

# Modelling the fear effect in prey predator ecosystem incorporating prey patches

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## Abstract

**In an ecosystem, the balance of prey-predator system is greatly influenced by the availability of prey and the fear imposed on its population.** In this paper, it is proposed that a prey-predator model in which prey is assumed to be able to detect the presence of predator and to counteract it by forming patches and incorporating the cost of fear into prey reproduction. **Equilibrium points are calculated and analysis of the local and global asymptotic behaviors of the system are done. Hopf-bifurcation is seen in case of adequate availability of prey. The system stabilizes in presence of high levels of fear.** Availability of prey act as a crucial role to change the dynamics of the system. **Numerical simulations showcases the relationship between prey patches and other related parameters like level of fear, conversion rate of predator and availability of prey. These simulations reveal the impact of fear on the prey-predator system and also justify the theoretical findings.** In the end, the bifurcation scenarios are derived when two different parameters switch together at a same time. Numerical simulations are justified the theoretical findings.

**Keywords:** Fear; Patches; Hunting Stability; Bifurcation.

## 1 Introduction

The survey of prey-predator dynamics is one of the blooming topics of ecosystem in last few decades. Predation process perform an indispensable part to maintain ecological balance. In real field application, the predator do not capture all the prey population due to refuge property of prey [1, 2]. In biomathematics, the research of prey refuge is one of the hot spot area. As a result, many researchers focus in this aspect [3, 4, 5]. Some experimental finding confirm that fear effect

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on predator may alter the behavior of prey [6, 7, 8]. Some theoretical studies have revealed that growth rate of prey need to improve through implementation of fear effects [9, 10, 11]. Recently, the authors in [12] studied the hunting cooperation and the fear factor among prey in a Leslie-Gower model. This study revealed that fear factor is more effective than hunting cooperation to stabilize the system. Also, the scientists in [13] proposed a Beddington-DeAngelis functional response of predator-prey model and investigated the impact of anti-predator activity on whole system. They noted that the system may exhibits multiple Hopf-bifurcation. The researchers in [14] investigated that chaotic system turned into stable system in presence of cost of fear in three species model. But very few numbers of researchers explored the combine effects of hunting cooperation and anti-predator activity in predator-prey system. In recent past, the authors in [15] studied the combine effects of hunting cooperation and fear factor in prey-predator system and observed that strong demographic Allee phenomenon. **Recently, the authors in [16] studies the influence of harvesting and allee effects in disease induced prey-predator system and reveals that allee effect and harvesting can be a handy technique for controlling the spread of disease. Fractional order mathematical models are a new research field in non-linear dynamics [17, 18]. The authors in [19] apply the homotopy analysis transform technique in prey-predator model to evaluate approximate solution which converges to the exact solution of time-fractional nonlinear subject to initial conditions.**

**Anti-grazing strategy is a vital part in prey-predator system to protect prey from predator.** In marine system, size of phytoplankton are very small compare to the predatory enemies but they can survive from consumes by using anti-grazing strategies like morphology [20] formation of colonies [21] which resist the grazing pressure by higher trophic organisms. Toxin ejected by phytoplankton is one of another anti-grazing strategies to protect from zooplankton [22]. The author in [23] studied the formulation of patches for defense mechanism and discussed the ability of releasing toxin chemicals. Thus, paired mechanism over with patching and poison release outcomes will act a crucial role for the coexistence species. Some experimental researches noted that the patch size depend on organism density and also proportional with it [24]. In real field, phytoplankton are allowed to form spherical patches or colonies and release toxin chemicals [25].

**Motivated by the above theoretical and experimental literatures,** the dynamics of such system in which hunting by predator and fear of prey is studied. The aim of the present study is to investigate the impact of hunting, fear effect and toxin effect due to formulation of patches. As per my knowledge, the combine effect of three above factors has not to explore yet. **The main target in present manuscript is to investigate the subsequent biological topics:**

- How does availability of prey density influence on the dynamics of prey-predator system.
- Can fear factor among prey influence to stabilize the prey-predator system.

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- How does patches influence the prey-predator dynamics.

It is considered that, birth rate of prey population is reduced due to fear of hunting by predator. In the next section, proposed model is developed with incorporate prey patches. Section 2 represents the construction of mathematical model based on some assumptions. Basic properties such as boundedness is discussed in Section 3. Analytical results based on the model and **global stability** are discussed in Section 4. Section 5 represents the local bifurcation such as Hopf and transcritical-bifurcation analysis. Numerical simulations and discussion are illustrated in Section 6 & 7. Finally, the paper summarize with a brief conclusion.

## 2 Basic assumptions and model formulation

Let us consider the assumption to construct the following mathematical model: Let  $x(t)$  and  $y(t)$  be the density of prey and predator population at time  $t > 0$  respectively. Here  $r$  and  $r_1$  be the intrinsic growth rate and the intra-species competition rate of prey.  $c$  and  $e$  represent the predation rate and conversion rate of predator. Here  $(1-k_1)$  terms represents the amount of availability of prey for predation by the predator where,  $k_1 \in (0, 1]$ . It is assumed that predation term is the Holling-II functional form. According to literature review, a fraction part  $k_1$  of prey aggregate to form  $N$  patches. Therefore, each patches represent as  $\frac{1}{N}k_1x$ . It is assume that the three dimensional patch is roughly spherical in ocean. Therefore, the radius of patch is proportional to  $[\frac{1}{N}k_1x]^{1/3}$ . As a result the surface of patch is proportional to  $[\frac{1}{N}k_1x]^{2/3} = \rho x^{2/3}$ , where  $\rho = [\frac{1}{N}k_1]^{2/3}$ . **The effect of fear has a direct impact on prey reproduction [26, 27, 28].** In presence of predator, intrinsic growth of prey becomes a function of the predator density like  $F(y; K) = \frac{r}{1+Ky}$  in which  $K$  is defined as level of fear of the prey according to anti-predator response. **This above function follows some conditions:**

- (i)  $F(y; 0) = r$ : in the absence of fear effect, the prey reproduction rate remain unaltered.
- (ii)  $F(0; K) = r$ : in the absence of predator, **the prey reproduction rate remain unaltered.**
- (iii)  $\lim_{K \rightarrow \infty} F(y; K) = 0$ : **extremely fearful prey fails to reproduce.**
- (iv)  $\lim_{y \rightarrow \infty} F(y; K) = 0$ : **at a extremely higher predator density, prey fails to reproduce.**
- (v)  $\frac{\partial F(y; K)}{\partial K} < 0$ : the prey reproduction rate low with high amount of fear effect.
- (vi)  $\frac{\partial F(y; K)}{\partial y} < 0$ : the prey reproduction rate low with high amount of predator density.

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$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+Ky} - r_1x^2 - \frac{c(1-k_1)xy}{1+a(1-k_1)x} \equiv G_1(x,y) \\ \frac{dy}{dt} &= \frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y \equiv G_2(x,y). \end{aligned} \tag{1}$$

The system (1) will be analyzed with the following initial conditions,

$$x(0) \geq 0, y(0) \geq 0. \tag{2}$$

### 3 Mathematical preliminaries

**Theorem 1.** *All non negative solutions  $(x(t), y(t))$  of the system (1) initiate in  $R_+^2 - \{0, 0\}$  are uniformly bounded.*

*Proof.* Let us choose a function  $\Theta = x + y$ .  
Therefore,

$$\frac{d\Theta}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \frac{rx}{1+Ky} - r_1x^2 - \frac{c(1-k_1)xy}{1+a(1-k_1)x} + \frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y.$$

Let us consider a positive constant  $\zeta$  such that  $\zeta \leq d$ . Therefore,

$$\begin{aligned} \frac{d\Theta}{dt} + \zeta\Theta &\leq r_0x - r_1x^2 + \zeta x - \frac{(1-k_1)(c-e)}{1+a(1-k_1)x} - y(d - \zeta) - e\rho x^{2/3}y \\ &\leq (r_0 + \zeta)x - r_1x^2 \leq \frac{(r_0 + \zeta)^2}{4r_1}. \end{aligned}$$

By choosing  $\Gamma = \frac{(r_0 + \zeta)^2}{4r_1}$ , we obtain

$$0 \leq \Theta(x(t), y(t)) \leq \frac{\Gamma}{\zeta}(1 - e^{-\zeta t}) + \Theta(x(0), y(0))e^{-\zeta t},$$

which indicates that  $0 \leq \Theta(x(t), y(t)) \leq \frac{\Gamma}{\zeta}$  as  $t \rightarrow \infty$ . Therefore, all non negatives solutions of the system (1) are originated from  $R_+^2 - \{0, 0\}$  will be restricted in the region  $\nabla = \{(x, y) \in R_+^2 : x(t) + y(t) \leq \frac{\Gamma}{\zeta} + \varepsilon\}$ .

In ecology, it means that the system act in a specified manner. Boundedness of the system implies that none of the two interacting species grow unexpectedly or exponentially for a long period of time. Clearly, as a result of limited resource, numbers of each species is surely bounded.  $\square$

**From the ecological point of view, let us first consider the following region  $R_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$ . Here, the function  $G_1(x, y) = xf(x, y)$  and  $G_2 = yg(x, y)$  of the system (1) are continuously differentiable and locally Lipschitz in  $R_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$ . Therefore, Theorem A.4, page 423 in H. R. Thieme's book [29] implies that the solutions of the initial value problem with non-negative initial conditions exist on the interval  $[0, S)$  and unique, where  $S$  is a sufficiently large number.**

### 4 Equilibria: Existence and stability

All possible equilibria are catalogued below:

(i) The predator free equilibrium  $E_1 = (\frac{r}{r_1}, 0)$ .

(ii) The positive coexistence equilibrium  $E^* = (x^*, y^*)$ ,

while  $x^*$  is ensured by solving  $\{a(1 - k_1)\}^3 e^3 \rho^3 x^{*5} + 3\{a(1 - k_1)\}^2 e^3 \rho^3 x^{*4} + [3e^3 \rho^3 a(1 - k_1) - \{(1 - k_1)(e - da)\}^3] x^{*3} + [e^3 \rho^3 + 3\{(1 - k_1)(e - da)\}^2 d] x^{*2} - 3\{(1 - k_1)(e - da)\} d^2 x^* + d^3 = 0$ .

Also,  $y^*$  is ensured by solving  $cK(1 - k_1)y^2 + [c(1 - k_1) + r_1x^*(1 + a(1 - k_1)x^*)K]y^* - (1 + a(1 - k_1)x^*)(r - r_1x^*) = 0$ .

Thus the condition for the existence of the interior equilibrium point  $E^*(x^*, y^*)$  is given by,  $x^* > 0, y^* > 0$ .

Explicitly, general form of the Jacobian matrix at  $\bar{E} = (\bar{x}, \bar{y})$  is defined as

$$\bar{J} = \begin{bmatrix} \frac{r}{(1+K\bar{y})} - 2r_1\bar{x} - \frac{c(1-k_1)\bar{y}}{(1+a(1-k_1)\bar{x})^2} & -\frac{rK\bar{x}}{(1+K\bar{y})^2} - \frac{c(1-k_1)\bar{x}}{1+a(1-k_1)\bar{x}} \\ \frac{e(1-k_1)\bar{y}}{(1+a(1-k_1)\bar{x})^2} - \frac{2}{3}e\rho\bar{y}\frac{1}{\bar{x}^{1/3}} & \frac{e(1-k_1)\bar{x}}{1+a(1-k_1)\bar{x}} - d - e\rho\bar{x}^{2/3} \end{bmatrix}. \tag{3}$$

There exists a feasible predator free steady state  $E_1$  of the system (1) which is unstable if  $\frac{d}{e} + \rho\frac{r}{r_1}^{2/3} < \frac{(1-k_1)r}{a(1-k_1)r+r_1}$ .

The Jacobian matrix at  $E^*$  can be written as

$$J^* = \begin{bmatrix} \frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} & -\frac{rKx^*}{(1+Ky^*)^2} - \frac{c(1-k_1)x^*}{1+a(1-k_1)x^*} \\ \frac{e(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} - \frac{2}{3}e\rho\frac{y^*}{x^{*1/3}} & 0 \end{bmatrix}.$$

Thus the eigenvalues in this case are obtained as roots of the quadratic  $\lambda^2 - tr(J^*) + det(J^*) = 0$ ,

$$tr(J^*) = \frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2},$$

$$det(J^*) = [\frac{rK}{(1+Ky^*)^2} + \frac{c(1-k_1)}{1+a(1-k_1)x^*}] [\frac{e(1-k_1)}{(1+a(1-k_1)x^*)^2} - \frac{2}{3}e\rho\frac{1}{x^{*1/3}}] x^* y^*.$$

Now  $tr(J^*) < 0$  if  $\frac{r}{(1+Ky^*)} < 2r_1x^* + \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2}$  as well as  $det(J^*) > 0$  if  $\rho < \frac{27}{8} \frac{(1-k_1)^3 x^*}{(1+a(1-k_1)x^*)^6}$ .

Therefore, according to the Routh–Hurwitz criterion we can admit that  $E^*$  is locally asymptotically stable providing the above two conditions are fulfilled.

**Theorem 2.** *If the non negative equilibrium  $E^*$  exists, then  $(x^*, y^*)$  is globally asymptotically stable in the  $x - y$  plane if  $r_1 > \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*}$ .*

*Proof.* Let us consider a Lyapunov function about  $E^*$

$$V = x - x^* - x^* \ln \frac{x}{x^*} + \frac{c}{e} (1 + a(1 - k_1)x^*) (y - y^* - y^* \ln \frac{y}{y^*}).$$

Differentiating  $V$  with respect to  $t$  of the system (1), we get

$$\frac{dV}{dt} = (x - x^*) (\frac{r}{1+Ky} - r_1x - \frac{c(1-k_1)y}{1+a(1-k_1)x}) + \frac{c}{e} (1 + a(1 - k_1)x^*) (y - y^*) (\frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y)$$

$$= (x - x^*) \left( \frac{rK(y - y^*)}{(1+Ky)(1+Ky^*)} - r_1(x - x^*) + \frac{c(1-k_1)(y - y^*)}{1+a(1-k_1)x} + \frac{c(1-k_1)^2 a(x - x^*)}{[1+a(1-k_1)x][1+a(1-k_1)x^*]} \right) + \frac{c}{e} (1 + a(1 - k_1)x^*) (y - y^*) \left[ \frac{e(1-k_1)(x - x^*)}{(1+a(1-k_1)x)(1+a(1-k_1)x^*)} - e\rho(x^{\frac{2}{3}} - x^{*\frac{2}{3}}) \right].$$

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After some calculation and simplification we get

$$\leq - \left[ r_1 - \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*} \right] (x - x^*)^2 - \frac{rK}{(1+Ky)} (x - x^*)(y - y^*).$$

Clearly,  $\dot{V}$  is negative definite if  $r_1 > \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*}$ . Therefore by LaSalle's theorem [30]  $E^*$  is globally asymptotically stable in  $x - y$  plane.  $\square$

## 5 Local bifurcation

### 5.1 Hopf-Bifurcation

**Theorem 3.** *The necessary and sufficient conditions for Hopf bifurcation of the system (1) around  $E^*$  at  $k_1 = k_1^c$  are  $[tr(J^*)]_{k_1=k_1^c} = 0$ ,  $[det(J^*)]_{k_1=k_1^c} > 0$  and  $\frac{d}{dk_1}[tr(J^*)]_{k_1=k_1^c} \neq 0$ .*

*Proof.* The condition  $[tr(J^*)]_{k_1=k_1^c} = 0$  gives  $\frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} = 0$ , in which  $[tr(J^*)]_{k_1=k_1^c} = 0$ .

Now  $[det(J^*)]_{k_1=k_1^c} > 0$  which is equivalent to the characteristic equation  $\lambda^2 + [det(J^*)]_{k_1=k_1^c} = 0$  whose roots are purely imaginary, For  $k_1 = k_1^c$ , the characteristic can be written as

$$\chi^2 + \omega = 0, \tag{4}$$

where  $\omega = [det(J^*)]_{k_1=k_1^c} > 0$ . Therefore, the above equation has two roots of the form  $\chi_1 = +i\sqrt{\omega}$  and  $\chi_2 = -i\sqrt{\omega}$ . Let at any neighbouring point  $k_1$  of  $k_1^c$ , we can express the above roots in general form like  $\chi_{1,2} = \theta_1(k_1) + \pm i\theta_2(k_1)$ ,

where  $\theta_1(k_1) = \frac{tr(J^*)}{2}$  and  $\theta_2(k_1) = \sqrt{det(J^*) - \frac{tr(J^*)^2}{4}}$ . Now it is to be verified the transversality condition  $\frac{d}{dk_1}(Re(\chi_j(k_1)))_{k_1=k_1^c} \neq 0$  for  $j = 1, 2$ .

Substituting  $\chi_1 = \theta_1(k_1) + i\theta_2(k_1)$  in (4) and calculate the derivative, we have

$$\begin{aligned} 2\theta_1(k_1)\theta_1'(k_1) - 2\theta_2(k_1)\theta_2'(k_1) + \omega' &= 0, \\ 2\theta_2(k_1)\theta_1'(k_1) + 2\theta_1(k_1)\theta_2'(k_1) &= 0. \end{aligned} \tag{5}$$

Solving (5), we get

$\frac{d}{dk_1}(Re(\chi_j(k_1)))_{k_1=k_1^c} = \frac{-2\theta_1\omega'}{2(\theta_1^2 + \theta_2^2)} \neq 0$ , i.e.,  $\frac{d}{dk_1}[tr(J^*)]_{k_1=k_1^c} \neq 0$ , which satisfy the transversality condition. This implies that the system undergoes a Hopf-bifurcation at  $k_1 = k_1^c$ .  $\square$

### 5.2 Transcritical-bifurcation

**Theorem 4.** *System (1) undergoes a transcritical bifurcation when the system parameters satisfy the restriction  $k_1 = k_1^{TC}$ . Here,  $k_1$  is seen as the bifurcation parameter.*

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*Proof.* For  $k_1 = k_1^{TC}$ , the Jacobian matrix  $J_1$  of the system (1) around  $E_1$  has one zero eigenvalue. Let  $U_1$  and  $V_1$  be the eigenvectors of the matrix  $J_1$  and  $(J_1)^T$  corresponding to zero eigenvalue respectively. Therefore, we obtain  $U_1 = \left(-\frac{r}{r_1} + \frac{c(1-k_1)}{r_1+a(1-k_1)r}, 1\right)^T$  and  $V_1 = (0 \ 1)^T$ . We have  $F_{k_1}(x, y) = \left(0 \ -y\right)^T$ ,  $F_{k_1}(E_1; k_1 = k_1^{TC}) = (0 \ 0)^T$  and  $(V_1)^T F_{k_1}(E_1; k_1 = k_1^{TC}) = 0$ .

Also,  $DF_{k_1}(E_1; k_1 = k_1^{TC})U_1 = (0 \ -1)^T$ .

Therefore, we obtain  $(V_1)^T [DF_{k_1}(E_1; k_1 = k_1^{TC})(U_1)] = -1$ .

Further,  $(V_1)^T D^2F(E_1; k_1 = k_1^{TC})(U_1, U_1)$

$$= -2e \left[ \frac{r_1^2(1-k_1)}{(r_1+a(1-k_1)r)^2} - \frac{2e\rho}{3} \left(\frac{r_1}{r}\right)^{1/3} \right] \left[ \frac{r_1}{r} + \frac{e(1-k_1)}{r_1+a(1-k_1)r} \right] < 0.$$

By applying Sotomayor's theorem [31] we can conclude that the system experiences a transcritical bifurcation at  $E_1$  when  $k_1$  crosses  $k_1^{TC}$ .

□

## 6 Numerical simulations

In order to visualize the analytical finding, we perform the numerical simulation over the set of parametric values [32, 33, 34]

$$\begin{aligned} r &= 1.2, \quad r_1 = 0.05, \quad K = 0.1, \quad k_1 = 0.7, \\ c &= 0.45, \quad e = 0.25, \quad a = 0.3, \quad d = 0.1, \quad \rho = 0.15. \end{aligned} \tag{6}$$

It is noted that the system (1) shows stable dynamics around at  $E^*(3.06, 5.74)$  (cf. Fig. 1(a)).

### 6.1 Effect of $k_1$

It is observed that when availability of prey species is high for predation, i.e., the low value of  $k_1$ , the dynamical system switches to unstable behavior (viz.  $k_1 = 0.66$ ). But high level of fear can stabilize the system (1) (viz.  $K = 0.2$ ). It is illustrated in Fig. 1(b). **Thus, the fear effect can prevent the occurrence of limit cycle oscillation and increase the stability of the system.** Fig. 2(a-b) depicts various steady state behavior of prey and predator for the parameter  $k_1$ . Here, it is noted that a Hopf point are situated (H) at  $k_1 = 0.673026$  with eigenvalue  $\pm 0.284862i$  and one Limit point (LP) and a Branch point (BP) coincide at  $k_1 = 0.864180$  with eigenvalue  $(0. - 1.2)$ . **Branch point (BP) indicates that at that particular point, predator goes to extension and the transcritical bifurcation occurs. The Limit point (LP) is a collision and disappearance of two equilibria in the dynamical system. The system switches from stable to unstable or unstable to stable behavior after crossing the Hopf point(H).** It is observed that the first Lyapunov coefficient being  $-2.654148e^{-03}$  at Hopf point (H) which confirm that a family of stable limit cycle generate from H (viz. Fig. 3(a)). **It is clearly indicates that increasing the amount of prey refuge**

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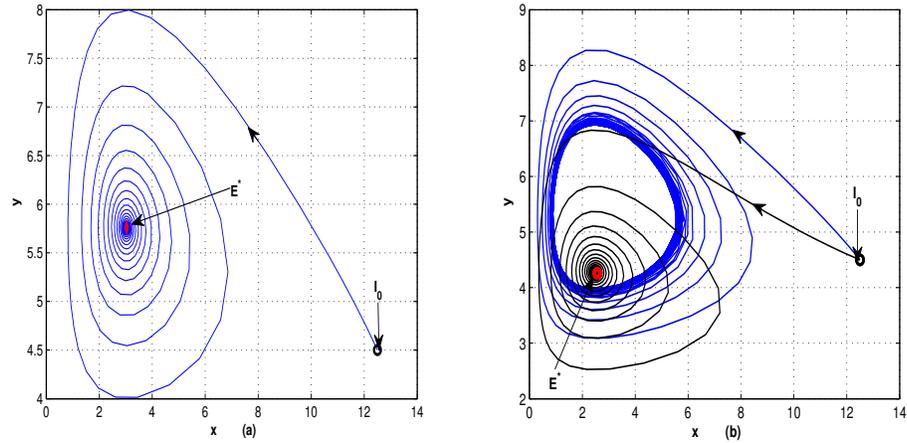


Figure 1: (a) The equilibrium point  $E^*$  is stable for the set of parametric values. (b) The figure depicts oscillatory behavior around at  $E^*$  of system (1) for  $k_1 = 0.66$  and  $K = 0.1$ (blue line), stable behaviour at  $E^*$  for  $k_1 = 0.66$  and  $K = 0.2$ (black line).

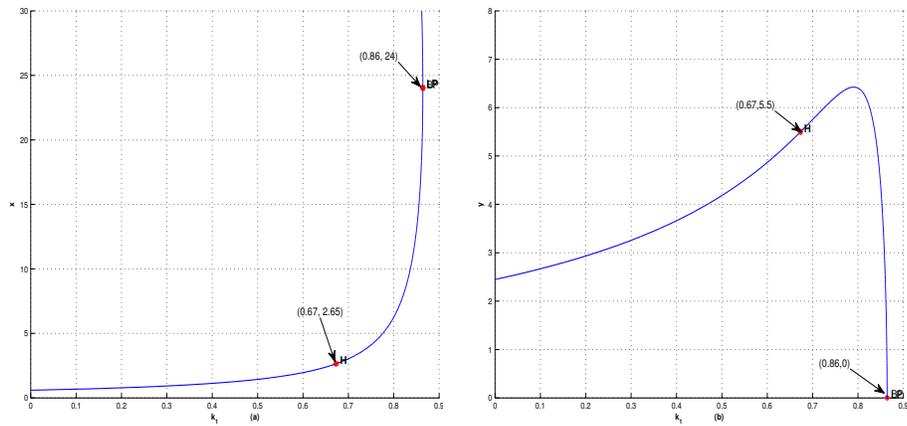


Figure 2: (a-b) The trajectory represents the different dynamical behaviors of prey and predator respectively for  $k_1$ .

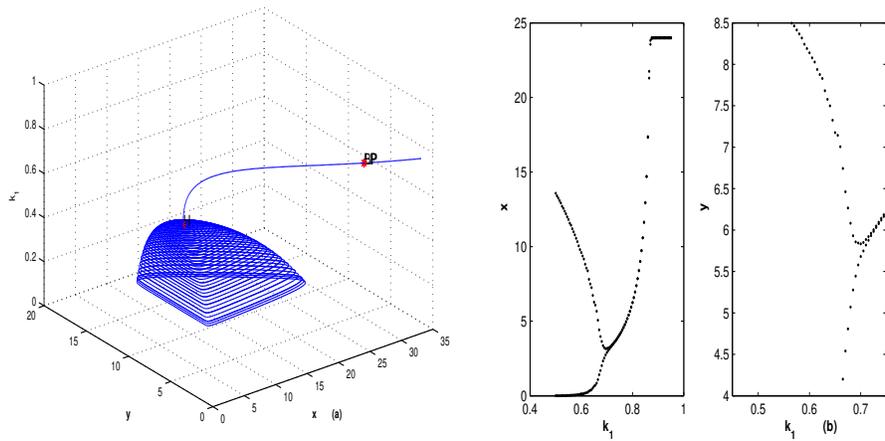


Figure 3: (a) The trajectory represents a family of stable limit cycles generate from Hopf (H) point for  $k_1$  in  $x - y - k_1$  plane. (b) Bifurcation diagram for  $k_1$ .

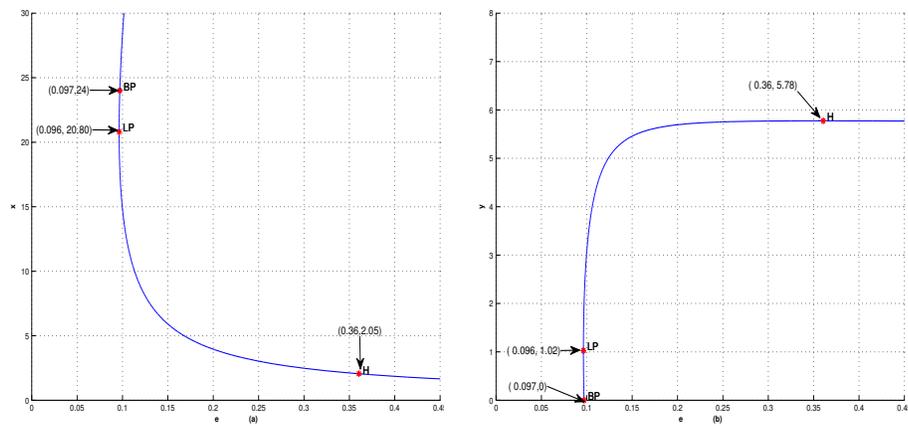


Figure 4: (a-b) The trajectory represents the different dynamical behaviors of prey and predator respectively for  $e$ .

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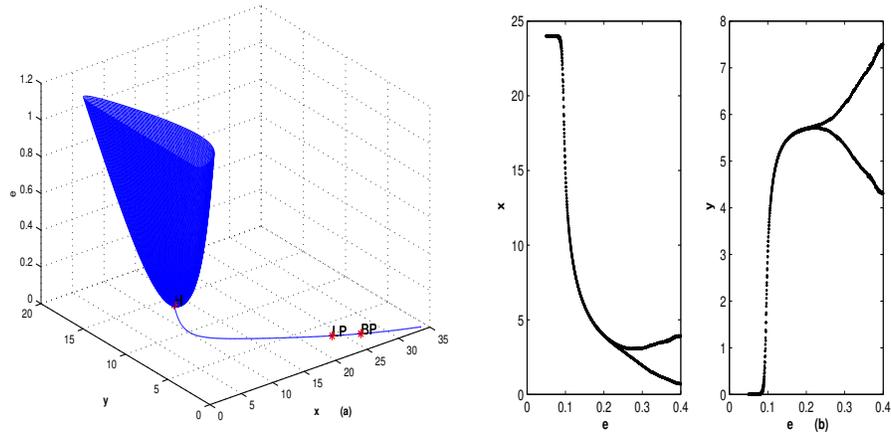


Figure 5: (a) The trajectory represents a family of stable limit cycles generate from Hopf (H) point for  $e$  in  $x - y - e$  plane. (b) Bifurcation diagram for  $e$ .

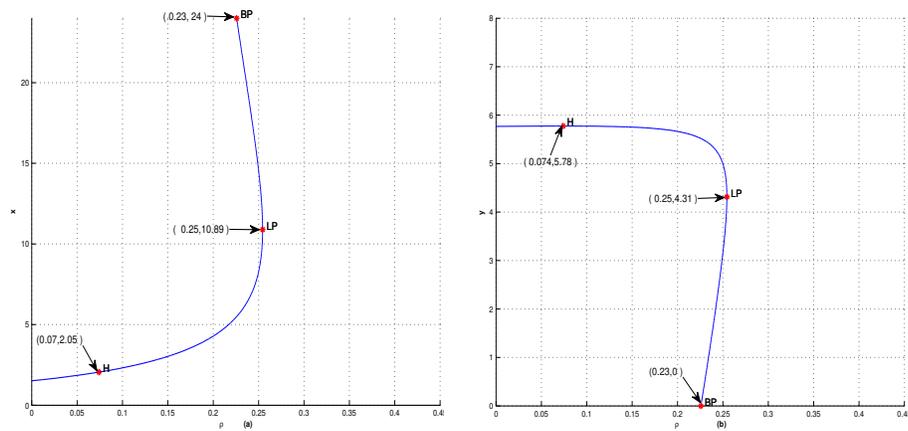


Figure 6: (a-b) The trajectory represents the different dynamical behaviors of prey and predator respectively for  $\rho$ .

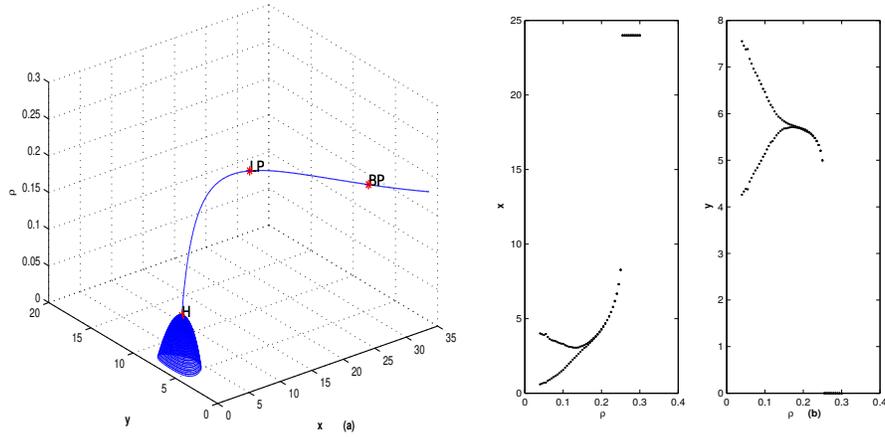


Figure 7: (a) The trajectory represents a family of stable limit cycles generate from Hopf (H) point for  $e$  in  $x - y - \rho$  plane.. (b) Bifurcation diagram for  $\rho$ .

can increase both densities of prey and predator. On the other hand, when  $k_1$  reaches a high risk threshold of the prey refuge the predator goes to extinct and the equilibrium  $E_1$  is globally asymptotically stable.

### 6.2 Effect of $e$

Fig. 4(a-b) indicates that predator’s conversion rate ( $e$ ) play a crucial role to switch the prey and predator natures. Here, we have one Hopf point ( $e = 0.360577$ ), Branch point ( $e = 0.097047$ ) and a Limit point ( $e = 0.096319$ ). Further, the system experiences a family of stable limit cycle generate from Hopf point (viz. Fig. 5(a)).

### 6.3 Effect of $\rho$

It is observed that the prey patches play a big impact in the system (1). From Fig. 6(a-b) & Fig. 7(a) it follow several stability behaviour and family of stable limit cycle for the free parameter  $\rho$  respectively. At  $\rho = 1.416971$ , the system experiences a super critical bifurcation with first Lyapunov coefficient  $-2.031921e^{-03}$  and predator becomes extinct at  $\rho = 0.225770$  i.e., at BP point. Also, a Limit point (LP) is obtained at  $\rho = 0.254407$ .

### 6.4 Bifurcation

The bifurcation diagrams (cf. Fig. 3(b), Fig. 5(b) and Fig. 7(b)) illustrate the complete dynamic pictures of the system (1) for the effect of parameter  $k_1, e$

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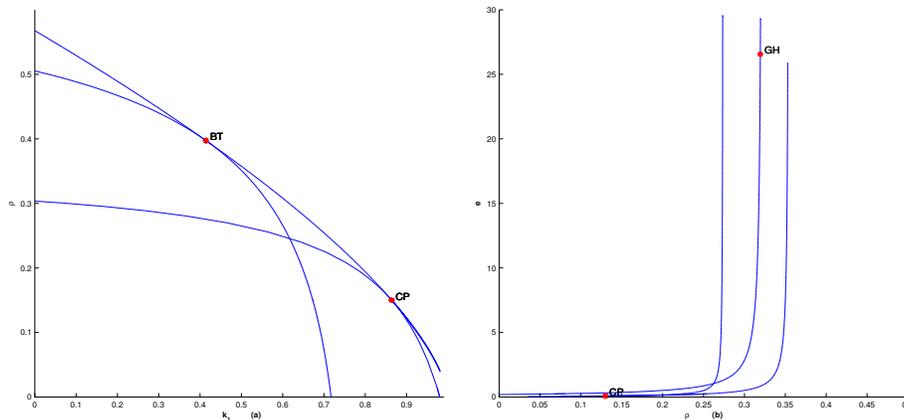


Figure 8: (a) Two parameters bifurcation diagram for  $k_1 - \rho$ . (b) Two parameters bifurcation diagram for  $\rho - e$ .

and  $\rho$  respectively. Fig. 5(a-b) display the two parameters bifurcation diagram for  $k_1 - \rho$  and  $\rho - e$  respectively. In this case, we see a Bogdanov-Takens (BT), Cusp bifurcation (CP) and Generalized Hopf (GH). **Generalized Hopf separates branches of sub-and supercritical Andronov-Hopf bifurcations in the two parameter plain.** The It is clearly indicates that a saddle-node bifurcation curve meet at transcritical curve at Cusp point(CP), i.e., **SN-TC point and saddle-node and Hopf bifurcation curve touch at BT point.** Also, the bifurcation curve exhibits a Generalized Hopf point (GH) where the 1<sup>st</sup> Lyapunov coefficient turn out to be zero. All the numerical finding are summarized in Table 1.

## 7 Discussion

In this present article, a prey-predator model is designed by incorporating patches, prey refuge and fear effect to discover the dynamics of prey-predator systems. **It is assumed that prey population grows logistically and predators consume prey population under Holling II functional response.** Firstly, some basic properties are analyzed and verified which are ecologically well behaved such as boundedness and properties of existence of equilibria. **The local stability behavior of the system is carried out around each equilibrium. In order to explore the dynamics of proposed system, it is identified that, the system (1) has two equilibrium point such as axial ( $E_1$ ) and coexistence equilibrium ( $E^*$ ). We also perform the global stability of coexistence equilibrium by choosing a suitable Lyapunov function.** Throughout the analysis, availability of prey, i.e., the parameter  $k_1$  play crucial role to exhibit Hopf bifurcation and stability

Table 1: Natures of equilibrium points.

Parameters	Values	Eigenvalues	Equilibrium points
$k_1$	0.673026	$(\pm 0.284862i)$	Hopf (H)
	0.864180	$(0, -1.2029)$	Limit Point (LP)
	0.864180	$(0, -1.2029)$	Branch Point (BP)
$e$	0.360577	$(\pm 0.305907i)$	Hopf (H)
	0.096319	$(0, -1.00886)$	Limit Point(LP)
	0.097047	$(0, -1.2)$	Branch Point(BP)
$\rho$	0.074021	$(\pm 0.289879i)$	Hopf (H)
	0.225770	$(0, -1.2)$	Branch Point(BP)
	0.254407	$(0, -0.398958)$	Limit Point(BP)
$(k_1, \rho)$	(0.4146440.397248)	$(\approx \pm 0.00)$	Bogdanov-Takens (BT)
	(0.8639100.150199)	$(0, -1.20027)$	Cusp bifurcation (CP)
$(\rho, e)$	(0.0835480.129990)	$(0, -1.2)$	Cusp bifurcation (CP)
	(0.319445, 26.549989)	$(\pm 1.53468i)$	Generalized Hopf (GH)

switching behavior. Numerically, we observe that when  $k_1 < k_1^c = 0.673026$ , the system exhibits oscillatory behavior and each population shows stable coexistence between  $0.673026 < k_1 < 0.864180$ . When processed further, coexistence equilibrium loses stability via transcritical bifurcation i.e., branch point and the predator population will die out. Similar characteristic nature of prey and predator have been seen for the effect of conversion rate of predator and toxicity level due to patches. Further, to study the impact of fear effect, prey shows anti-predator behaviours. **Several two parameter bifurcations are drawn that show different stability nature of dynamics.** It is observed that high value of fear level can stabilize the whole system in presence of high availability of prey species for predation. **So, availability of prey species, conversion rate of predator, prey patches and fear level acts an crucial roles in in determining the long-term population dynamics. We hope that this study will contribute in understanding the impact of fear, effect of conversion rate of predator and toxicity level due to patches. The system (1) can also be modified further for two prey and one or two predator which may be more significant to the biological diversity.**

## 8 Conclusion

In this article, we consider fear effect prey-predator model and a prey refuge with forming patches. By examining the characteristic equation of the corresponding linearized system we obtain the threshold conditions for the stability of system. It is observed that level of fear, availability of prey due to refuge mechanism, conversion rate of predator and toxicity level due to patches play major role to stabilize the system. We find that combined effects of more than one parameters results in complex dynamical behaviour.

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