrace{n,n,\ldots,n}_{m};\end{array}1\right]-$$

$$-\sum_{k=0}^{m}\frac{k!\binom{m}{k}(\ell n[2])^{m-k}}{2^{n-1}(n-1)^{k+1}}\ {}_{k+1}F_k\left[\begin{array}{c}\overbrace{n-1,n-1,\ldots,n-1}^{k+1};\\ \underbrace{n,n,\ldots,n}_{k};\end{array}\frac{1}{2}\right]. \tag{6.3}$$

Independent proof of the integral (6.1):
The integral (6.1) can be solved by substituting $1+z=e^{-t}$ and using the result (2.7) of Laplace transforms.
Independent proof of the integral (6.2):

$$\text{Suppose}\qquad L_5=\int_{0}^{1}(\ell n[1+z])^m\mathrm{d}z.$$

Put $1+z=e^t$, then we have

$$L_5=\int_{0}^{\ell n[2]}t^m e^t\mathrm{d}t.$$

Now integrating by parts, we get

$$L_5=\Bigg[\binom{m}{0}(-1)^0t^me^t+\binom{m}{1}(-1)^1t^{m-1}e^t+\binom{m}{2}(-1)^2t^{m-2}e^t2!+\binom{m}{3}(-1)^3t^{m-3}e^t3!+\ldots$$

$$+\binom{m}{m-1}(-1)^{m-1}te^t(m-1)!+\binom{m}{m}(-1)^me^tm!\Bigg]_0^{\ell n[2]}$$

$$L_5=(-1)^{m+1}m!+2\sum_{k=0}^{m}\binom{m}{k}(\ell n[2])^{m-k}(-1)^kk!$$

$$\text{or}\quad L_5 = (-1)^{m+1}m! + 2(\ell n[2])^m\ {}_2F_0\left[\begin{matrix} -m, 1; \\ -; \end{matrix}\ (\ell n[2])^{-1}\right].$$

Independent proof of the integral (6.3):

$$\text{Suppose}\qquad L_6 = \int_0^1 \frac{(\ell n[1+z])^m}{z^n}\mathrm{d}z$$

Now substitute $1+z=e^t$, then we have

$$L_6 = \int_0^{\ell n[2]} \frac{t^m e^t}{(e^t-1)^n}\mathrm{d}t = \int_0^{\ell n[2]} t^m e^{-t(n-1)}(1-e^{-t})^{-n}\mathrm{d}t = \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\int_0^{\ell n[2]} t^m e^{-t(n+r-1)}\mathrm{d}t$$

Further, integrate by parts with $\alpha = n+r-1$, we get

$$L_6 = \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\Bigg[-\binom{m}{0}\frac{t^m e^{-\alpha t}}{(\alpha)} - \binom{m}{1}\frac{t^{m-1}e^{-\alpha t}}{(\alpha)^2} - \binom{m}{2}\frac{t^{m-2}e^{-\alpha t}2!}{(\alpha)^3} - \binom{m}{3}\frac{t^{m-3}e^{-\alpha t}3!}{(\alpha)^4} - \dots$$
$$- \binom{m}{m-1}\frac{te^{-\alpha t}(m-1)!}{(\alpha)^m} - \binom{m}{m}\frac{e^{-\alpha t}m!}{(\alpha)^{m+1}}\Bigg]_0^{\ell n[2]}$$

$$= -\sum_{r=0}^{\infty}\frac{(n)_r}{r!}\binom{m}{0}\frac{t^m e^{-\alpha t}}{(\alpha)} - \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\binom{m}{1}\frac{t^{m-1}e^{-\alpha t}}{(\alpha)^2} - \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\binom{m}{2}\frac{t^{m-2}e^{-\alpha t}2!}{(\alpha)^3} - \dots$$
$$- \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\binom{m}{m-1}\frac{\ell n[2](m-1)!}{2^\alpha(\alpha)^m} - \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\frac{(m)!}{2^\alpha(\alpha)^{m+1}} + \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\frac{(m)!}{(\alpha)^{m+1}}$$
$$= \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\frac{(m)!}{(n+r-1)^{m+1}} - \sum_{r=0}^{\infty}\frac{(n)_r}{r!}\left[\sum_{k=0}^{m}\frac{k!\binom{m}{k}(\ell n[2])^{m-k}}{2^{n+r-1}(n+r-1)^{k+1}}\right]$$

$$= \sum_{r=0}^{\infty}\frac{\{(n-1)_r\}^{m+1}}{(n-1)^{m+1}r!}\frac{(m)!}{\{(n)_r\}^m} - \sum_{k=0}^{m}\frac{k!\binom{m}{k}(\ell n[2])^{m-k}}{2^{n-1}}\sum_{r=0}^{\infty}\frac{(n)_r}{2^r(n+r-1)^{k+1}r!}.$$

Now using the well-known definition of generalized hypergeometric function ${}_pF_q$, we obtain the desired result.

7. Evaluation of $\int_a^b \frac{(\ell n[1+z])^c}{z^d}\mathrm{d}z$ in terms of Kummer's confluent hypergeometric functions; where $b>a>0$, $\Re(d)>1$, $\Re(c+1)>0$

Theorem 7.1. *The following general result holds true:*

$$L_7 = \int_a^b \frac{(\ell n[1+z])^c}{z^d}dz = \sum_{r=0}^{\infty}\frac{(d)_r}{r!}\Bigg\{\frac{(\ell n[1+b])^{c+1}}{c+1}{}_1F_1\left[\begin{matrix} c+1; \\ c+2; \end{matrix}\ -(d+r-1)\ell n[1+b]\right] -$$
$$-\frac{(\ell n[1+a])^{c+1}}{c+1}{}_1F_1\left[\begin{matrix} c+1; \\ c+2; \end{matrix}\ -(d+r-1)\ell n[1+a]\right]\Bigg\}, \qquad (7.1)$$

where $\Re(c+1)>0$ *and* $\Re(d)>1$.

Independent proof of the integral (7.1):

$$\text{Suppose} \qquad L_7 = \int_a^b \frac{(\ell n[1+z])^c}{z^d} \mathrm{d}z$$

Now substitute $1+z=e^t$, then we have

$$L_7 = \int_{\ell n[1+a]}^{\ell n[1+b]} t^c e^{-t(d-1)} (1-e^{-t})^{-d} \mathrm{d}t = \sum_{r=0}^{\infty} \frac{(d)_r}{r!} \int_{\ell n[1+a]}^{\ell n[1+b]} t^c e^{-t(d+r-1)} \mathrm{d}t.$$

Further put $t(d+r-1)=x$, we get

$$L_7 = \sum_{r=0}^{\infty} \frac{(d)_r}{r!(d+r-1)^{c+1}} \int_{(d+r-1)\ell n[1+a]}^{(d+r-1)\ell n[1+b]} e^{-x} x^c \mathrm{d}x$$

$$L_7 = \sum_{r=0}^{\infty} \frac{(d)_r}{r!(d+r-1)^{c+1}} \left\{ \int_0^{(d+r-1)\ell n[1+b]} e^{-x} x^c \mathrm{d}x - \int_0^{(d+r-1)\ell n[1+a]} e^{-x} x^c \mathrm{d}x \right\}.$$

Now using the definition of incomplete Gamma function, we obtain

$$L_7 = \sum_{r=0}^{\infty} \frac{(d)_r}{r!(d+r-1)^{c+1}} \left\{ \gamma(c+1, (d+r-1)\ell n[1+b]) - \gamma(c+1, (d+r-1)\ell n[1+a]) \right\}.$$

Now expressing incomplete Gamma in terms of hypergeometric notation (1.1), we get the right hand side of equation (7.1) which is always convergent, in view of convergence conditions of hypergeometric function ${}_pF_q(z)$ when $p=q$ then $|z|<\infty$.

8. Concluding remarks and Future scope

In this paper we have obtained some results involving hypergeometric functions and harmonic sums. We conclude our present investigation by observing that several other theorems of the similar types integrals related with other mathematical functions, different from the following integrals, are obtained in an analogous manner:

$$\int_a^b \frac{(\ell n[1+z])^c}{z^d} \mathrm{d}z;$$

$$\int_{-a}^{-b} \frac{(\ell n[1-z])^c}{z^d} \mathrm{d}z = \frac{1}{(-1)^{d+1}} \int_a^b \frac{(\ell n[1+z])^c}{z^d} \mathrm{d}z;$$

$$\int_0^1 z^m (\ell n[1-z])^k (\ell n[1+z])^\ell \mathrm{d}z = (-1)^m \int_{-1}^0 z^m (\ell n[1+z])^k (\ell n[1-z])^\ell \mathrm{d}z$$

and generalized Nielsen integrals with special cases.
Moreover the results derived in this paper are quite significant and are expected to be beneficial for the researchers in the field of applied mathematics, mathematical sciences and other branches of science and engineering. The interested readers and researchers can consult an appendix (Section 7 on Mellin Transforms) of the beautiful paper by "Blümlein and Kurth" [4, pp.27-39] for the Mellin integrals, which are different from the present paper.

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