Generalization of Hermite-Hadamard inequality for differentiable convex and quasi-convex function

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Abstract

In this paper, the generalization of Simpson's identity has been derived. This generalized identity has been used to obtain new Hermite-Hadamard inequalities for differentiable convex and quasi-convex functions. Also, the validation of the derived inequalities has been established using suitable examples.

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1 Introduction

The theory of inequality has many applications in mathematics, physical sciences and engineering fields. It includes the study of various inequalities such as Holder's inequality, Jensen' inequality, Azuma's inequality, Boole's inequality, Hermite-Hadamard inequality and many more well known inequalities. Hermite-Hadamard inequality is one of the most famous inequality in mathematics. It was derived independently by Charles Hermite and Jacques Hadamard. It is involved with the convexity of function. In 1998, Dragomir and Agarwal[6] derived the inequality associated with the right hand side of Hermite-Hadamard inequality for differentiable convex function. Later on this estimate was improved by Pearce and Pecaric[21]. Kirmaci[17] discovered the inequality linked with the left hand side of Hermite-Hadamard inequality. By using the work of Dragomir et al. and Kirmaci many researcher have derived the inequalities associated with left side and right side of Hermite-Hadamard inequality. The Hermite-Hadamard integral inequality for convex functions is used in Kirmaci's work to present a number of inequalities for differentiable convex functions. Kirmaci's work employs the Hermite-Hadamard integral inequality holding for convex functions to describe a few inequalities for differentiable convex functions. Additionally, certain applications to unique real number means were offered, and some midway formula error estimates were discovered. Later, the inequality related to right hand side of Hermite-Hadamard inequality for quasi-convex function was discovered by D. A Ion.[15]

Before discussing the the main findings of the paper, some prilimianary concepts that are useful for the better understanding of the research. We begin with the Hermite-Hadamard inequality.

$$\varrho\left(\frac{y_1+z_1}{2}\right) \leq \frac{1}{z_1-y_1}\int_{y_1}^{z_1} \varrho(s)ds \leq \frac{\varrho(y_1)+\varrho(z_1)}{2}. \tag{1}$$

Next, we define convex and quasi convex function.

Definition 1. *A function $\varrho : \mathcal{I} \to \mathbb{R}$ is said to be convex if*

$$\varrho(y_1\varkappa + (1-\varkappa)z_1) \leq \varkappa\varrho(y_1) + (1-\varkappa)\varrho(z_1),$$

for all $y_1, z_1 \in \mathcal{I}$ and $0 < \varkappa < 1$.

Definition 2. *A function $\varrho : \mathcal{I} \to \mathbb{R}$ is said to be quasi-convex if*

$$\varrho(y_1\varkappa + (1-\varkappa)z_1) \leq max\{\varrho(y_1), \varrho(z_1)\},$$

for all $y_1, z_1 \in \mathcal{I}$ and $0 < \varkappa < 1$.

In [20](page 3 , Lemma 1), Alomari et al. has derived the following identity.

Lemma 1. *Let $L[y_1, z_1]$ denote the class of all Lebesgue integrable functions on $[y_1, z_1]$. Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $\varrho' \in L[y_1, z_1]$, then*

$$\left(\frac{z_1-y_1}{3}\right)\left[\frac{\varrho(y_1)+\varrho(z_1)}{2} + 2f\left(\frac{y_1+z_1}{2}\right)\right] - \int_{y_1}^{z_1} \varrho(s)ds$$
$$= \int_0^1 \left[\left(\varkappa - \frac{1}{3}\right)\varrho'\left(\varkappa\left(\frac{y_1+z_1}{2}\right) + (1-\varkappa)y_1\right) + \left(\varkappa - \frac{2}{3}\right)\varrho'\left(\varkappa z_1 + (1-\varkappa)\left(\frac{y_1+z_1}{2}\right)\right)\right]d\varkappa. \tag{2}$$

2 Main Results

In this section, we generalize the identity obtained by Alomari et al.[20]. Also, with the help of this generalized identity, several Hermite-Hadamard-type inequalities have been derived. Also, the validity of derived inequalities has been derived.

Theorem 1. *Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $\varrho' \in L[y_1, z_1]$, then the following equality holds:*

$$\frac{2(z_1-y_1)\varrho(x)}{3} + \frac{(x-y_1)\varrho(y_1)}{3} + \frac{(z_1-x)\varrho(z_1)}{3} - \int_{y_1}^{z_1} \varrho(s)ds$$
$$= (x-y_1)^2 \int_0^1 \left(\varkappa - \frac{1}{3}\right)\varrho'(\varkappa x + (1-\varkappa)y_1)d\varkappa + (z_1-x)^2 \int_0^1 \left(\varkappa - \frac{2}{3}\right)\varrho'(\varkappa z_1 + (1-\varkappa)x)d\varkappa. \tag{3}$$

Proof. By applying integration by parts two times,

$$\begin{aligned} I_1 &= \int_0^1 \left(\varkappa - \frac{1}{3}\right)\varrho'(\varkappa x + (1-\varkappa)y_1)d\varkappa \\ &= \left(\varkappa - \frac{1}{3}\right)\frac{\varrho(\varkappa x + (1-\varkappa)y_1)}{x-y_1}\Bigg|_0^1 - \frac{1}{(x-y_1)}\int_0^1 \varrho(\varkappa x + (1-\varkappa)y_1)d\varkappa \\ &= \frac{2\varrho(x)}{3(x-y_1)} + \frac{\varrho(y_1)}{3(x-y_1)} - \frac{1}{(x-y_1)}\int_0^1 \varrho(\varkappa x + (1-\varkappa)y_1)d\varkappa. \end{aligned} \tag{4}$$

Making use of change of the variable $s = \varkappa x + (1-\varkappa)y_1$ and multiplying by $(x-y_1)^2$ both sides, we have

$$(x-y_1)^2 I_1 = \frac{2}{3}(x-y_1)\varrho(x) + \frac{1}{3}(x-y_1)\varrho(y_1) - \int_{y_1}^{x} \varrho(s)ds. \tag{5}$$

Similarly,

$$(x-z_1)^2 I_2 = \frac{2}{3}(z_1-x)\varrho(x) + \frac{1}{3}(z_1-x)\varrho(z_1) - \int_{x}^{z_1} \varrho(s)ds. \tag{6}$$

By adding (5) and (6) we have required identity. □

Remark 1. *By setting $x = \frac{y_1+z_1}{2}$ in Theorem1, the identity (3) becomes the identity (2)*

Next, the certain estimates associated with RHS of (3) are given.

Theorem 2. *Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $|\varrho'|$ is convex on $[y_1, z_1]$, then the following inequality holds:*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|$$
$$\leq (x-y_1)^2\left(\frac{29|\varrho'(x)|}{162}+\frac{8|\varrho'(y_1)|}{81}\right)+(z_1-x)^2\left(\frac{8|\varrho'(z_1)|}{81}+\frac{29|\varrho'(x)|}{162}\right). \tag{7}$$

Proof. Using Theorem 1 and the convexity of $|\varrho'|$, we have

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|$$
$$\leq (x-y_1)^2\int_0^1|\varkappa-\frac{1}{3}||\varrho'(\varkappa x+(1-\varkappa)y_1)|d\varkappa+(z_1-x)^2\int_0^1|\varkappa-\frac{2}{3}||\varrho'(\varkappa z_1+(1-\varkappa)x)|d\varkappa.$$
$$\leq (x-y_1)^2\int_0^1|\varkappa-\frac{1}{3}|\big(\varkappa|\varrho'(x)|+(1-\varkappa)|\varrho'(y_1)|\big)d\varkappa$$
$$+(z_1-x)^2\int_0^1|\varkappa-\frac{2}{3}|\big(\varkappa|\varrho'(z_1)|+(1-\varkappa)|\varrho'(x)|d\varkappa$$
$$=(x-y_1)^2\left[|\varrho'(x)|\int_0^1\varkappa|\varkappa-\frac{1}{3}|d\varkappa+|\varrho'(y_1)|\int_0^1(1-\varkappa)|\varkappa-\frac{1}{3}|d\varkappa\right]$$
$$+(z_1-x)^2\left[|\varrho'(z_1)|\int_0^1\varkappa|\varkappa-\frac{2}{3}|d\varkappa+|\varrho'(x)|\int_0^1(1-\varkappa)|\varkappa-\frac{2}{3}|d\varkappa\right]$$
$$=(x-y_1)^2\left(\frac{29|\varrho'(x)|}{162}+\frac{8|\varrho'(y_1)|}{81}\right)+(z_1-x)^2\left(\frac{8|\varrho'(z_1)|}{81}+\frac{29|\varrho'(x)|}{162}\right).$$

This completes the proof. ∎

Example 1. *Let the function f be defined as $f(x) = x^6$. Then the function f is convex on $[1, 2]$. We have*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|=\left|\frac{2x^6}{3}+\frac{2}{15}+\frac{508}{21}\right| \tag{8}$$

and

$$(x-y_1)^2\left(\frac{29|\varrho'(x)|}{162}+\frac{8|\varrho'(y_1)|}{81}\right)+(z_1-x)^2\left(\frac{8|\varrho'(z_1)|}{81}+\frac{29|\varrho'(x)|}{162}\right)$$
$$=(x-1)^2\left(\frac{16}{27}+\frac{29|x|^5}{27}\right)+(2-x)^2(\frac{512}{27}+\frac{29|x|^5}{27}). \tag{9}$$

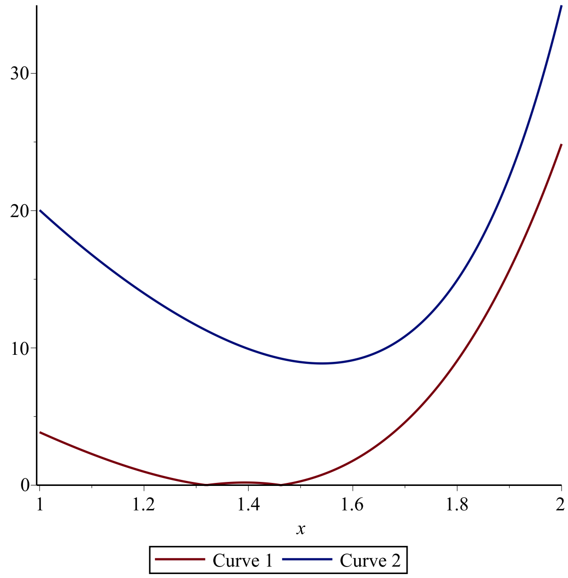

Figure 1:

Here Curve 1 and Curve 2 represents the expression (8) and (9) respectively. Figure 1 depicts that the Curve 1 is below Curve 2. Hence, it also refers to our calculation where the value of the expression (8) is less than the expression (9). This validates the inequality (7).

Theorem 3. *Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $|\varrho'|^q$ is convex on $[y_1, z_1]$, then the following inequality holds:*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|$$
$$\leq\left(\frac{2^{p+1}+1}{(3p+3)3^p}\right)^{\frac{1}{p}}\frac{1}{2^{\frac{1}{q}}}\left[(x-y_1)^2\left(|\varrho'(y_1)|^q+|\varrho'(x)|^q\right)^{\frac{1}{q}}+(z_1-x)^2\left(|\varrho'(x)|^q+|\varrho'(z_1)|^q\right)^{\frac{1}{q}}\right]. \quad (10)$$

Proof. Using Theorem 1, Holder's inequality and the convexity of $|\varrho'|^q$, we have

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|$$
$$\leq(x-y_1)^2\int_0^1|\varkappa-\frac{1}{3}||\varrho'(\varkappa x+(1-\varkappa)y_1)|d\varkappa+(z_1-x)^2\int_0^1|\varkappa-\frac{2}{3}||\varrho'(\varkappa z_1+(1-\varkappa)x)|d\varkappa.$$
$$\leq(x-y_1)^2\left(\int_0^1\left|\varkappa-\frac{1}{3}\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1|\varrho'(\varkappa x+(1-\varkappa)y_1)|^q d\varkappa\right)^{\frac{1}{q}}$$
$$+(z_1-x)^2\left(\int_0^1\left|\varkappa-\frac{2}{3}\right|^p\right)^{\frac{1}{p}}\left(\int_0^1|\varrho'(\varkappa z_1+(1-\varkappa)x)|^q d\varkappa\right)^{\frac{1}{q}}$$
$$\leq(x-y_1)^2\left(\frac{2^{p+1}}{(3p+3)3^p}\right)^{\frac{1}{p}}\left(\int_0^1\left(\varkappa|\varrho'(x)|^q+(1-\varkappa)|\varrho'(y_1)|^q\right)d\varkappa\right)^{\frac{1}{q}}$$
$$+(z_1-x)^2\left(\frac{2^{p+1}+1}{(3p+3)3^p}\right)^{\frac{1}{p}}\left(\int_0^1\left(\varkappa|\varrho'(z_1)|^q+(1-\varkappa)|\varrho'(x)|^q d\varkappa\right)\right)^{\frac{1}{q}}$$
$$=\left(\frac{2^{p+1}+1}{(3p+3)3^p}\right)^{\frac{1}{p}}\frac{1}{2^{\frac{1}{q}}}\left[(x-y_1)^2\left(|\varrho'(y_1)|^q+|\varrho'(x)|^q\right)^{\frac{1}{q}}+(z_1-x)^2\left(|\varrho'(x)|^q+|\varrho'(z_1)|^q\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Example 2. *Let the function f be defined as* $f(x) = x^4$. *Then the function* f *is convex on* $[0, 1]$. *We have*

$$\left|\frac{2(z_1 - y_1)\varrho(x)}{3} + \frac{(x - y_1)\varrho(y_1)}{3} + \frac{(z_1 - x)\varrho(z_1)}{3} - \int_{y_1}^{z_1} \varrho(s)ds\right| = \left|\frac{2x^4}{3} + \frac{2}{15} - \frac{x}{3}\right| \tag{11}$$

and

$$\begin{aligned}&\leq \left(\frac{2^{p+1}+1}{(3p+3)3^p}\right)^{\frac{1}{p}} \frac{1}{2^{\frac{1}{q}}}\left[(x - y_1)^2\left(|\varrho'(y_1)|^q + |\varrho'(x)|^q\right)^{\frac{1}{q}} + (z_1 - x)^2\left(|\varrho'(x)|^q + |\varrho'(z_1)|^q\right)^{\frac{1}{q}}\right]\\&= \frac{17^{\frac{1}{3}}2^{\frac{2}{3}}3^{\frac{2}{3}}}{9}(|x|^5 + (|x|^{\frac{9}{2}} + 1)^{\frac{2}{3}}x^2 - 2((|x|^5 + (|x|^{\frac{9}{2}} + 1)^{\frac{2}{3}})x + (|x|^5 + (|x|^{\frac{9}{2}} + 1)^{\frac{2}{3}}).\end{aligned} \tag{12}$$

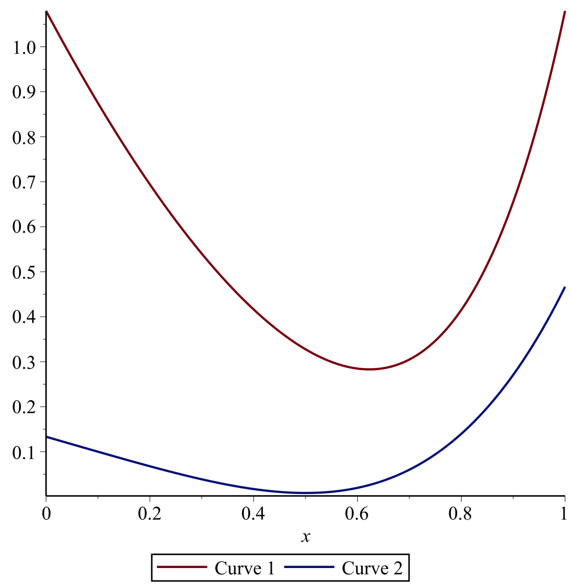

Figure 2:

Here Curve 1 and Curve 2 represents the expression (11) and (12) respectively. Figure 2 depicts that the Curve 1 is below Curve 2. Hence, it also refers to our calculation where the value of the expression (11) is less than the expression (12). This validates the inequality (10).

Theorem 4. *Let* $\varrho : [y_1, z_1] \to \mathbb{R}$ *be a differentiable function on* (y_1, z_1) *with* $y_1 < z_1$. *If* $|\varrho'|^q$ *is convex on* $[y_1, z_1]$, *then the following inequality holds:*

$$\begin{aligned}&\left|\frac{2(z_1 - y_1)\varrho(x)}{3} + \frac{(x - y_1)\varrho(y_1)}{3} + \frac{(z_1 - x)\varrho(z_1)}{3} - \int_{y_1}^{z_1} \varrho(s)ds\right|\\&\leq \left(\frac{5}{18}\right)^{\frac{1}{p}}\left[(x - y_1)^2\left(\frac{8|\varrho'(y_1)|^q}{81} + \frac{29|\varrho'(x)|^q}{162}\right)^{\frac{1}{q}} + (z_1 - x)^2\left(8\frac{|\varrho'(x)|^q}{81} + \frac{29|\varrho'(z_1)|^q}{162}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{13}$$

Proof. Using Theorem 1, Power-mean inequality and the convexity of $|\varrho'|^q$, we have

$$\begin{aligned}
&\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right| \\
&\leq (x-y_1)^2\int_0^1\left|\varkappa-\frac{1}{3}\right||\varrho'(\varkappa x+(1-\varkappa)y_1)|d\varkappa+(z_1-x)^2\int_0^1\left|\varkappa-\frac{2}{3}\right||\varrho'(\varkappa z_1+(1-\varkappa)x)|d\varkappa. \\
&\leq (x-y_1)^2\left(\int_0^1\left|\varkappa-\frac{1}{3}\right|d\varkappa\right)^{\frac{1}{p}}\times\left(\int_0^1\left|\varkappa-\frac{1}{3}\right||\varrho'(\varkappa x+(1-\varkappa)y_1)|^q d\varkappa\right)^{\frac{1}{q}} \\
&+(z_1-x)^2\left(\int_0^1\left|\varkappa-\frac{2}{3}\right|d\varkappa\right)^{\frac{1}{p}}\times\left(\int_0^1\left|\varkappa-\frac{2}{3}\right||\varrho'(\varkappa z_1+(1-\varkappa)x)|^q d\varkappa\right)^{\frac{1}{q}} \\
&\leq (x-y_1)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\int_0^1\left|\varkappa-\frac{1}{3}\right|\left(\varkappa|\varrho'(x)|^q+(1-\varkappa)|\varrho'(y_1)|^q\right)d\varkappa\right)^{\frac{1}{q}} \\
&\quad+(z_1-x)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\int_0^1\left|\varkappa-\frac{2}{3}\right||\left(\varkappa|\varrho'(z_1)|^q+(1-\varkappa)|\varrho'(x)|^q d\varkappa\right)^{\frac{1}{p}}d\varkappa \\
&=(x-y_1)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\int_0^1\varkappa\left|\varkappa-\frac{1}{3}\right||\varrho'(x)|^q d\varkappa+\int_0^1(1-\varkappa)\left|\varkappa-\frac{1}{3}\right||\varrho'(y_1)|^q d\varkappa\right)^{\frac{1}{q}} \\
&\quad+(z_1-x)^2\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\int_0^1\varkappa\left|\varkappa-\frac{2}{3}\right||\varrho'(z_1)|^q d\varkappa+\int_0^1(1-\varkappa)\left|\varkappa-\frac{2}{3}\right||\varrho'(x)|^q d\varkappa\right)^{\frac{1}{q}} \\
&=\left(\frac{5}{18}\right)^{\frac{1}{p}}\left[(x-y_1)^2\left(\frac{8|\varrho'(y_1)|^q}{81}+\frac{29|\varrho'(x)|^q}{162}\right)^{\frac{1}{q}}+(z_1-x)^2\left(8\frac{|\varrho'(x)|^q}{81}+\frac{29|\varrho'(z_1)|^q}{162}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Example 3. *Let the function f be defined as $f(x)=x^3$. Then the function f is convex on $[0,1]$. We have*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|=\left|\frac{4x^3}{3}-\frac{2x}{3}\right| \tag{14}$$

and

$$\begin{aligned}
&\left(\frac{5}{18}\right)^{\frac{1}{p}}\left[(x-y_1)^2\left(\frac{8|\varrho'(y_1)|^q}{81}+\frac{29|\varrho'(x)|^q}{162}\right)^{\frac{1}{q}}+(z_1-x)^2\left(8\frac{|\varrho'(x)|^q}{81}+\frac{29|\varrho'(z_1)|^q}{162}\right)^{\frac{1}{q}}\right] \\
&=\frac{5^{\frac{1}{4}}3^{\frac{1}{2}}}{27}(x+1)^2(29|x|^{\frac{8}{3}}+16)^{\frac{3}{4}}
\end{aligned} \tag{15}$$

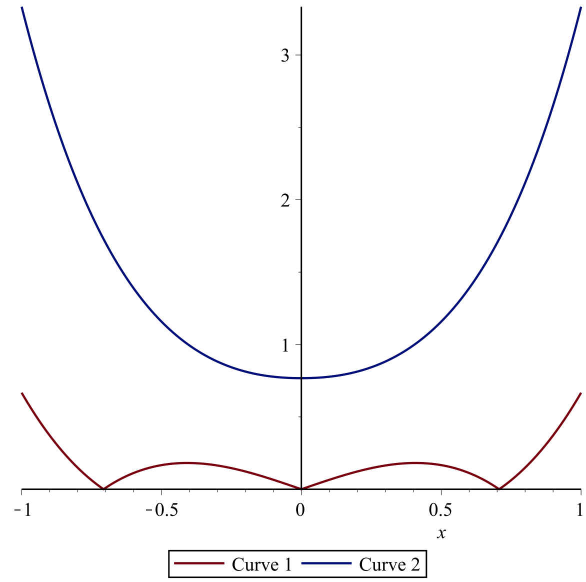

Figure 3:

Here Curve 1 and Curve 2 represents the expression (14) and (15) respectively. Figure 3 depicts that the Curve 1 is below Curve 2. Hence, it also refers to our calculation where the value of the expression (14) is less than the expression (15). This validates the inequality (13).

Theorem 5. *Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $|\varrho'|$ is quasi-convex on $[y_1, z_1]$, then the following inequality holds:*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right| \\ \leq \frac{5(x-y_1)^2}{18}max\{|\varrho'(x)|,|\varrho'(y_1)|\}+\frac{5(z_1-x)^2}{18}max\{|\varrho'(z_1),|\varrho'(x)|\}. \tag{16}$$

Proof. Using Theorem 1 and the quasi-convexity of $|\varrho'|$, we have

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|$$
$$\leq (x-y_1)^2\int_0^1\left|\varkappa-\frac{1}{3}\right||\varrho'(\varkappa x+(1-\varkappa)y_1)|d\varkappa+(z_1-x)^2\int_0^1\left|\varkappa-\frac{2}{3}\right||\varrho'(\varkappa z_1+(1-\varkappa)x)|d\varkappa.$$
$$\leq (x-y_1)^2\int_0^1\left|\varkappa-\frac{1}{3}\right|max\{|\varrho'(x)|,|\varrho'(y_1)|\}d\varkappa+(z_1-x)^2\int_0^1\left|\varkappa-\frac{2}{3}\right|max\{|\varrho'(z_1),|\varrho'(x)|\}d\varkappa$$
$$=\frac{5(x-y_1)^2}{18}max\{|\varrho'(x)|,|\varrho'(y_1)|\}+\frac{5(z_1-x)^2}{18}max\{|\varrho'(z_1),|\varrho'(x)|\}$$

This completes the proof. □

Example 4. *Let the function f be defined as $f(x)=x^5$. Then the function f is convex on $[-3,5]$. We have*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right| = |\frac{16x^5}{3}-\frac{3368x}{3}+\frac{7448}{3}| \tag{17}$$

and

$$\frac{5(x-y_1)^2}{18}max\{|\varrho'(x)|,|\varrho'(y_1)|\}+\frac{5(z_1-x)^2}{18}max\{|\varrho'(z_1),|\varrho'(x)|\} \\ =\frac{5(x+3)^2max\{405,5x^4\}}{18}+\frac{5(5-x)^2max\{3125,5x^4\}}{18}. \tag{18}$$

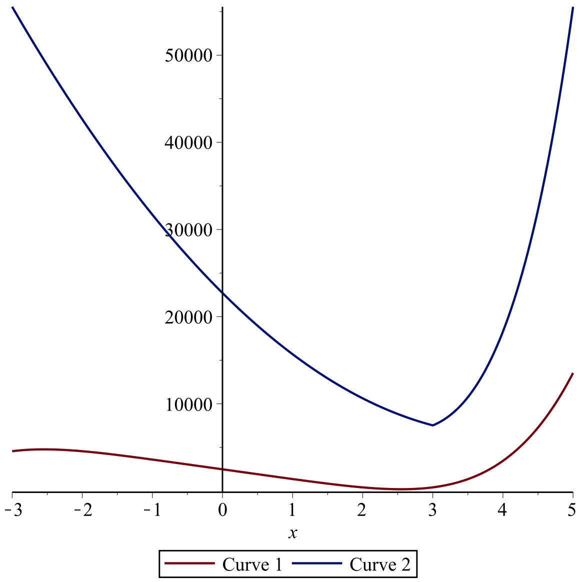

Figure 4:

Here Curve 1 and Curve 2 represents the expression (17) and (18) respectively. Figure 4 depicts that the Curve 1 is below Curve 2. Hence, it also refers to our calculation where the value of the expression (17) is less than the expression (18). This validates the inequality (16).

Theorem 6. *Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $|\varrho'|^q$ is quasi-convex on $[y_1, z_1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|\\&\le (x-y_1)^2\left(\frac{2^{p+1}+1}{(3p+1)3^p}\right)^{\frac{1}{p}}\left(max\{|\varrho'(x)|^q,|\varrho'(y_1)|^q\}\right)^{\frac{1}{q}}\\&\quad+(z_1-x)^2\left(\frac{2^{p+1}+1}{(3p+1)3^p}\right)^{\frac{1}{p}}\left(max\{|\varrho'(z_1)|^q,|\varrho'(x)|^q\}\right)^{\frac{1}{q}}. \qquad (19)\end{aligned}$$

Proof. Using Theorem 1, Holder's inequality and the quasi-convexity of $|\varrho'|^q$, we have

$$\begin{aligned}&\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|\\&\le (x-y_1)^2\int_0^1\left|\varkappa-\frac{1}{3}\right||\varrho'(\varkappa x+(1-\varkappa)y_1)|d\varkappa+(z_1-x)^2\int_0^1\left|\varkappa-\frac{2}{3}\right||\varrho'(\varkappa z_1+(1-\varkappa)x)|d\varkappa.\\&\le (x-y_1)^2\left(\int_0^1\left|\varkappa-\frac{1}{3}\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1|\varrho'(\varkappa x+(1-\varkappa)y_1)|^q d\varkappa\right)^{\frac{1}{q}}\\&\quad+(z_1-x)^2\left(\int_0^1\left|\varkappa-\frac{2}{3}\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1|\varrho'(\varkappa z_1+(1-\varkappa)x)|^q d\varkappa\right)^{\frac{1}{q}}\\&\le (x-y_1)^2\left(\frac{2^{p+1}+1}{(3p+1)3^p}\right)^{\frac{1}{p}}\left(max\{|\varrho'(x)|^q,|\varrho'(y_1)|^q\}\right)^{\frac{1}{q}}\\&\quad+(z_1-x)^2\left(\frac{2^{p+1}+1}{(3p+1)3^p}\right)^{\frac{1}{p}}\left(max\{|\varrho'(z_1)|^q,|\varrho'(x)|^q\}\right)^{\frac{1}{q}}. \qquad (20)\end{aligned}$$

This completes the proof. □

Example 5. *Let the function f be defined as $f(x) = x^7$. Then the function f is convex on $[-2, 1]$. We have*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3} + \frac{(x-y_1)\varrho(y_1)}{3} + \frac{(z_1-x)\varrho(z_1)}{3} - \int_{y_1}^{z_1} \varrho(s)ds\right| = |2x^7 - 43x - \frac{425}{8}| \tag{21}$$

and

$$\begin{aligned}(x-y_1)^2\left(\frac{2^{p+1}+1}{(3p+1)3^p}\right)^{\frac{1}{p}}\left(max\{|\varrho'(x)|^q, |\varrho'(y_1)|^q\}\right)^{\frac{1}{q}} \\ + (z_1-x)^2\left(\frac{2^{p+1}+1}{(3p+1)3^p}\right)^{\frac{1}{p}}\left(max\{|\varrho'(z_1)|^q, |\varrho'(x)|^q\}\right)^{\frac{1}{q}} \\ = \frac{1}{3}(x+2)^2(max\{200704, 49x^12\})^{\frac{1}{2}} + \frac{1}{3}(1-x)^2(max\{49, 49x^12\})^{\frac{1}{2}}.\end{aligned} \tag{22}$$

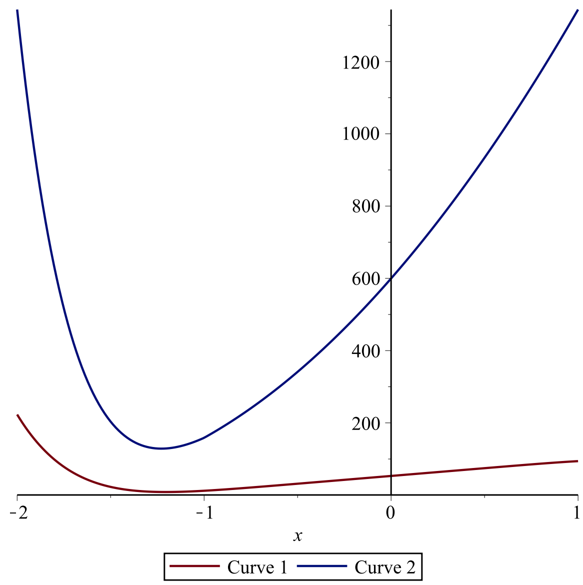

Figure 5:

Here Curve 1 and Curve 2 represents the expression (21) and (22) respectively. Figure 5 depicts that the Curve 5 is below Curve 2. Hence, it also refers to our calculation where the value of the expression (21) is less than the expression (22). This validates the inequality (19).

Theorem 7. *Let $\varrho : [y_1, z_1] \to \mathbb{R}$ be a differentiable function on (y_1, z_1) with $y_1 < z_1$. If $|\varrho'|^q$ is quasi-convex on $[y_1, z_1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{2(z_1-y_1)\varrho(x)}{3} + \frac{(x-y_1)\varrho(y_1)}{3} + \frac{(z_1-x)\varrho(z_1)}{3} - \int_{y_1}^{z_1} \varrho(s)ds\right| \\ &\leq \frac{5(x-y_1)^2}{18}\left(max\{|\varrho'(x)|^q, |\varrho'(y_1)|^q\}\right)^{\frac{1}{q}} + \frac{5(z_1-x)^2}{18}\left(max\{|\varrho'(z_1)|^q, |\varrho'(x)|^q\}\right)^{\frac{1}{q}}.\end{aligned} \tag{23}$$

Proof. Using Theorem 1, Power-mean inequality and the quasi-convexity of $|\varrho'|^q$, we have

$$\begin{aligned}
&\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right| \\
&\leq (x-y_1)^2\int_0^1\left|\varkappa-\frac{1}{3}\right||\varrho'(\varkappa x+(1-\varkappa)y_1)|d\varkappa+(z_1-x)^2\int_0^1\left|\varkappa-\frac{2}{3}\right||\varrho'(\varkappa z_1+(1-\varkappa)x)|d\varkappa. \\
&\leq (x-y_1)^2\left(\int_0^1\left|\varkappa-\frac{1}{3}\right|d\varkappa\right)^{\frac{1}{p}}\times\left(\int_0^1\left|\varkappa-\frac{1}{3}\right||\varrho'(\varkappa x+(1-\varkappa)y_1)|^q d\varkappa\right)^{\frac{1}{q}} \\
&\qquad+(z_1-x)^2\left(\int_0^1\left|\varkappa-\frac{2}{3}\right|d\varkappa\right)^{\frac{1}{p}}\times\left(\int_0^1|\left|\varkappa-\frac{2}{3}\right|\varrho'(\varkappa z_1+(1-\varkappa)x)|^q d\varkappa\right)^{\frac{1}{q}} \\
&\leq (x-y_1)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\int_0^1\left|\varkappa-\frac{1}{3}\right|(max\{|\varrho'(x)|^q,|\varrho'(y_1)|^q\}d\varkappa\right)^{\frac{1}{q}} \\
&\quad+(z_1-x)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\int_0^1\left|\varkappa-\frac{2}{3}\right|(max\{|\varrho'(z_1)|^q,|\varrho'(x)|^q\}d\varkappa\right)^{\frac{1}{q}} \\
&= (x-y_1)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\frac{5}{18}\right)^{\frac{1}{q}}\left(max\{|\varrho'(x)|^q,|\varrho'(y_1)|^q\}\right)^{\frac{1}{q}} \\
&\quad+(z_1-x)^2\left(\frac{5}{18}\right)^{\frac{1}{p}}\left(\frac{5}{18}\right)^{\frac{1}{q}}\left(max\{|\varrho'(z_1)|^q,|\varrho'(x)|^q\}\right)^{\frac{1}{q}} \\
&= \frac{5(x-y_1)^2}{18}\left(max\{|\varrho'(x)|^q,|\varrho'(y_1)|^q\}\right)^{\frac{1}{q}}+\frac{5(z_1-x)^2}{18}\left(max\{|\varrho'(z_1)|^q,|\varrho'(x)|^q\}\right)^{\frac{1}{q}}. \qquad (24)
\end{aligned}$$

This completes the proof. □

Example 6. *Let the function f be defined as $f(x)=x^3$. Then the function f is convex on $[-8,-2]$. We have*

$$\left|\frac{2(z_1-y_1)\varrho(x)}{3}+\frac{(x-y_1)\varrho(y_1)}{3}+\frac{(z_1-x)\varrho(z_1)}{3}-\int_{y_1}^{z_1}\varrho(s)ds\right|=|4x^3-168x-340| \qquad (25)$$

and

$$\begin{aligned}
&\frac{5(x-y_1)^2}{18}\left(max\{|\varrho'(x)|^q,|\varrho'(y_1)|^q\}\right)^{\frac{1}{q}}+\frac{5(z_1-x)^2}{18}\left(max\{|\varrho'(z_1)|^q,|\varrho'(x)|^q\}\right)^{\frac{1}{q}} \\
&=\frac{5^{\frac{4}{5}}(65)^{\frac{1}{5}}}{54}\left((x+8)^2max\{384(2^{\frac{1}{2}})(3^{\frac{1}{4}}),3^{\frac{5}{4}}|x|^{\frac{5}{2}}\}^{\frac{4}{5}}+(2+x)^2max\{12,3^{\frac{5}{4}|x|^{\frac{5}{2}}}\}\right). \qquad (26)
\end{aligned}$$

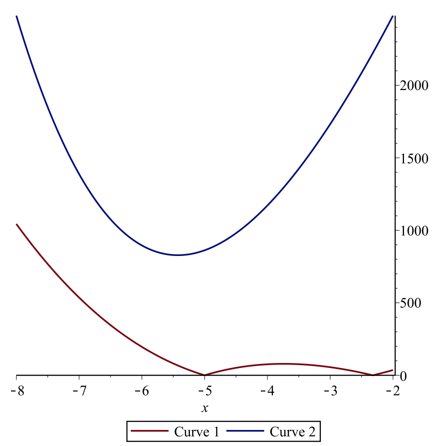

Figure 6:

Here Curve 1 and Curve 2 represents the expression (25) and (26) respectively. Figure 6 depicts that the Curve 1 is below Curve 2. Hence, it also refers to our calculation where the value of the expression (25) is less than the expression (26). This validates the inequality (23).

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