Multivariate Ostrowski type inequalities for several Banach algebra valued functions

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Abstract

Here we are dealing with several smooth functions from a compact convex set of $\mathbb{R}^k$, $k \geq 2$ to a Banach algebra. For these we prove general multivariate Ostrowski type inequalities with estimates in norms $\|\cdot\|_p$, for all $1 \leq p \leq \infty$. We provide also interesting applications.

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1 Introduction

In 1938, A Ostrowski [5] proved the following famous inequality:

Theorem 1 *(1938, Ostrowski [6]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \quad (1)$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This article is also greatly motivated by the following result:

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Theorem 2 *(see [1]) Let* $f \in C^1\left(\prod_{i=1}^{k}[a_i,b_i]\right)$*, where* $a_i < b_i$*;* $a_i, b_i \in \mathbb{R}$*,* $i = 1,...,k$*, ans let* $\overrightarrow{x_0} := (x_{01},...,x_{0k}) \in \prod_{i=1}^{k}[a_i,b_i]$ *be fixed. Then*

$$\left|\frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_i}^{b_i}...\int_{a_k}^{b_k} f(z_1,...,z_k)\,dz_1...dz_k - f\left(\overrightarrow{x_0}\right)\right| \leq \tag{2}$$

$$\sum_{i=1}^{k}\left(\frac{(x_{0i}-a_i)^2+(b_i-x_{0i})^2}{2(b_i-a_i)}\right)\left\|\frac{\partial f}{\partial z_i}\right\|_\infty.$$

Inequality (2) is sharp, here the optimal function is

$$f^*(z_1,...,z_k) := \sum_{i=1}^{k}|z_i-x_{0i}|^{\alpha_i}, \quad \alpha_i > 1.$$

Clearly inequality (2) generalizes inequality (1) to multidimension.

We are inspired also by [2].

In this article we establish multivariate Ostrowski type inequalities for several smooth functions from a compact convex subset of $\mathbb{R}^k$, $k \geq 2$, to a Banach algebra. These involve the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

2 About Banach Algebras

All here come from [6].

We need

Definition 3 *([6], p. 245) A complex algebra is a vector space A over the complex field* $\mathbb{C}$ *in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{3}$$

$$(x+y)z = xz+yz, \quad x(y+z) = xy+xz, \tag{4}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{5}$$

for all x, y *and* z *in* A *and for all scalars* α*.*

Additionally if A *is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \leq \|x\|\,\|y\| \quad (x \in A,\ y \in A) \tag{6}$$

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and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{7}$$

and

$$\|e\| = 1, \tag{8}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 4 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (7).

Inequality (6) makes multiplication to be continuous, more precisely left and right continuous, see [6], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [6], p. 247-248, § 10.3.

We also make

Remark 5 *Next we mention about integration of A-valued functions, see [6], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [6], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \; d\mu = \int_Q x f(p) \; d\mu(p) \tag{9}$$

and

$$\left(\int_Q f \; d\mu \right) x = \int_Q f(p)\, x \; d\mu(p). \tag{10}$$

The vector integrals we will involve in our article follow (9) and (10).

3 Vector Analysis Background

(see [8], pp. 83-94)

Let $f(t)$ be a function defined on $[a,b] \subseteq \mathbb{R}$ taking values in a real or complex normed linear space $(X, \|\cdot\|)$, Then $f(t)$ is said to be differentiable at a point $t_0 \in [a,b]$ if the limit

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h} \tag{11}$$

exists in X, the convergence is in $\|\cdot\|$. This is called the derivative of $f(t)$ at $t = t_0$.

We call $f(t)$ differentiable on $[a,b]$, iff there exists $f'(t) \in X$ for all $t \in [a,b]$.

Similarly and inductively are defined higher order derivatives of f, denoted $f'', f^{(3)}, ..., f^{(k)}$, $k \in \mathbb{N}$, just as for numerical functions.

For all the properties of derivatives see [8], pp. 83-86.

Let now $(X, \|\cdot\|)$ be a Banach space, and $f : [a,b] \to X$.

We define the vector valued Riemann integral $\int_a^b f(t)\,dt \in X$ as the limit of the vector valued Riemann sums in X, convergence is in $\|\cdot\|$. The definition is as for the numerical valued functions.

If $\int_a^b f(t)\,dt \in X$ we call f integrable on $[a,b]$. If $f \in C([a,b], X)$, then f is integrable, [8], p. 87.

For all the properties of vector valued Riemann integrals see [8], pp. 86-91.

We define the space $C^n([a,b], X)$, $n \in \mathbb{N}$, of n-times continuousky differentiable functions from $[a,b]$ into X; here continuity is with respect to $\|\cdot\|$ and defined in the usual way as for numerical functions·.

Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n([a,b], X)$, then we have the vector valued Taylor's formula, see [8], pp. 93-94, and also [7], (IV, 9; 47).

It holds

$$f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 - ... - \frac{1}{(n-1)!} f^{(n-1)}(x)(y-x)^{n-1}$$
$$= \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} f^{(n)}(t)\,dt, \quad \forall\, x, y \in [a,b]. \tag{12}$$

In particular (12) is true when $X = \mathbb{R}^m, \mathbb{C}^m$, $m \in \mathbb{N}$, etc.

A function $f(t)$ with values in a normed linear space X is said to be piecewise continuous (see [8], p. 85) on the interval $a \le t \le b$ if there exists a partition $a = t_0 < t_1 < t_2 < ... < t_n = b$ such that $f(t)$ is continuous on every open interval $t_k < t < t_{k+1}$ and has finite limits $f(t_0+0)$, $f(t_1-0)$, $f(t_1+0)$, $f(t_2-0)$, $f(t_2+0)$, ..., $f(t_n-0)$.

Here $f(t_k-0) = \lim_{t \uparrow t_k} f(t)$, $f(t_k+0) = \lim_{t \downarrow t_k} f(t)$.

The values of $f(t)$ at the points t_k can be arbitrary or even undefined.

A function $f(t)$ with values in normed linear space X is said to be piecewise smooth on $[a,b]$, if it is continuous on $[a,b]$ and has a derivative $f'(t)$ at all but a finite number of points of $[a,b]$, and if $f'(t)$ is piecewise continuous on $[a,b]$ (see [8], p. 85).

Let $u(t)$ and $v(t)$ be two piecewise smooth functions on $[a,b]$, one a numerical function and the other a vector function with values in Banach space X. Then we have the following integration by parts formula

$$\int_a^b u(t)\,dv(t) = u(t)\,v(t)\,|_a^b - \int_a^b v(t)\,du(t), \tag{13}$$

see [8], p. 93.

We mention also the mean value theorem for Banach space valued functions.

Theorem 6 *(see [4], p. 3) Let $f \in C([a,b],X)$, where X is a Banach space. Assume f' exists on $[a,b]$ and $\|f'(t)\| \le K$, $a<t<b$, then*

$$\|f(b) - f(a)\| \le K(b-a). \tag{14}$$

Here the multiple Riemann integral of a function from a real box or a real compact and convex subset to a Banach space is defined similarly to numerical one however convergence is with respect to $\|\cdot\|$. Similarly are defined the vector valued partial derivatives as in the numerical case.

We mention the equality of vector valued mixed partiasl derivatives.

Proposition 7 *(see Proposition 4.11 of [3], p. 90) Let $Q=(a,b)\times(c,d)\subseteq\mathbb{R}^2$ and $f \in C(Q,X)$, where $(X,\|\cdot\|)$ is a Banach space. Assume that $\frac{\partial}{\partial t}f(s,t)$, $\frac{\partial}{\partial s}f(s,t)$ and $\frac{\partial^2}{\partial t\partial s}f(s,t)$ exist and are continuous for $(s,t)\in Q$, then $\frac{\partial^2}{\partial s\partial t}f(s,t)$ exists for $(s,t)\in Q$ and*

$$\frac{\partial^2}{\partial s\partial t}f(s,t) = \frac{\partial^2}{\partial t\partial s}f(s,t), \text{ for } (s,t)\in Q. \tag{15}$$

4 Main Results

We present general Ostrowski type inequalities results regarding several Banach algebra valued functions.

Theorem 8 *Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$; $(A,\|\cdot\|)$ a Banach algebra and $f_i \in C^{n+1}(Q,A)$, $i=1,...,r$; $r\in\mathbb{N}$, $n\in\mathbb{Z}_+$, and fixed $\overrightarrow{x_0}\in Q\subset\mathbb{R}^k$, $k\ge 2$, where Q is a compact and convex subset. Here all vector partial derivatives $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, where $\alpha=(\alpha_1,...,\alpha_k)$, $\alpha_\lambda\in\mathbb{Z}^+$, $\lambda=1,...,k$, $|\alpha| = \sum_{\lambda=1}^k \alpha_\lambda = j$, $j=1,...,n$, fulfill $f_{i\alpha}(\overrightarrow{x_0})=0$, $i=1,...,r$.*

Denote

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \|\|f_{i\alpha}\|\|_{\infty,Q}, \tag{16}$$

$i=1,...,r$, and

$$\|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1} := \sum_{\lambda=1}^k |z_\lambda - x_{0\lambda}|. \tag{17}$$

Then

$$\left\| \sum_{i=1}^{r} \int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho (\overrightarrow{z}) \right) f_i (\overrightarrow{z}) \, d\overrightarrow{z} - \sum_{i=1}^{r} \left(\int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho (\overrightarrow{z}) \right) d\overrightarrow{z} \right) f_i (\overrightarrow{x_0}) \right\| \leq \tag{18}$$

$$\frac{\max_{i \in \{1,...,r\}} D_{n+1} (f_i)}{(n+1)!} \sum_{i=1}^{r} \left(\int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \| f_\rho (\overrightarrow{z}) \| \right) \| \overrightarrow{z} - \overrightarrow{x_0} \|_{l_1}^{n+1} \, d\overrightarrow{z} \right) \leq$$

$$\frac{\max_{i \in \{1,...,r\}} D_{n+1} (f_i)}{(n+1)!} \min \Bigg\{ \left(\int_Q \| \overrightarrow{z} - \overrightarrow{x_0} \|_{l_1}^{n+1} \, d\overrightarrow{z} \right) \left[\sum_{i=1}^{r} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \| \| f_\rho \| \|_{\infty, Q} \right) \right],$$

$$\left\| \| \cdot - \overrightarrow{x_0} \|_{l_1}^{n+1} \right\|_{\infty, Q} \left[\sum_{i=1}^{r} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \| f_\rho \| \right) \right\|_{L_1(Q,A)} \right],$$

$$\left\| \| \cdot - \overrightarrow{x_0} \|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left[\sum_{i=1}^{r} \left[\left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \| f_\rho \| \right) \right\|_{L_q(Q,A)} \right] \right] \Bigg\}. \tag{19}$$

Proof. Take $g_{i\overrightarrow{z}} (t) := f_i (\overrightarrow{x_0} + t (\overrightarrow{z} - \overrightarrow{x_0}))$, $0 \leq t \leq 1$; $i = 1, ..., r$. Notice that $g_{i\overrightarrow{z}} (0) = f_i (\overrightarrow{x_0})$ and $g_{i\overrightarrow{z}} (1) = f_i (\overrightarrow{z})$. The jth derivative of $g_{i\overrightarrow{z}} (t)$, based on Proposition 7, is given by

$$g_{i\overrightarrow{z}}^{(j)} (t) = \left[\left(\sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (x_{01} + t (z_1 - x_{01}), ..., x_{0k} + t (z_k - x_{0k})) \tag{20}$$

and

$$g_{i\overrightarrow{z}}^{(j)} (0) = \left[\left(\sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (\overrightarrow{x_0}), \tag{21}$$

for $j = 1, ..., n+1$; $i = 1, ..., r$.

Let $f_{i\alpha}$ be a partial derivative of $f_i \in C^{n+1} (Q, A)$. Because by assumption of the theorem we have $f_{i\alpha} (\overrightarrow{x_0}) = 0$ for all $\alpha : |\alpha| = j$, $j = 1, ..., n$, we find that

$$g_{i\overrightarrow{z}}^{(j)} (0) = 0, \quad j = 1, ..., n; \; i = 1, ..., r.$$

Hence by vector Taylor's theorem (12) we see that

$$f_i (\overrightarrow{z}) - f_i (\overrightarrow{x_0}) = \sum_{j=1}^{n} \frac{g_{i\overrightarrow{z}}^{(j)} (0)}{j!} + R_{in} (\overrightarrow{z}, 0) = R_{in} (\overrightarrow{z}, 0), \tag{22}$$

where

$$R_{in}\left(\overrightarrow{z},0\right):=\int_{0}^{1}\left(\int_{0}^{t_{1}}...\left(\int_{0}^{t_{n-1}}\left(g_{i\overrightarrow{z}}^{(n)}\left(t_{n}\right)-g_{i\overrightarrow{z}}^{(n)}\left(0\right)\right)dt_{n}\right)...\right)dt_{1}, \quad (23)$$

$i=1,...,r.$

Therefore,

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\int_{0}^{1}\left(\int_{0}^{t_{1}}...\left(\int_{0}^{t_{n-1}}\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\left(\xi\left(t_{n}\right)\right)\right\|\right\|_{\infty}t_{n}dt_{n}\right)...\right)dt_{1}, \quad (24)$$

by the vector mean value Theorem 6 applied on $g_{i\overrightarrow{z}}^{(n)}$ over $(0,t_n)$. Moreover, we get

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}\int_{0}^{1}\int_{0}^{t_{1}}...\left(\int_{0}^{t_{n-1}}t_{n}dt_{n}\right)...dt_{1}$$

$$=\frac{\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}}{(n+1)!}. \quad (25)$$

However, there exists a $t_{i0}\in[0,1]$ such that $\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}=\left\|g_{i\overrightarrow{z}}^{(n+1)}\left(t_{i0}\right)\right\|$.

That is

$$\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}=\left\|\left[\left(\sum_{\lambda=1}^{k}\left(z_{\lambda}-x_{0\lambda}\right)\frac{\partial}{\partial z_{\lambda}}\right)^{n+1}f_{i}\right]\left(\overrightarrow{x_{0}}+t_{i0}\left(\overrightarrow{z}-\overrightarrow{z_{0i}}\right)\right)\right\|$$

$$\leq\left[\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\left\|\frac{\partial}{\partial z_{\lambda}}\right\|\right)^{n+1}f_{i}\right]\left(\overrightarrow{x_{0}}+t_{i0}\left(\overrightarrow{z}-\overrightarrow{z_{0i}}\right)\right).$$

I.e.,

$$\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}\leq\left[\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\left\|\left\|\frac{\partial}{\partial z_{\lambda}}\right\|\right\|_{\infty}\right)^{n+1}f_{i}\right], \quad (26)$$

$i=1,...,r.$

Hence by (26) we get

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\frac{\left[\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\left\|\left\|\frac{\partial}{\partial z_{\lambda}}\right\|\right\|_{\infty}\right)^{n+1}f_{i}\right]}{(n+1)!}\leq$$

$$\frac{D_{n+1}\left(f_{i}\right)}{(n+1)!}\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\right)^{n+1}=\frac{D_{n+1}\left(f_{i}\right)}{(n+1)!}\left\|\overrightarrow{z}-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}, \quad (27)$$

$i=1,...,r.$

Therefore it holds

$$\left\| R_{in}\left(\overrightarrow{z},0\right)\right\| \leq \frac{\max\limits_{i\in\{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!}\left\|\overrightarrow{z}-\overrightarrow{x_0}\right\|_{l_1}^{n+1}, \tag{28}$$

for $i=1,...,r$.

By (22) we get that

$$\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)-\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right)=\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right), \tag{29}$$

for all $i=1,...,r$.

Hence

$$\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)-\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right)$$

$$=\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right). \tag{30}$$

Therefore we find

$$E\left(f_1,...,f_r\right)\left(x_0\right):=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z}-\sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)d\overrightarrow{z}. \tag{31}$$

Consequently, we have that

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|=$$

$$\left\|\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z}-\sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)\right\|=$$

$$\left\| \sum_{i=1}^{r} \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z}, 0\right) d\overrightarrow{z} \right\| \leq \tag{32}$$

$$\sum_{i=1}^{r} \left\| \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z}, 0\right) d\overrightarrow{z} \right\| \leq$$

$$\sum_{i=1}^{r} \left(\int_{Q} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z}, 0\right) \right\| d\overrightarrow{z} \right) \overset{(6)}{\leq}$$

$$\sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| R_{in}\left(\overrightarrow{z}, 0\right) \right\| d\overrightarrow{z} \right) \overset{(28)}{\leq} \tag{33}$$

$$\frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right).$$

So far we have proved

$$\left\| E\left(f_1, ..., f_r\right)\left(x_0\right) \right\| \leq$$

$$\frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) =: (\xi). \tag{34}$$

Furthermore it holds

$$(\xi) \leq \frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \left(\int_{Q} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[\sum_{i=1}^{r} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| \left\| f_{\rho} \right\| \right\|_{\infty,Q} \right) \right], \tag{35}$$

and

$$(\xi) \leq \frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[\sum_{i=1}^{r} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_{\rho} \right\| \right) \right\|_{L_1(Q,A)} \right], \tag{36}$$

and finally

$$(\xi) \leq \frac{\max\limits_{i \in \{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \left[\sum_{i=1}^{r} \left[\left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \|f_\rho\| \right) \right\|_{L_q(Q,A)} \right] \right] \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{L_p(Q,A)}, \tag{37}$$

proving (18), (19). ■

We give

Corollary 9 *(to Theorem 8) All as in Theorem 8, with* $f_1 = ... = f_r = f$, $r \in \mathbb{N}$. *Then*

$$\left\| \int_Q f^r(\overrightarrow{z})\, d\overrightarrow{z} - \left(\int_Q f^{r-1}(\overrightarrow{z})\, d\overrightarrow{z} \right) f(\overrightarrow{x_0}) \right\| \leq$$

$$\frac{D_{n+1}(f)}{(n+1)!} \left(\int_Q \|f(\overrightarrow{z})\|^{r-1} \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq \tag{38}$$

$$\frac{D_{n+1}(f)}{(n+1)!} \min\left\{ \left(\int_Q \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left(\|\|f\|\|_{\infty,Q} \right)^{r-1}, \right.$$

$$\left. \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{\infty,Q} \left\| \|f\|^{r-1} \right\|_{L_1(Q,A)}, \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left\| \|f\|^{r-1} \right\|_{L_q(Q,A)} \right\}. \tag{39}$$

We also give

Corollary 10 *(to Theorem 8) All as in Theorem 8, with* $(A, \|\cdot\|)$ *being a commutative Banach algebra. Then*

$$\left\| r \int_Q \left(\prod_{\rho=1}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} - \sum_{i=1}^{r} \left(\int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} \right) f_i(\overrightarrow{x_0}) \right\| \leq$$

$$\textit{Right hand side of (18)} \leq \textit{Right hand side of (19)}. \tag{40}$$

We make

Remark 11 *Of great interest are applications of Theorem 8 when* $Q = \prod\limits_{\lambda=1}^{k} [a_\lambda, b_\lambda]$, *where* $[a_\lambda, b_\lambda] \subset \mathbb{R}$, $\lambda = 1, ..., k$.

We observe that by the multinomial theorem we get:

$$\int_{\prod_{\lambda=1}^{k} [a_\lambda, b_\lambda]} \left(\sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}| \right)^{n+1} dz_1 ... dz_k = \sum_{\rho_1 + \rho_2 + ... \rho_k = n+1} \frac{(n+1)!}{\rho_1! \rho_2! ... \rho_k!}$$

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} |z_1-x_{01}|^{\rho_1}|z_2-x_{02}|^{\rho_2}...|z_k-x_{0k}|^{\rho_k}dz_1...dz_k = \tag{41}$$

$$\sum_{\rho_1+\rho_2+...\rho_k=n+1}\frac{(n+1)!}{\rho_1!\rho_2!...\rho_k!}\prod_{\lambda=1}^{k}\left(\int_{a_\lambda}^{b_\lambda}|z_\lambda-x_{0\lambda}|^{\rho_\lambda}dz_\lambda\right)=$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\int_{a_\lambda}^{x_{0\lambda}}(x_{0\lambda}-z_\lambda)^{\rho_\lambda}dz_\lambda+\int_{x_{0\lambda}}^{b_\lambda}(z_\lambda-x_{0\lambda})^{\rho_\lambda}dz_\lambda\right)=$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\frac{(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}+(b_\lambda-x_{0\lambda})^{\rho_\lambda+1}}{\rho_\lambda+1}\right). \tag{42}$$

We have found that

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1}^{n+1}d\overrightarrow{z}= \tag{43}$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\frac{(b_\lambda-x_{0\lambda})^{\rho_\lambda+1}+(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}}{\rho_\lambda+1}\right).$$

Based on (18), (19) and (43) we conclude:

Theorem 12 *Let $(A,\|\cdot\|)$ a Banach algebra and $f_i\in C^{n+1}\left(\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda],A\right)$, $i=1,...,r$; $r\in\mathbb{N}$, $n\in\mathbb{Z}_+$, and fixed $\overrightarrow{x_0}\in\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]\subset\mathbb{R}^k$, $k\geq 2$. Here all vector partial derivatives $f_{i\alpha}:=\frac{\partial^\alpha f_i}{\partial z^\alpha}$, where $\alpha=(\alpha_1,...,\alpha_k)$, $\alpha_\lambda\in\mathbb{Z}^+$, $\lambda=1,...,k$, $|\alpha|=\sum_{\lambda=1}^{k}\alpha_\lambda=j$, $j=1,...,n$, fulfill $f_{i\alpha}(\overrightarrow{x_0})=0$, $i=1,...,r$.*

Denote

$$D_{n+1}(f_i):=\max_{\alpha:|\alpha|=n+1}\|\|f_{i\alpha}\|\|_{\infty,\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}, \tag{44}$$

$i=1,...,r$.

Then

$$\left\|\sum_{i=1}^{r}\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}f_\rho(\overrightarrow{z})\right)f_i(\overrightarrow{z})d\overrightarrow{z}-\right.$$

$$\left.\sum_{i=1}^{r}\left(\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}f_\rho(\overrightarrow{z})\right)d\overrightarrow{z}\right)f_i(\overrightarrow{x_0})\right\|\leq \tag{45}$$

$$\left(\max_{i\in\{1,\dots,r\}} D_{n+1}(f_i)\right)\left[\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\||\|f_\rho\||\|_{\infty,\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\right)\right]$$

$$\left[\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{1}{\prod_{\lambda=1}^{k}\rho_\lambda!\prod_{\lambda=1}^{k}(\rho_\lambda+1)}\prod_{\lambda=1}^{k}\left((b_\lambda-x_{0\lambda})^{\rho_\lambda+1}+(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}\right)\right].$$

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