# Gap Formula for the Mexican hat wavelet transform

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**Abstract**

In this paper, we study the Mexican hat wavelet formulated from the Gaussian function. The Mexican hat wavelet transform (MHWT) is defined using this basic wavelet. A standard method is introduced to obtain the gap formula for the MHWT. Further, an example for the gap formula is also presented.

**Key words**: Fourier transform; Wavelet transform; Schwartz distributions; Tempered Boehmians

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## 1 Introduction

By utilizing the theory of distributional as well as classical Fourier and Hilbert transforms, the theory of wavelet transform in $L^p$-spaces $(1 \leq p \leq \infty)$ is formulated. The wavelet transform has been rising as a major mathematical tool for the past two decades and its contribution to signal analysis is significant. The major reason for this is the representation of functions in a time-frequency plane is possible with wavelet transform. Hence, the wavelet transform can be treated as an operator which localizes time and frequency. Moreover, one can regulate wavelets within a fixed time period to acquire varied frequency components that are useful in enhancing the study of signals having localized impulses and oscillations. Based on the idea of wavelets as a family of functions, the mother wavelet $\psi_{b,a}(t)$ is defined by dilating and translating the function $\psi \in L^2(\mathbb{R})$ and is given by

$$\psi_{b,a}(u) = (\sqrt{a})^{-1}\psi\left(\frac{u-b}{a}\right), \quad b, u \in \mathbb{R}, a \in \mathbb{R}_+ = (0, \infty), \tag{1.1}$$

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where $a$ is the dilation, which calculates the level of compression, and $b$ is called shifting parameter, which works out the wavelet's time location. If $|a|< 1$, then (1.1) is the compressed version of the mother wavelet and represents higher frequencies.
For a square integrable function $f$, the wavelet transform with respect to $\psi_{b,a}$ is defined by [5],

$$W(b,a) = \int_{-\infty}^{\infty} f(u)\overline{\psi_{b,a}}(u)du \;\text{ for } a \in T_+ \text{ and } u, b \in \mathbb{R}. \tag{1.2}$$

The inversion formula for (1.2) is given as follows:

$$f(x) = \frac{2}{C_\psi}\int_0^\infty \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} W(b,a)\psi\left(\frac{x-b}{a}\right) db\right]\frac{da}{a^2}, \quad x \in \mathbb{R} \tag{1.3}$$

where

$$\frac{1}{2}C_\psi = \int_0^\infty \frac{|\hat{\psi}(u)|^2}{|u|}du = \int_0^\infty \frac{|\hat{\psi}(-u)|^2}{|u|}du < \infty \qquad [1, \text{p. } 64].$$

Recently among very many authors, the researches carried out by R. S. Pathak *et al.* [4-10] have investigated the theory of wavelet transform to distributions and ultradistribution spaces. Singh *et al.* have extended the theory for distributional wavelet and mexican hat wavelet transform [11-14]. Further, inversion formulae for the same are established in the sense of distributions and ultradistributions.

Mexican hat wavelet that is formulated by taking the second derivative of Gaussian function is defined by

$$\psi(u) = exp\left(\frac{-u^2}{2}\right)\,(1-u^2) = -\frac{d^2}{du^2}\,exp\left(\frac{-u^2}{2}\right). \tag{1.4}$$

Therefore,

$$\psi_{b,a}(u) = -a^{3/2}D_u^2 exp\left(-\frac{(b-u)^2}{2a^2}\right), \qquad \left(D_u = \frac{d}{du}\right). \tag{1.5}$$

Thus from (1.2), we have

$$W(b,a) = -a^{3/2}\int_{-\infty}^{\infty} f(t)\; D_t^2\; exp\left(-\frac{(b-t)^2}{2a^2}\right)dt, \qquad a > 0. \tag{1.6}$$

Then, under certain conditions on $f$, we have

$$W(b,a) = -a^{3/2}\int_{-\infty}^{\infty} f^{(2)}(t)\; exp\left(-\frac{(b-t)^2}{2a^2}\right)dt, \qquad a > 0. \tag{1.7}$$

From the above two equations we can consider the MHWT as the Weierstrass transform of $\left(\frac{d}{du}\right)^2 f(u)$. This relation can further be utilized to explore various

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properties of $W(b,a)$. Also, as Weierstrass transform is defined for complex values of $b$, therefore, the definition of the MHWT can be extended for $b$ being complex, whenever required.

Now for $a \in (0,\infty)$ and $b \in \mathbb{C}$, we define

$$k(b,a) = \frac{1}{\sqrt{2\pi a}} exp\left(\frac{-b^2}{2a}\right). \tag{1.8}$$

Clearly,

$$D_u^2 k(b-u,a^2) = \frac{1}{\sqrt{2\pi}a} D_u^2 \left(exp\left(\frac{-(b-u)^2}{2a^2}\right)\right). \tag{1.9}$$

Hence the Mexican hat wavelet transform of a function $f(t)$ is given by [7]

$$W(b,a) = a^{3/2} \int_{-\infty}^{\infty} f^{(2)}(u) exp\left(\frac{-(b-u)^2}{2a^2}\right) du. \tag{1.10}$$

# 2 Gap formula for Mexican hat wavelet transform

The gap formula which is also known as the jump operator provides a unified approach to obtain a relation between the determining function at a given point in terms of the transform. Here, it acts as an operator which gives $f^{(2)}(b+) - f^{(2)}(b-)$ in terms of $W(b,a)$ where $W(b,a)$ and $f^{(2)}(b)$ are related by (1.10). Such representations have been obtained for various integral transform like Laplace transform, Stieltjes transform, Weierstrass transform, and many more [2, 15, 16]. In the next theorem, we present Gap formula for the Mexican hat wavelet transform.

**Theorem 2.1.** *Let $f^{(2)}(y) \in L_1(m,n)$ for any finite interval such that the integral (1.10) relating $W(b,a)$ to $f^{(2)}(y)$ converges for $m < b < n$. Also, there exists numbers $f^{(2)}(b \pm 0)$ satisfying*

$$\int_0^h [f^{(2)}(b \pm u) - f^{(2)}(b \pm 0)] du = o(h), \quad h \to 0.$$

*Then for $d$ satisfying $m < d < n$ we have for $-\infty < b < \infty$,*

$$\lim_{a^2 \to 1-} -i(1-a^2)^{3/2} a \int_{d-i\infty}^{d+i\infty} (s-b) \exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1) ds = f^{(2)}(b+0) - f^{(2)}(b-0).$$

*Proof.* Let $\alpha(u) = \int_0^u f^{(2)}(v) dv, \ \forall d \in (m,n)$. Also, let $\alpha(u)$ be locally bounded variation, such that

$$|\alpha(u)| = \begin{cases} M \exp\left(\frac{(u-\eta)^2}{2}\right), & u > x, \\ M \exp\left(\frac{(u-\xi)^2}{2}\right), & u < x. \end{cases} \tag{2.1}$$

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Then the MHWT of $f(v)$ is defined by

$$W(b,1) = \int_{-\infty}^{\infty} k(b-u,1) f^{(2)}(v)dv. \tag{2.2}$$

Now, using integration by parts on (2.2), we get

$$W(b,1) = \int_{-\infty}^{\infty} k_1(b-u,1)\alpha(u)du, \tag{2.3}$$

where

$$k_1(b-u,1) = \frac{\partial}{\partial b} k(b-u,1).$$

Consider

$$\begin{aligned} I &= -i(1-a^2)^{13/2} \int_{d-i\infty}^{d+i\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1)ds \\ &= -i(1-a^2)^{3/2} \int_{d-i\infty}^{d+i\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) \int_{-\infty}^{\infty} k_1(s-u,1)\alpha(u)du \\ &= -i(1-a^2)^{3/2}\sqrt{2\pi}a \int_{-\infty}^{\infty} \alpha(u)du \int_{d-i\infty}^{d+i\infty} \frac{(s-b)}{\sqrt{2\pi}a} \exp\left(\frac{(s-b)^2}{2a^2}\right) k_1(s-u,1)ds. \end{aligned}$$

Let us consider

$$\begin{aligned} J &= \frac{-i}{\sqrt{2\pi}a} \int_{d-i\infty}^{d+i\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) k_1(s-u,1)ds \\ &= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} (d+iy-b)\exp\left(\frac{(d+iy-b)^2}{2a^2}\right) k_1(d+iy-u,1)dy, \quad (s=d+iy) \\ &= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} i(y-i(d-b))\exp\left(\frac{-(y-i(d-b))^2}{2a^2}\right) k_1(iy+d-u,1)dy \\ &= \int_{-\infty}^{\infty} k(d+iy-b,a^2)k_2(d+iy-u,1)dy, \end{aligned}$$

where

$$k_2(s-u,1) = \frac{\partial^2 k(s-u,1)}{\partial s^2} = (s-u)k_1(s-u,1).$$

By [7, Theorem 2.1], we have

$$\begin{aligned} J &= \int_{-\infty}^{\infty} k(d+iy-b,a^2)k_2(d+iy-u,1)dy \\ &= k_2(d+iy-u-d-iy+b, 1-a^2) \\ &= k_2(b-u,1-a^2). \end{aligned} \tag{2.4}$$

Hence, we obtain $J = k_2(b-u,1-a^2)$, by combining (2.4) with Corollary 2.2 of [3], where $f^{(2)}(b) = k_2(b-u,1-a^2)$. Further, breaking the integral $I$ into

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4 parts, corresponding to the intervals $(-\infty, b-\delta)$, $(b-\delta, b)$, $(b, b+\delta)$ and $(b+\delta, \infty)$, we have

$$\begin{aligned} I &= (1-a^2)^{3/2}(2\pi)^{1/2}a\left\{\int_{-\infty}^{b-\delta}+\int_{b-\delta}^{b}+\int_{b}^{b+\delta}+\int_{b+\delta}^{\infty}\right\}\alpha(u)(u)k_2(b-u,1-a^2)du \\ &= I_1(a)+I_2(a)+I_3(a)+I_4(a). \end{aligned}$$

For $I_2(a)$, we can choose a $\delta > 0$ so that $|f^{(2)}(u) - f^{(2)}(b-)| < \epsilon$ for $b-\delta < u < b$ and therefore,

$$\begin{aligned} |I_2(a)+f^{(2)}(b-)| &= \left|\int_{b-\delta}^{b} k_1(b-u,1-a^2)[f^{(2)}(u)-f^{(2)}(b-)]du\right| + o(1) \\ &= \left|\int_{b-\delta}^{b} k_2(b-u,1-a^2)\beta(u)du\right| + o(1) \\ &\leq \epsilon\int_{b-\delta}^{b} k_2(b-u,1-a^2)|s-u|du + o(1) \\ &\leq \epsilon M + o(1) \quad as\ a^2 \to 1-. \end{aligned}$$

Similarly $|I_3(a) - f^{(2)}(b+)| \leq \epsilon M + o(1)$.

For $\epsilon$ being arbitrary, we have $I_2(a) \approx -f^{(2)}(b-)$ and $I_3(a) \approx f^{(2)}(b+)$.

For $I_1(a)$ and $I_4(a)$ by Lemma 2.1c of [3], for some $\xi$ and $\eta$ such that $m < \xi < \eta < n$, at $a = 1$

$$\begin{aligned} f^{(2)}(u) &= o\left[exp\left(\frac{(u-\eta)^2}{2}\right)\right], \quad u \to \infty, \\ f^{(2)}(u) &= o\left[exp\left(\frac{(u-\xi)^2}{2}\right)\right], \quad u \to \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_1(a)| &= \lim_{a^2\to 1-}\left|(2\pi)^{1/2}(1-a^2)^{3/2}\int_{-\infty}^{b-\delta} k_1(b-u,1-a^2)f^{(2)}(u)du\right| \\ &\leq \lim_{a^2\to 1-}(1-a^2)^{-3/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)|f^{(2)}(u)|du \\ &\leq \lim_{a^2\to 1-} M(1-a^2)^{-3/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)\exp\left(\frac{-(u-\xi)^2}{2}\right)du \\ &= \mathrm{o}(1). \end{aligned}$$

Hence, $I_1(a) = \mathrm{o}(1)$ and similarly $I_4(a) = \mathrm{o}(1)$ as $a^2 \to 1-$, which concludes the proof of the theorem.

□

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**Example 2.2.** As a simple example take the MHWT at $a = 1$,

$$\begin{aligned} W(s,1) &= \int_{-\infty}^{\infty} k_1(s-u,1)\alpha(u)du \\ &= \exp\left(\frac{-s^2}{2}\right), \end{aligned} \tag{2.5}$$

where

$$\alpha(u) = \int_0^u f^{(2)}(v)dv = \begin{cases} 0 & u < 0 \\ 1 & u > 0. \end{cases}$$

Since the integral (1.10) converges always, therefore by Theorem 2.1, we have

$$\begin{aligned} &= \lim_{a^2\to 1-} -i(1-a^2)^{3/2} \int_{-\infty}^{\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1)ds \\ &= \lim_{a^2\to 1-} -i(1-a^2)^{3/2} \int_{-\infty}^{\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) \exp\left(\frac{-s^2}{2}\right) ds \\ &= \lim_{a^2\to 1-} \frac{i(1-a^2)^{3/2}\sqrt{2\pi}a^4}{(a^2-1)^{3/2}} \exp\left(\frac{-b^2}{2(1-a^2)}\right) \\ &= \begin{cases} 1 & b = 0, \\ 0 & otherwise. \end{cases} \end{aligned} \tag{2.6}$$

# Conclusions

In this article, we studied the conditions needed to obtain a relation between the determining function at a point of discontinuity with its MHWT. As the Gaussian function derives the Mexican hat wavelet, therefore it satisfies the Gaussian decays in both frequency and space. Further, as the MHWT has localization in both space and frequency, it has a strong appeal to applications in space-frequency analysis, mixed boundary value problems, approximation theory, mathematical modeling, other digital modulation.

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