Generalized Canavati Fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Using generalized Canavati fractional left and right vectorial Taylor formulae we prove corresponding left and right fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions. We cover also the sequential fractional case. We finish with applications.

2020 Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: generalized Canavati fractional derivative, generalized Canavati fractional inequalities, Hilbert-Pachpatte inequalities, Banach algebra.

1 Introduction

Motivation follows:

We need

Definition 1 *(see [5]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ (diam =diameter) for every i. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diam C_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \tag{1}$$

Then there exists $\lim_{\delta\to 0} h_\alpha^\delta(A) = \sup_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field

1

of all Borel subsets of T. If $T=\mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C)=\mu(C)$ for every Borel set $C\subseteq\mathbb{R}$, where μ is the Lebesgue measure.

We also need

Definition 2 *([2], Ch. 1) Let $[a,b]\subset\mathbb{R}$, X be a Banach space, $\nu>0$; $n:=\lceil\nu\rceil\in\mathbb{N}$, $\lceil\cdot\rceil$ is the ceiling of the number, $f:[a,b]\to X$. We assume that $f^{(n)}\in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order ν:*

$$\left(D_{*a}^{\nu}f\right)(x):=\frac{1}{\Gamma(n-\nu)}\int_a^x(x-t)^{n-\nu-1}f^{(n)}(t)\,dt,\quad\forall\ x\in[a,b].\tag{2}$$

If $\nu\in\mathbb{N}$, we set $D_{*a}^{\nu}f:=f^{(\nu)}$ the ordinary X-valued derivative, and also set $D_{*a}^{0}f:=f$. Here Γ is the gamma function and integrals are of Bochner type [3].

By [2], Ch. 1, $\left(D_{*a}^{\nu}f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^{\nu}f\in L_1([a,b],X)$.

If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}<\infty$, then by [2], Ch. 1, $D_{*a}^{\nu}f\in C([a,b],X)$.

We are motivated by a Hilbert-Pachpatte left fractional inequality:

Theorem 3 *([2], Ch. 1) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\nu_1>\frac{1}{q}$, $\nu_2>\frac{1}{p}$, $n_i:=\lceil\nu_i\rceil$, $i=1,2$. Here $[a_i,b_i]\subset\mathbb{R}$, $i=1,2$; X is a Banach space. Let $f_i\in C^{n_i-1}([a_i,b_i],X)$, $i=1,2$. Set*

$$F_{x_i}(t_i):=\sum_{j_i=0}^{n_i-1}\frac{(x_i-t_i)^{j_i}}{j_i!}f_i^{(j_i)}(t_i),\tag{3}$$

$\forall\ t_i\in[a_i,x_i]$, where $x_i\in[a_i,b_i]$; $i=1,2$. Assume that $f_i^{(n_i)}$ exists outside a μ-null Borel set $B_{x_i}\subseteq[a_i,x_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right)=0,\ \ \forall\ x_i\in[a_i,b_i];\ i=1,2.\tag{4}$$

We also assume that $f_i^{(n_i)}\in L_1([a_i,b_i],X)$, and

$$f_i^{(k_i)}(a_i)=0,\ \ k_i=0,1,...,n_i-1;\ i=1,2,\tag{5}$$

and

$$\left(D_{*a_1}^{\nu_1}f_1\right)\in L_q([a_1,b_1],X),\quad\left(D_{*a_2}^{\nu_2}f_2\right)\in L_p([a_2,b_2],X).\tag{6}$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\|f_1(x_1)\|\,\|f_2(x_2)\|\,dx_1dx_2}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|D_{*a_1}^{\nu_1}f_1\right\|_{L_q([a_1,b_1],X)}\left\|D_{*a_2}^{\nu_2}f_2\right\|_{L_p([a_2,b_2],X)}.\tag{7}$$

We need

Definition 4 *([2], Ch. 2) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\, x \in [a,b]. \tag{8}$$

We observe that $D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $D_{b-}^{0} f(x) = f(x)$.

By [2], Ch. 2, $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by [2], Ch. 2, $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We are motivated also by the following Hilbert-Pachpatte right fractional inequality:

Theorem 5 *([2], Ch. 2) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\alpha_1 > \frac{1}{q}$, $\alpha_2 > \frac{1}{p}$, $m_i := \lceil \alpha_i \rceil$, $i = 1,2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1,2$; X is a Banach space. Let $f_i \in C^{m_i - 1}([a_i, b_i], X)$, $i = 1,2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{m_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{9}$$

$\forall\, t_i \in [x_i, b_i]$, where $x_i \in [a_i, b_i]$; $i = 1,2$. Assume that $f_i^{(m_i)}$ exists outside a μ-null Borel set $B_{x_i} \subseteq [x_i, b_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right) = 0, \quad \forall\, x_i \in [a_i, b_i]; \ i = 1,2. \tag{10}$$

We also assume that $f_i^{(m_i)} \in L_1([a_i, b_i], X)$, and

$$f_i^{(k_i)}(b_i) = 0, \quad k_i = 0, 1, ..., , m_i - 1; \ i = 1,2, \tag{11}$$

and

$$\left(D_{b_1-}^{a_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{b_2-}^{\alpha_2} f_2\right) \in L_p([a_2, b_2], X). \tag{12}$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \|f_2(x_2)\|\, dx_1 dx_2}{\left(\frac{(b_1-x_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(b_2-x_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)} \right)} \leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\| D_{b_1-}^{\alpha_1} f_1 \right\|_{L_q([a_1,b_1],X)} \left\| D_{b_2-}^{\alpha_2} f_2 \right\|_{L_p([a_2,b_2],X)}. \tag{13}$$

In this work we derive Hilbert-Pachpatte inequalities for Banach algebra valued functions with respect to their Canavati type generalized left and right fractional derivatives. We cover also the sequential fractional case. We finish with applications.

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [2], pp. 109-115 and [1].

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a, b])$, and $g^{-1} \in C^n([g(a), g(b)])$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a Banach space. Let $f \in C^n([a, b], X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq X$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g(a), g(b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$(J_\nu^{z_0} h)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z - t)^{\nu - 1} h(t)\, dt, \tag{14}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)}^{\nu}([g(a), g(b)], X) =$$
$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \tag{15}$$

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^{\nu} h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \tag{16}$$

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)}^{\nu} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{17}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{18}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D_{g(x_0)}^{n}\left(f \circ g^{-1} \right) = \left(f \circ g^{-1} \right)^{(n)}$ and $D_{g(x_0)}^{0}\left(f \circ g^{-1} \right) = f \circ g^{-1}$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ the following left generalized g-fractional, of Canavati type,X-valued Taylor's formula:

Theorem 6 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{19}$$

for all $x_0 \leq x \leq b$.

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{20}$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J^{\nu}_{z_0-}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{21}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J^{0}_{z_0-}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J^{1-\alpha}_{g(x_0)-}h^{([\nu])} \in C^{1}([g(a), g(x_0)], X)\right\}. \tag{22}$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-}h := (-1)^{n-1}\left(J^{1-\alpha}_{g(x_0)-}h^{([\nu])}\right)'. \tag{23}$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{24}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{25}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^{n}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{26}$$

and $\left(D^{0}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 7 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{27}$$

for all $a \leq x \leq x_0$,

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{28}$$

all $a \leq x \leq x_0$.

III) Denote by

$$D^{m\nu}_{g(x_0)} = D^{\nu}_{g(x_0)} D^{\nu}_{g(x_0)} \ldots D^{\nu}_{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{29}$$

We mention the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 8 *Let $f \in C^1([a, b], X)$, $g \in C^1([a, b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a, b]$, for $i = 0, 1, \ldots, m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{30}$$

all $x_0 \leq x \leq b$.

6

IV) Denote by

$$D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu} ... D_{g(x_0)-}^{\nu} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{31}$$

We mention the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 9 *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D_{g(x_0)-}^{i\nu}(f \circ g^{-1})\right) \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D_{g(x_0)-}^{(m+1)\nu}(f \circ g^{-1})\right)(z)\, dz, \tag{32}$$

all $a \le x \le x_0 \le b$.

3 Banach Algebras background

All here come from [4].

We need

Definition 10 *([4], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{33}$$

$$(x+y)z = xz + yz,\ \ x(y+z) = xy + xz, \tag{34}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{35}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \le \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{36}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{37}$$

and

$$\|e\| = 1, \tag{38}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 11 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (37).

Inequality (36) makes multiplication to be continuous, more precisely left and right continuous, see [4], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [4], p. 247-248, § 10.3.

We also make

Remark 12 *Next we mention about integration of A-valued functions, see [4], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [4], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{39}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p) x \ d\mu(p). \tag{40}$$

The Bochner integrals we will involve in our article follow (39) and (40). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [2], p. 3, f is Bochner integrable.

4 Main Results

We start with a left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 13 *Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, and $(A, \|\cdot\|)$ is a Banach algebra; and $i = 1, 2$. Let also $x_{0i} \in [a_i, b_i] \subset \mathbb{R}$, $\nu_i \geq 1$, $n_i = [\nu_i]$, $f_i \in C^{n_i}([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$, with $\left(f_i \circ g_i^{-1}\right)^{(k_i)}(g_i(x_{0i})) = 0$, $k_i = 0, 1, ..., n_i - 1$. Assume further that $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(\nu_2 - 1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq$$

8

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \tag{41}$$

$$\left\|\left\|D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)}.$$

Proof. By (19) and assumptions we get that

$$\left(f_i\circ g_i^{-1}\right)\left(z_i\right)=\frac{1}{\Gamma\left(\nu_i\right)}\int_{g_i(x_{0i})}^{z_i}\left(z_i-t_i\right)^{\nu_i-1}\left(D_{g_i(x_{0i})}^{\nu_i}\left(f_i\circ g_i^{-1}\right)\right)\left(t_i\right)dt_i, \tag{42}$$

for all $g_i\left(x_{0i}\right)\leq z_i\leq g_i\left(b_i\right)$; $i=1,2$.

By Hölder's inequality we obtain

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\int_{g_1(x_{01})}^{z_1}\left(z_1-t_1\right)^{\nu_1-1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|dt_1\leq$$

$$\frac{1}{\Gamma\left(\nu_1\right)}\left(\int_{g_1(x_{01})}^{z_1}\left(z_1-t_1\right)^{p(\nu_1-1)}dt_1\right)^{\frac{1}{p}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}. \tag{43}$$

That is

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}, \tag{44}$$

for all $g_1\left(x_{01}\right)\leq z_1\leq g_1\left(b_1\right)$.

Similarly, we prove that

$$\left\|\left(f_2\circ g_2^{-1}\right)\left(z_2\right)\right\|\leq\frac{1}{\Gamma\left(\nu_2\right)}\frac{\left(z_2-g_2\left(x_{02}\right)\right)^{\frac{q(\nu_2-1)+1}{q}}}{\left(q\left(\nu_2-1\right)+1\right)^{\frac{1}{q}}}$$

$$\left(\int_{g_2(x_{02})}^{z_2}\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)\left(t_2\right)\right\|^p dt_2\right)^{\frac{1}{p}}, \tag{45}$$

for all $g_2\left(x_{02}\right)\leq z_2\leq g_2\left(b_2\right)$.

Therefore we have

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}, \tag{46}$$

for all $g_1(x_{01})\leq z_1\leq g_1(b_1)$;

and

$$\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_2)}\frac{(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}, \tag{47}$$

for all $g_2(x_{02})\leq z_2\leq g_2(b_2)$.

Hence we get that

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)(p(\nu_1-1)+1)^{\frac{1}{p}}(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}} \tag{48}$$
$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}\leq$$

(using Young's inequality for $a,b\geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)$$
$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq \tag{50}$$

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq \tag{51}$$

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$
$$\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

The denominators in (50), (51) can be zero only when both $z_1=g_1(x_{01})$ and $z_2=g_2(x_{02})$.

Therefore we obtain (41), by integrating (50), (51) over $[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$. ■

We continue with a right generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 14 *All as in Theorem 13, however now it is* $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $i = 1, 2$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{52}$$

$$\left\| \left\| D^{\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. Similar to Theorem 13, by using now (27). ■

Next comes a sequential left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 15 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})}\left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(b_1) - g_1(x_{01}))(g_2(b_2) - g_2(x_{02}))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{53}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

Proof. Using (30), as similar to Theorem 13 the proof is omitted. ■

The right side analog of Theorem 15 follows:

Theorem 16 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})-}\left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{54}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. Using (32), as similar to Theorem 13 is omitted. ■

5 Applications

We give

Corollary 17 *(to Theorem 13) All as in Theorem 13 for* $g_i(t) = e^t$, $i = 1, 2$. *Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{e^{x_{02}}}^{e^{b_2}} \frac{\|(f_1 \circ \log)(z_1)(f_2 \circ \log)(z_2)\| \, dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p(\nu_1 - 1)+1}}{p(p(\nu_1 - 1)+1)} + \frac{(z_2 - e^{x_{02}})^{q(\nu_2 - 1)+1}}{q(q(\nu_2 - 1)+1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\left(e^{b_2} - e^{x_{02}}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{55}$$

$$\left\|\left\|D_{e^{x_{01}}}^{\nu_1}(f_1 \circ \log)\right\|\right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\|\left\|D_{e^{x_{02}}}^{\nu_2}(f_2 \circ \log)\right\|\right\|_{L_p\left(\left[e^{x_{02}}, e^{b_2}\right], A\right)}.$$

We finish with

Corollary 18 *(to Theorem 15) All as in Theorem 15 for* $[a_1, b_1] \subset \mathbb{R}$, $[a_2, b_2] \subset (0, \infty)$, *and* $g_1(t) = e^t$ *and* $g_2(t) = \log t$. *Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{\log(x_{02})}^{\log(b_2)} \frac{\|(f_1 \circ \log)(z_1)(f_2 \circ e^t)(z_2)\| \, dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1 - 1)+1)} + \frac{(z_2 - \log(x_{02}))^{q((m_2+1)\nu_2 - 1)+1}}{q(q((m_2+1)\nu_2 - 1)+1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\log(b_2/x_{02})}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{56}$$

$$\left\|\left\|D_{e^{x_{01}}}^{(m_1+1)\nu_1}(f_1 \circ \log)\right\|\right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\|\left\|D_{\log(x_{02})}^{(m_2+1)\nu_2}\left(f_2 \circ e^t\right)\right\|\right\|_{L_p([\log(x_{02}), \log(b_2)], A)}.$$

References

[1] G.A. Anastassiou, *Strong mixed and generalized fractional calculus for Banach space valued functions*, Mat. Vesnik, 69(3) (2017), 176-191.

[2] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[3] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[4] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.

[5] C. Volintiru, *A proof of the fundamental theorem of Calculus using Hausdorff measures*, Real Analysis Exchange, 26 (1), 2000/2001, 381-390.