Asymptotic behavior of solutions of a class of time-varying systems with periodic perturbation

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Abstract This paper deals with stability of nonlinear differential equations with parameter with periodic perturbation. We determine values of the parameter under which the solutions of the perturbed systems could be uniformly exponentially stable. Sufficient conditions for global uniform asymptotic stability and/or practical stability in terms of Lyapunov-like functions are obtained in the sense that the trajectories converge to a small ball centered at the origin. Moreover, to illustrate the applicability of our result, we study the stabilization problem for a class of control system.

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1 Introduction

The investigation of stability analysis of nonlinear uncertain systems is an important topic in systems theory. The problem of stability analysis of nonlinear time-varying systems has attracted the attention of several researchers and has produced a vast body of important results (see [2]-[26], [29], [32], [33], [34] and the references therein). There have been a number of interesting developments in searching the stability criteria for nonlinear differential systems, but most have been restricted to finding the asymptotic stability conditions for some classes of certain systems. In particular, parametric stability for nonlinear systems is an interesting area of research, and it naturally arises in diverse fields such as population biology, economics, neural networks, and chemical processes.

Basically, parametric stability for nonlinear systems addresses the stability of equilibria for nonlinear systems with real parametric uncertainty, especially the feasibility of equilibria and the stability nature of the equilibria with respect to small variations of the real parametric uncertainty (see [25]). Dynamic systems governed by ordinary differential equations with periodically varying coefficients have been studied since one and a half centuries ago (see [12], [14], [19] and the references therein).
Mathieu [31] introduced a differential equation with periodic coefficient and Hill [24] presented the first ever solution technique of linear periodic equations. Lyapunov [30] demonstrated the Lyapunov-Floquet transformation for autonomous systems which is a linear periodic system into a dynamically equivalent time-invariant form. Unlike the differential systems without parameters, studying stability of differential parametric systems with periodic coefficients may not be easily verified ([16]-[17]).
It is well known that for linear parametric systems of the form: $\dot{x} = A(\alpha)x$, α is a real parameter which can be constant or depending on time. For technical reasons, it is important to distinguish between constant and time-varying parameters. Constant parameters have a fixed value that is known only approximately. In this case, the underlying dynamical linear system is time invariant. Time-varying parameter $\alpha(t)$ is a certain function which varies in some range and the resulting system is then time-varying. Kharitonov's theorem (see [27]) gives a simple necessary and sufficient condition for parametric system where a quadratic Lyapunov function is used to solve the problem of stability. Barmish in [3] introduced the notion of parameter dependent Lyapunov functions for continuous-time linear systems whose dynamic matrices are affected by bounded uncertain time-varying parameters. Floquet [20] developed the complete study for stability of linear time-periodic differential equations. Based on Floquet theory the stability of the linear system with time-periodic coefficients can be determined from the eigenvalues of a certain matrix. These eigenvalues are often called Floquet multipliers. He proved that, if all Floquet multipliers have magnitude less than one, the linear system with time-periodic coefficient is asymptotically stable. In general to solve the problem of stability the usual techniques are related to some linear matrices inequalities that finding an adequate Lyapunov matrix to solve a system of Lyapunov inequalities which is a convex program. Perturbation theory is a pertinent discipline for the applications of time parametric dynamics which is a compilation of methods systematically used to evaluate the global behavior of solutions to differential equations. This motivates us to study the problem of uniform exponential stability of perturbed systems by assuming that the nominal associated system is globally uniformly asymptotically stable by imposing some restrictions on the size of perturbations in particular that are periodic in time.
The goal is to obtain estimates for the solutions of perturbed differential equations and to get uniform boundedness and uniform convergence to a small neighborhood of the origin. The notion of practical stability, (see [6]), is introduced in a special case. We determine values of parameters under which the systems are uniformly practically exponentially stable where some estimates on the decay rate of solutions at infinity are obtained. Finally, we give an application for the stabilization a class of control parametric system.

2 General definitions

Consider the non-autonomous system

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

where $f : [0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous in t and locally Lipschitz in x on $[0, \infty) \times \mathbb{R}^n$. The origin is an equilibrium point for (1), if $f(t, 0) = 0, \quad \forall t \geq 0$.

Definition 1. ***(Exponential stability)*** *The zero solution of system (1) is exponentially stable if there exist positive constants c, μ, and λ such that*

$$\|x(t)\| \leq \mu \|x(t_0)\| e^{-\lambda(t-t_0)}, \quad \forall \ \|x(t_0)\| < c \tag{2}$$

and globally exponentially stable if (2) *is satisfied for any initial state $x(t_0) \in \mathbb{R}^n$.*

The exponential stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, in particular when $f(t, 0) \neq 0$, thus the notion of *practical stability* is more suitable in several situations than Lyapunov stability, it means that the trajectories converge to a small neighborhood of the origin, in the sense of uniform stability and uniform attractivity of system (1) with respect a certain ball $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$.

Definition 2. ***(Uniform stability of** B_r**)*** *B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0. \tag{3}$$

Definition 3. ***(Uniform attractivity of** B_r**)*** *B_r is uniformly attractive, if for $\varepsilon > r$, $t_0 > 0$ and $x(t_0) \in D$, there exists $T(\varepsilon, x(t_0)) > 0$, such that*

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, x(t_0)). \tag{4}$$

B_r is globally uniformly attractive if (4) *is satisfied for all $x(t_0) \in \mathbb{R}^n$.*

Definition 4. ***(Practical stability)*** *System (1) is said uniformly practically asymptotically stable, if there exists $B_r \subset \mathbb{R}^n$, such that B_r is uniformly stable and uniformly attractive. It is globally uniformly practically asymptotically stable if $x(t_0) \in \mathbb{R}^n$.*

Definition 5. *System (1) is said uniformly exponentially convergent to B_r, if there exist $\gamma > 0$ and $k \geq 0$, such that*

$$\|x(t)\| \leq k\|x(t_0)\| \exp(-\gamma(t - t_0)) + r, \quad \forall t \geq t_0, \ \ \forall x(t_0) \in \mathbb{R}^n. \tag{5}$$

If $x(t_0) \in \mathbb{R}^n$, the system is globally uniformly exponentially convergent to B_r.
We say that the system is globally uniformly practically exponentially stable if for $r > 0$, it is globally uniformly exponentially convergent to B_r.

Here, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| \ x(t) \ \| -r$, so that if $r = 0$ we find the classical definition of the uniform asymptotic or exponential stability of the origin viewed as an equilibrium point.

3 Problem formulation

We consider the following system of differential equations

$$\frac{dx}{dt} = \mu(A(\alpha(t)) + B(t))x + \nu\varphi(t,x), \quad t \geq 0, \tag{6}$$

where $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is a matrix given by $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, with $\alpha_1(t) + \alpha_2(t) = 1, \alpha_i(t) \in \mathbb{R}^+,\ \forall t \geq 0,\ B(t) \in \mathbb{R}^{n\times n}$ is T-periodic matrix, $\mu, \nu \in \mathbb{R}$ are parameters and $\varphi(t,x)$ is a smooth vector function such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$

$$\varphi(t+T,x) = \varphi(t,x)$$

and

$$\|\varphi(t,x)\| \leq k\|x\|^{1+\delta} + r, \quad \delta \geq 0,\ k > 0,\ r > 0. \tag{7}$$

Suppose that the spectrum of matrices A_1 and A_2 belong to the left half-plane $\{\lambda \in \mathbb{C}, \mathcal{R}(\lambda) < 0\}$ and

$$\int_0^T B(t)d(t) = 0. \tag{8}$$

Throughout this paper, we indicate the following domains:

$$I_1 = \{\mu \in \mathbb{R}, 0 < \mu < \mu_0\}, \qquad I_2 = \{\nu \in \mathbb{R}, |\nu| < \nu_0\},$$

such that the system (6) is practically uniformly exponentially stable for $\mu \in I_1$, $\nu \in I_2$. Moreover, we obtain estimates on the solutions of (6) that guarantee exponential decay when $t \longrightarrow +\infty$ to a certain ball $B(0, r_i)$ with a radius $r_i, i = 1,2$.

Remark For $\mu = \nu = 1$, the system (6) can be seen as a perturbed system (see [8], [9]).

Notations: The following notations will be used throughout this paper. For a matrix X, the notation X^* denotes the transpose of matrix X. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and the maximum eigenvalues of X respectively.

Since

$$spect(A_i)_{i=1,2} \subset \{\lambda \in \mathbb{C}, Re(\lambda) < 0\},$$

then, there exist symmetric and positive definite matrices H_1 and H_2 solutions of the matrices Lyapunov equations (see [26] for the existence and uniqueness of the matrices $H_i,\ i = 1,2$),

$$H_1A_1 + A_1^*H_1 = -I \tag{9}$$

and

$$H_2A_2 + A_2^*H_2 = -I. \tag{10}$$

The matrices $H_i,\ i = 1,2$ satisfy:

$$H_i = \int_0^\infty e^{sA_i^*}e^{sA_i}ds.$$

In many cases, it is hard to find a common positive-definite matrix $H = H_1 = H_2$. In fact, the existence of a common positive-definite matrix depends on the difference of the two matrices $A_i, i = 1, 2$. In order to solve these problems, many scholars have made many further investigations. For example, in [28], the authors showed that, if the matrices A_1 and A_2 are real Hurwitz matrices, and that their difference is rank one, then A_1 and A_2 have a common quadratic Lyapunov function if and only if the product A_1A_2 has no real negative eigenvalue. We can solve this problem, in the special case when $A_1 + A_1^* = A_2 + A_2^*$, we get

$$H = \int_0^\infty e^{sA_1^*}e^{sA_1}ds = \int_0^\infty e^{sA_2^*}e^{sA_2}ds.$$

To facilitate our task, we will suppose that, (9) and (10) have a unique solution $H = H^* > 0$.
We have

$$\gamma_1\|x\|^2 \leq \langle Hx, x\rangle \leq \|H\|\|x\|^2,$$

where $\gamma_1 = \lambda_{\min}(H)$.

Now, In order to study the asymptotic behavior of solutions, we shall impose some conditions on the parameters under which the system (6) can be practically uniformly exponentially stable.

Theorem 1. *Let*

$$\beta_1 = \max_{\tau\in[t_0,t_0+T]} \|H\int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH\|,$$

$$\beta_2 = \max_{\tau\in[t_0,t_0+T]} \|(H\int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH)(A_1 + B(\tau))\|,$$

$$\beta_3 = \max_{\tau\in[t_0,t_0+T]} \|(H\int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH)(A_2 + B(\tau))\|,$$

and

$$\mu_0 = \min\{\frac{\gamma_1}{\beta_1}, \frac{1}{2\beta}\} \quad \textit{where } \beta = \max\{\beta_2, \beta_3\}.$$

Let H be a solution to the matrices Lyapunov equations (9) and (10) and $\delta = 0$. Then, for parameters μ and ν such that

$$0 < \mu < \mu_0 \qquad \textit{and} \quad 2\mu\beta + 2|\nu|\ k\left(\frac{\|H\|}{\mu} + \beta_1\right) < 1,$$

and for any initial data $x(t_0) \in \mathbb{R}^n$, the solutions of system (6) converge exponentially towards the ball $B(0, r_1)$ whose radius is given by

$$r_1 = 2|\nu|r\frac{(\frac{\|H\|}{\mu} + \beta_1)^2}{(\frac{\gamma_1}{\mu} - \beta_1)\left(1 - 2\mu\beta - 2|\nu|k(\frac{\|H\|}{\mu} + \beta_1)\right)}.$$

Remark Note that, if $\nu = \nu(t)$ with $|\nu(t)| \longrightarrow 0$ as $t \longrightarrow +\infty$, then the solution of system (6) tend to zero when t tends to infinity.

Proof Define the following matrix

$$H(t,\mu) = \frac{1}{\mu}H - H\int_{t_0}^{t} B(s)ds - \int_{t_0}^{t} B^*(s)ds\ H. \tag{11}$$

Since $H = H^*$, it follows that

$$H(t,\mu) = H^*(t,\mu)$$

and by (8), the matrix $H(t,\mu)$ is T-periodic, i.e.

$$H(t+T,\mu) = H(t,\mu).$$

Let $x(t)$ be a solution to (6), then the function

$$h(t,\mu,\nu) = \langle H(t,\mu)x(t), x(t)\rangle$$

is continuously differentiable on t. It follows that, the derivative of $h(t,\mu,\nu)$ is given by

$$\frac{d}{dt}h(t,\mu,\nu) = \langle \frac{d}{dt}H(t,\mu)x(t), x(t)\rangle + \langle H(t,\mu)\frac{d}{dt}x(t), x(t)\rangle + \langle H(t,\mu)x(t), \frac{d}{dt}x(t)\rangle.$$

Since

$$\frac{d}{dt}H(t,\mu) = -HB(t) - B^*(t)H,$$

then

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) &= -\langle (HB(t) + B^*(t)H)x(t), x(t)\rangle \\ &\quad +\langle \mu H(t,\mu)(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\ &\quad +\langle \mu(A(\alpha(t))^* + B^*(t))H(t,\mu)x(t), x(t)\rangle \\ &\quad +\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle + \nu\langle H(t,\mu)x(t), \varphi(t,x)\rangle.\end{aligned}$$

Using the definition of matrix $H(t,\mu)$, we obtain

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) &= \langle (-HB(t) - B^*(t)H)x(t), x(t)\rangle + \langle H(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\ &\quad -\mu\langle (H\int_{t_0}^{t} B(s)ds + \int_{t_0}^{t} B^*(s)ds\ H)(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\ &\quad +\langle (A(\alpha(t))^* + B^*(t))Hx(t), x(t)\rangle \\ &\quad -\mu\langle (A(\alpha(t))^* + B^*(t))(H\int_{t_0}^{t} B(s)ds + \int_{t_0}^{t} B^*(s)ds\ H)x(t), x(t)\rangle \\ &\quad +2\left(\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle\right).\end{aligned}$$

Replacing $A(\alpha(t))$ by its value and multiplying $B(t)$ by $(\alpha_1(t)+\alpha_2(t))$, we get

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) &= \langle \alpha_1(t)(HA_1+A_1^*H)+\alpha_2(t)(HA_2+A_2^*H)x(t),x(t)\rangle \\
&\quad -\alpha_1(t)\mu\left(\langle (H\int_{t_0}^{t}B(s)ds+\int_{t_0}^{t}B^*(s)ds\ H)(A_1+B(t))x(t),x(t)\rangle\right. \\
&\quad \left. +\ \langle (A_1+B(t))^*(H\int_{t_0}^{t}B(s)ds+\int_{t_0}^{t}B^*(s)ds\ H)x(t),x(t)\rangle\right) \\
&\quad -\alpha_2(t)\mu\left(\langle (H\int_{t_0}^{t}B(s)ds+\int_{t_0}^{t}B^*(s)ds\ H)(A_2+B(t))x(t),x(t)\rangle\right. \\
&\quad \left. +\ \langle (A_2+B(t))^*(H\int_{t_0}^{t}B(s)ds+\int_{t_0}^{t}B^*(s)ds\ H)x(t),x(t)\rangle\right) \\
&\quad +2\left(\nu\langle H(t,\mu)\varphi(t,x),x(t)\rangle\right).
\end{aligned} \tag{12}$$

Taking into account (9) and (10) and using the fact that $0<\mu<\mu_0$, we obtain the following estimate

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) &\le -\|x(t)\|^2 \\
&\quad +2\mu\alpha_1(t)\max_{\tau\in[t_0,t_0+T]}\|(H\int_{t_0}^{\tau}B(s)ds+\int_{t_0}^{\tau}B^*(s)dsH)(A_1+B(\tau))\|\|x(t)\|^2 \\
&\quad +2\mu\alpha_2(t)\max_{\tau\in[t_0,t_0+T]}\|(H\int_{t_0}^{\tau}B(s)ds+\int_{t_0}^{\tau}B^*(s)dsH)(A_2+B(\tau))\|\|x(t)\|^2 \\
&\quad +2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|^2+2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\| \\
&\le -\left(1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)\right)\|x(t)\|^2 \\
&\quad +2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|.
\end{aligned}$$

Since the matrix $H(t,\mu)$ is positive definite for $0<\mu<\mu_0$, it follows that

$$0<(\frac{1}{\mu}\gamma_1-\beta_1)I\le H(t,\mu)\le(\frac{1}{\mu}\|H\|+\beta_1)I.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) &\le -\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\frac{1}{\mu}\|H\|+\beta_1}h(t,\mu,\nu) \\
&\quad +\ 2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\sqrt{h(t,\mu,\nu)}.
\end{aligned}$$

Let $\mathcal{H}(t,\mu,\nu)=\sqrt{h(t,\mu,\nu)}$, it follows that,

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t,\mu,\nu) &\leq -\frac{1-2\mu\beta-2|\nu|\;k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\frac{\|H\|}{\mu}+\beta_1)}\mathcal{H}(t,\mu,\nu)\\ &+ |\nu| r\frac{\frac{\|H\|}{\mu}+\beta_1}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{H}(t,\mu,\nu) &\leq \mathcal{H}(t_0,\mu,\nu)\exp\left(-\frac{1-2\mu\beta-2|\nu|\;k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\mu(t-t_0)\right)\\ &\quad +2|\nu| r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\left(1-2\mu\beta-2|\nu| k(\frac{\|H\|}{\mu}+\beta_1)\right)}\\ &\leq \sqrt{\frac{\|H\|}{\mu}}\|x(t_0)\|\exp\left(-\frac{1-2\mu\beta-2|\nu|\;k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\mu(t-t_0)\right)\\ &\quad +2|\nu| r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\left(1-2\mu\beta-2|\nu| k(\frac{\|H\|}{\mu}+\beta_1)\right)}\end{aligned}$$

and consequently

$$\begin{aligned}\|x(t)\| &\leq \sqrt{\frac{\|H\|}{\gamma_1-\mu\beta_1}}\exp\left(-\frac{1-2\mu\beta-2|\nu|\;k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\;\mu(t-t_0)\right)\|x(t_0)\|\\ &+ 2|\nu| r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{(\frac{\gamma_1}{\mu}-\beta_1)\left(1-2\mu\beta-2|\nu| k(\frac{\|H\|}{\mu}+\beta_1)\right)}.\end{aligned}$$

Thus, we obtain an estimation as in Definition 5. Hence, the solutions of system (6) converge exponentially towards the ball $B(0,r_1)$ whose radius is given by

$$r_1=2|\nu| r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{(\frac{\gamma_1}{\mu}-\beta_1)\left(1-2\mu\beta-2|\nu| k(\frac{\|H\|}{\mu}+\beta_1)\right)}.$$

Remark A simple verification shows that $r_1>0$.

In the next part of this paper, a new class of functions appears: functions that depend on a set of constant parameters, that is, $f=f(t,x,\varepsilon)$, where $\varepsilon\in\mathbb{R}^p$. The constant parameters could represent physical parameters of the system and the study of perturbation of these parameters accounts for modeling errors or changes in the parameter values due to aging. Let begin by introducing the following lemma.

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Lemma (see [26]) Let $f(t,x,\varepsilon)$ be continuous in (t,x,ε) and locally Lipschitz in x (uniformly in t and ε) on $[t_0,+\infty[\times\mathbb{R}^n\times\{\|\varepsilon-\varepsilon_0\|\leq c\}$. Let $y(t,\varepsilon_0)$ be a solution of $\dot{x}=f(t,x,\varepsilon_0)$ with $y(t_0,\varepsilon_0)=y_0\in\mathbb{R}^n$. Suppose $y(t,\varepsilon_0)$ is defined and belongs to $\mathbb{R}^n$ for all $t\geq t_0$. Then, given $\lambda>0$, there is $\gamma>0$ such that, if

$$\|z_0-y_0\|<\gamma \ and\ \|\varepsilon-\varepsilon_0\|<\gamma$$

then there is a unique solution $z(t,\varepsilon)$ of $\dot{x}=f(t,x,\varepsilon)$ defined for $t\geq t_0$, with $z(t_0,\varepsilon)=z_0$, and $z(t,\varepsilon)$ satisfies

$$\|z(t,\varepsilon)-y(t,\varepsilon_0)\|<\lambda,\quad \forall t\geq t_0.$$

Quite often when we study the state equation $\dot{x}=f(t,x,\varepsilon)$, where $\varepsilon\in\mathbb{R}^p$, we need to compute bounds on the solution $x(t)$ without computing the solution itself. That is why, in order to make our tache more easy, we will solve the differential equation $\dot{x}=f(t,x,\varepsilon_0)$ where ε_0 is a parameter sufficiently close to ε, i.e., $\|\varepsilon-\varepsilon_0\|$ sufficiently small and after that we will approximate the solution of $\dot{x}=f(t,x,\varepsilon)$.

Theorem 2. *Let H be a solution to the matrices Lyapunov equations (9) and (10). Let β_1, β_2, β_3, β and μ_0 be defined in the Theorem 1, let $\delta>0$, $\rho>0$ and*

$$\nu_0=\frac{\mu^{1-\delta/2}\,(\gamma_1-\mu\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}{2\,k(\|H\|+\mu\beta_1)^2\,(\sqrt{\frac{\|H\|}{\mu}}\rho+\gamma)^\delta}$$

with γ is some constant. Then, for $0<\mu<\mu_0$, $|\nu|<\nu_0$ and for any initial data

$$x(t_0)\in\mathbb{R}^n,\quad \|x(t_0)\|\leq\rho,$$

the system (6) is practically uniformly exponentially stable.

Proof Let $x(t)$ be a solution to system (6) and $H(t,\mu)$ be defined by (11). From the proof of Theorem 1, the function $h(t,\mu,\nu)$ satisfy the inequality (12). By the definition of matrix $H(t,\mu)$ and taking into account that $\|\varphi(t,x)\|\leq k\|x\|^{1+\delta}+r$, we obtain the following estimate

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) \;&\leq\; -(1-2\mu\beta)\|x(t)\|^2+2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|^{2+\delta}\\ &\quad +2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|.\end{aligned}$$

Since

$$\|x(t)\|^2\leq\frac{h(t,\mu,\nu)}{(\frac{1}{\mu}\gamma_1-\beta_1)}\ \ and\ \ \|x(t)\|^\delta\leq\frac{h(t,\mu,\nu)^{\delta/2}}{(\frac{1}{\mu}\gamma_1-\beta_1)^{\delta/2}},$$

then,

$$\|x(t)\|^{2+\delta}\leq\frac{h(t,\mu,\nu)^{1+\delta/2}}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}.$$

It follows that

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) \quad &\leq \quad -\frac{1-2\mu\beta}{\frac{1}{\mu}\|H\|+\beta_1}h(t,\mu,\nu)\\ &\quad +\frac{2|\nu|k\left(\frac{1}{\mu}\|H\|+\beta_1\right)}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}h(t,\mu,\nu)^{1+\delta/2}\\ &\quad +2|\nu|r\frac{\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\sqrt{\frac{1}{\mu}\gamma_1-\beta_1}}\sqrt{h(t,\mu,\nu)}.\end{aligned}$$

Introduce the following notation

$$\epsilon_1=\frac{1-2\mu\beta}{\frac{1}{\mu}\|H\|+\beta_1},\ \epsilon_2=\frac{2|\nu|k\left(\frac{1}{\mu}\|H\|+\beta_1\right)}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}\quad and\quad \epsilon_3=2|\nu|r\frac{\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\sqrt{\frac{1}{\mu}\gamma_1-\beta_1}},$$

hence

$$\frac{d}{dt}h(t,\mu,\nu)\leq-\epsilon_1h(t,\mu,\nu)+\epsilon_2h(t,\mu,\nu)^{1+\delta/2}+\epsilon_3\sqrt{h(t,\mu,\nu)}.$$

Let

$$z(t)=\sqrt{h(t,\mu,\nu)},$$

we have

$$\frac{d}{dt}z(t)\leq-\frac{\epsilon_1}{2}z(t)+\frac{\epsilon_2}{2}z(t)^{1+\delta}+\frac{\epsilon_3}{2}. \tag{13}$$

Let $z(t,\varepsilon)$ the solution of (13) where $\varepsilon=(\epsilon_1,\epsilon_2,\epsilon_3)\in\mathbb{R}^3_+$ and $y_1(t,\varepsilon_0)$ the solution of

$$\frac{d}{dt}z(t)\leq-\frac{\epsilon_1}{2}z(t)+\frac{\epsilon_2}{2}z(t)^{1+\delta} \tag{14}$$

where $\varepsilon_0=(\epsilon_1,\epsilon_2,0)\in\mathbb{R}^3_+$.

In order to solve (14), we can take $\eta=1+\delta$ and $w(t)=y_1(t,\varepsilon_0)^{1-\eta}=y_1(t,\varepsilon_0)^{-\delta}$. Thus,

$$\frac{d}{dt}w(t)=\frac{\epsilon_1\delta}{2}w-\frac{\epsilon_2\delta}{2}.$$

Solving the homogenous equation

$$\frac{d}{dt}w(t)=\frac{\epsilon_1\delta}{2}w,$$

we get

$$w(t)=L\ e^{\frac{\epsilon_1\delta}{2}t}.$$

Now, suppose that L is a function that depends on t, i.e. we have

$$w(t)=L(t)\ e^{\frac{\epsilon_1\delta}{2}t}.$$

A simple computation shows that

$$L(t) = \frac{\epsilon_2}{\epsilon_1}\, e^{-\frac{\epsilon_1\delta}{2}t} + \theta, \quad \forall \theta \geq 0,$$

and consequently

$$w(t) = \frac{\epsilon_2}{\epsilon_1} + \theta e^{\frac{\epsilon_1\delta}{2}t}$$

where

$$\theta = \left(w(t_0) - \frac{\epsilon_2}{\epsilon_1}\right)\, e^{-\frac{\epsilon_1\delta}{2}t_0}.$$

It follows that,

$$w(t) = \frac{\epsilon_2}{\epsilon_1} + \left(w(t_0) - \frac{\epsilon_2}{\epsilon_1}\right)\, e^{\frac{\epsilon_1\delta}{2}(t-t_0)}.$$

Since $y_1(t,\varepsilon_0) = w(t)^{-1/\delta}$ and $w(t_0) = y_1(t_0,\varepsilon_0)^{-\delta}$, we obtain

$$y_1(t,\varepsilon_0) = \left(y_1(t_0,\varepsilon_0)^{-\delta}\, e^{\frac{\epsilon_1\delta}{2}(t-t_0)} + \frac{\epsilon_2}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\, e^{\frac{\epsilon_1\delta}{2}(t-t_0)}\right)^{-1/\delta}.$$

If

$$\epsilon_2 y_1^{\delta}(t_0,\varepsilon_0) < \epsilon_1, \tag{15}$$

which will be verified later on, and using the fact that for all $a \geq 0$ and $b \geq 0$, we have

$$(a+b)^p \leq a^p(1+\frac{b}{a})^p, \quad \forall p \in \mathbb{R},$$

Thus,

$$y_1(t,\varepsilon_0) \leq y_1(t_0,\varepsilon_0)e^{-\frac{\epsilon_1}{2}(t-t_0)} \quad \times \quad \left(1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1} + y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}e^{-\frac{\epsilon_1}{2}\delta(t-t_0)}\right)^{-1/\delta}$$

yields,

$$y_1(t,\varepsilon_0) \leq y_1(t_0,\varepsilon_0)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left(1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right)^{-1/\delta}.$$

Then, by the Lemma, for $\|\epsilon_3\|_2 < \gamma$ and $\lambda > 0$, we get

$$\|z(t,\varepsilon) - y_1(t,\varepsilon_0)\| < \lambda,$$

which implies that

$$\begin{aligned}
\|z(t,\varepsilon)\| &< \lambda + \left\| y_1(t_0,\varepsilon_0)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left(1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right)^{-1/\delta}\right\| \\
&< \lambda + (\|z(t_0,\varepsilon)\| + \gamma)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left\|1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta} \\
&< \lambda + (\sqrt{\frac{\|H\|}{\mu}}\|x(t_0)\| + \gamma)e^{-\frac{\epsilon_1}{2}(t-t_0)}\left\|1 - y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}.
\end{aligned}$$

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Since

$$\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\|x(t)\| \leq z(t,\varepsilon) \leq \sqrt{\frac{\|H\|}{\mu}+\beta_1}\|x(t)\|,$$

then,

$$\begin{aligned}\|x(t)\| \quad &\leq \quad \sqrt{\frac{\|H\|}{\gamma_1-\mu\beta_1}}\left\|1-y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}\|x(t_0)\|e^{-\frac{\epsilon_1}{2}(t-t_0)}\\ &+\frac{\lambda}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}+\frac{\gamma}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\left\|1-y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}.\end{aligned}$$

The last inequality implies that the solutions of system (6) converge exponentially toward the ball $B(0,r_2)$ whose radius is given by

$$r_2=\frac{\lambda}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}+\frac{\gamma}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\left\|1-y_1^{\delta}(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}$$

which is clearly positive.

Finally, let verify the condition (15). Since $|\nu|<\nu_0$, $0<\mu<\mu_0$ and $\|x(t_0)\|\leq\rho$, then

$$\begin{aligned}\frac{\epsilon_2}{\epsilon_1}y_1^{\delta}(t_0,\varepsilon_0) \quad &\leq \quad \frac{2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)^2}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}(\|z(t_0,\varepsilon)\|+\gamma))^{\delta}\\ &\leq \quad \frac{2\nu_0k}{\mu^{1-\delta/2}}\,\frac{(\|H\|+\mu\beta_1)^2}{(\gamma_1-\mu\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}\left(\sqrt{\frac{\|H\|}{\mu}}\rho+\gamma\right)^{\delta}.\end{aligned}$$

Hence, according to the definition of ν_0, we have

$$\frac{\epsilon_2}{\epsilon_1}y_1^{\delta}(t_0,\varepsilon_0)<1.$$

4 Application to control

In this section we study the stabilization problem of a control system modeled by the same dynamic as (6).

Definition 6. *A function $\alpha:[0,a[\to[0,+\infty[$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0)=0$. It is of class $\mathcal{K}_\infty$ if, in addition, $a=+\infty$ and $\alpha(r)\to+\infty$ as $r\to+\infty$.*

Let as recall the following result (see [6]).

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Theorem 3. *Let consider system* (1) *and suppose that there exist a continuously differentiable real function $h(\cdot,\cdot)$ on $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, a $\mathcal{K}$ function $\alpha_3(\cdot)$ and a small positive real number ϱ such that the following inequalities hold for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$*

$$\alpha_1(\|x\|) \leq h(t,x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(t,x) \leq -\alpha_3(\|x\|) + \varrho.$$

Then the system is globally uniformly practically stable with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

When the function satisfying $f(t,0) \neq 0$ for certain $t \in \mathbb{R}_+$, we shall study the asymptotic stability of the system at a neighborhood of the origin viewed as a small ball centered at the origin. The state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. The following result gives sufficient conditions for practical global exponential stability.

Theorem 4. *Consider system* (1)*. Let $h : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that*

$$c_1\|x\|^2 \leq h(t,x) \leq c_2\|x\|^2$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(t,x) \leq -c_3 h(t,x) + \varrho$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where c_1, c_2 and c_3 are positive constants. Then B_r is globally uniformly exponentially stable, with $r = \sqrt{\varrho/c_1c_2}$.

Now we state the stabilizability problem associated with the following nonlinear time-varying control system:

$$\frac{dx}{dt} = f(t,x(t),u(t)), \quad t \geq 0, \tag{16}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Definition 7. *The feedback controller $u(t) = u(t,x(t))$, where $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$ stabilizes globally uniformly asymptotically or exponentially the control system* (16) *if the closed-loop system*

$$\frac{dx}{dt} = f(t,x(t),u(t,x(t))) \tag{17}$$

is globally uniformly asymptotic or exponential stable.

In the case where $f(t,0,0) \neq 0$ for a certain $t \geq 0$. We can formulate the above definition as:

Definition 8. *The feedback controller $u(t) = u(t,x(t))$ stabilizes globally uniformly asymptotically or exponentially the control system* (16) *with respect B_r, if the associated closed-loop system* (17) *is globally practically uniformly asymptotically or exponentially stable.*

From Theorem 3, one has the following result which concern the asymptotic stabilizability problem of system (16).

Theorem 5. *Suppose that there exist a stabilizing feedback controller* $u(t) = u(t, x(t))$ *for control system* (16) *and a continuously differentiable function* $h(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{K}_\infty$ *functions* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, *a* $\mathcal{K}$ *function* $\alpha_3(\cdot)$ *and a small positive real number* ϱ *such that the following inequalities hold for all* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leq h(t,x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial h}{\partial x}\ f(t,x,u(t,x(t))) \leq -\alpha_3(\|x\|) + \varrho.$$

Then system (16) *in closed-loop with the feedback controller* $u = u(t, x(t))$ *is globally uniformly practically asymptotically stable with* $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

Also, we can say that the control system (16) is globally uniformly exponentially stabilizable by the feedback control $u(t) = u(t, x(t))$, where $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$, if the closed-loop system (17) is globally uniformly exponentially stable.

Definition 9. B_r *is globally uniformly exponentially stabilizable by the feedback control* $u(t) = u(t, x(t))$ *if there exist* $\gamma > 0$ *and* $k > 0$ *such that for all* $t \geq t_0 \geq 0$ *and* $x_0 \in \mathbb{R}^n$, *the solution* $x(t)$ *of the closed-loop system* (17) *satisfies:*

$$\|x(t)\| \leq k\|x_0\|exp(-\gamma(t-t_0)) + r.$$

In this case, system (16) *is globally practically uniformly exponentially stabilizable by the feedback control* $u(t) = u(t, x(t))$.

One has the following result which concern the exponential stabilizability problem of system (16).

Theorem 6. *Let* $u = u(t, x(t))$ *an exponential stabilizing feedback law and*

$$h : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$$

be a continuously differentiable Lyapunov function such that

$$c_1\|x\|^2 \leq h(t,x) \leq c_2\|x\|^2$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\ f(t,x,u(t,x(t))) \leq -c_3 h(t,x) + \varrho$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^n$, *where* c_1, c_2 *and* c_3 *are positive constants. Then* B_r *is globally uniformly exponentially stable with* $r = \sqrt{\varrho/c_1c_2}$, *with respect the closed-loop system* (17).

Now, we will study the practical exponential stability problem a class of nonlinear systems of the form (6). It is worth to notice that the origin is not required to be an equilibrium point for the system (6). This may be in many situations meaningful from

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a practical point of view specially, when stability for control systems is investigated.

Consider the class of systems that can be modeled by:

$$\frac{dx}{dt} = \mu(A(\alpha(t)) + B(t))x + \nu\varphi(t, x, u), \quad t \geq 0, \tag{18}$$

where $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is a matrix given by $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, with $\alpha_1(t) + \alpha_2(t) = 1, \alpha_i(t) \in \mathbb{R}^+,\ \forall t \geq 0$, $B(t) \in \mathbb{R}^{n\times n}$ is T-periodic matrix, $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$ are parameters and $\varphi(t, x, u)$ is a smooth vector function. u denotes the control of the system. We suppose that there exists a stabilizing feedback control $u(t) = u(t, x(t))$, where the function u is a suitable feedback controller such that the condition (7) is replaced as follows: $\varphi(t, x, u)$ is a smooth vector function such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\varphi(t + T, x, u(t, x(t))) = \varphi(t, x, u(t, x(t)))$$

and

$$\|\varphi(t, x, u(t, x(t)))\| \leq k\|x\|^{1+\delta} + r, \quad \delta \geq 0,\ k > 0,\ r > 0.$$

The practical uniform exponential stability can therefore be established as in Theorem 2, an d an estimation as in Definition 9 can be obtained which gives that the system (18) in closed-loop with $u(t) = u(t, x(t))$ is practically globally uniformly exponentially stable.

5 Conclusion

Asymptotic stability of a class of parametric differential equations has been studied. New sufficient conditions for practical uniform asymptotic exponential stability of solutions for parametric systems with periodic coefficients are obtained. An application to control system is given.

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