The Atomic Solution for Fractional Wave Type Equation

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ABSTRACT

Sometimes, it is not possible to find a general solution for some differential equations using some classical methods, like separation of variables. In such a case, one can try to use theory of tensor product of Banach spaces to find certain solutions, called atomic solution. The aim of this paper is to find atomic solution for conformable non-linear wave equation.

Key Words: fractional wave type equation; conformable derivative; atomic solution.

1 Introduction

In [Khalil et al., 2014], a new definition called α-conformable fractional derivative was introduced as follows:
Letting $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty)$. Then for $x \in E$

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1}$$

If the limit exists then it is called the α-conformable fractional derivative of f at x.
For x=0, if f is α-differentiable on $(0,r)$ for some $r > 0$, and $\lim_{x \to 0} D^{\alpha} f(0)$ exists then we define $D^{\alpha} f(0) = \lim_{x \to 0} D^{\alpha} f(0)$. The new definition satisfies:

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, for all $a, b \in R$.

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2. $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$ and f,g are α-differentiable at a point t, with $g(t) \neq 0$. Then

1. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
2. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}, g(t) \neq 0$.

We list here the fractional derivatives of certain functions,

1. $T_\alpha(t^p) = pt^{p-\alpha}$.
2. $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
3. $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.
4. $T_\alpha e^{\frac{1}{\alpha}t^\alpha} = e^{\frac{1}{\alpha}t^\alpha}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for the ordinary derivatives.
One should notice that a function could be α-conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0) = 1$.
This is not the case for the known classical fractional derivatives, since $T_1(f)(0)$ does not exist.
A vast number of researcher dedicated so much of their work to study conformable derivatives and its applications. Among them, [Abdeljawad, 2015], [Abu Hammad and Khalil, 2014], [Aldarawi, 2018], [Alhabees and Aldarawi, 2020], [ALHabees, 2021], [ALHorani and Khalil, 2018], [Anderson et al., 2018], [Atangana et al., 2015], [Chung, 2015], [Hammad and Khalil, 2014], [Khalil et al., 2016], [Kilbas,], [Mhailan et al., 2020].

2 Atomic Solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$, defined by

$$T(x^*) = x^*(x)y \tag{2}$$

is bounded one rank linear operator. We write $x \otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products.

Atoms are used in theory of best approximation in Banach spaces, see [Al Horani et al., 2016]. According to [Khalil, 1985], one of the known results that we need in our paper is: if the sum of two atoms is an atom, then either the first components are dependent or the second are dependent.
For more on tensor product of Banach spaces we refer to [Deeb and Khalil, 1988] and [Khalil, 1985].
Our main object in this paper is to find an atomic solution of the equation

$$D_t^\alpha D_t^\alpha u = c^2 D_x^\beta D_x^\beta u + D_t^\alpha D_x^\beta u. \tag{3}$$

This is called the conformable non-linear wave equation, where c is constant. Let $c = 1$ for simplicity to get

$$D_t^\alpha D_t^\alpha u = D_x^\beta D_x^\beta u + D_t^\alpha D_x^\beta u. \tag{4}$$

If one tries to solve this equation via separation of variables, then it is not possible since the variables can not be separated.

3 Procedure

Let $u(x,t) = X(x)T(t)$. substitute in equation (4) to get:

$$X(x)T^{2\alpha}(t) = X^{2\beta}(x)T(t) + X^\beta(x)T^\alpha(t). \tag{5}$$

This can be written in tensor product form as:

$$X(x) \otimes T^{2\alpha}(t) = X^{2\beta}(x) \otimes T(t) + X^\beta(x) \otimes T^\alpha(t). \tag{6}$$

Let us consider the following conditions: $X(0) = 1$, $X^\beta(0) = 1$.
In equation (6), we have the situation: the sum of two atoms is an atom. Hence, we have two cases:

3.1 case I: $X^{2\beta}(x) = X^\beta(x)$

The situation of case I: $X^{2\beta}(x) = X^\beta(x)$, using the result in [Al-Horani et al., 2020], we get

$$X(x) = e^{\frac{x^\beta}{\beta}}. \tag{7}$$

Now, we substitute in (6) to get

$$
\begin{aligned}
e^{\frac{x^\beta}{\beta}} \otimes T^{2\alpha}(t) &= e^{\frac{x^\beta}{\beta}} \otimes T(t) + e^{\frac{x^\beta}{\beta}} \otimes T^{\alpha}(t). \\
e^{\frac{x^\beta}{\beta}} \otimes T^{2\alpha}(t) &= e^{\frac{x^\beta}{\beta}} \otimes [T(t) + T^{\alpha}(t)]. \\
T^{2\alpha}(t) &= T(t) + T^{\alpha}(t). \qquad (8)
\end{aligned}
$$

Hence, $T^{2\alpha}(t) = T(t)+T^{\alpha}(t)$. Again, using the result in [Al-Horani et al., 2020],

$$T(t) = c_1 e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + c_2 e^{(\frac{1-\sqrt{5}}{2})\frac{t^\alpha}{\alpha}}. \qquad (9)$$

Using the conditions $T(0) = T^{\alpha}(0) = 1$, we get

$$T(t) = \frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{\sqrt{5}-1}{2})\frac{t^\alpha}{\alpha}}. \qquad (10)$$

From (7) and (10), we obtain the atomic solution of (4) as follows:

$$u(x,t) = e^{\frac{x^\beta}{\beta}} \left(\frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{1-\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} \right). \qquad (11)$$

3.2 case II: $T(t) = T^{\alpha}(t)$

This is conformable linear differential equation. Hence, we can use the result in [Khalil, 1985], or use the fact that

$$T^{\alpha}(t) = t^{1-\alpha} T\prime(t). \qquad (12)$$

To get

$$
\begin{aligned}
T(t) &= t^{1-\alpha} T\prime(t) \\
\frac{dT(t)}{T(t)} &= t^{\alpha-1} dt \\
LnT(t) &= \frac{t^\alpha}{\alpha} + k. \qquad (13)
\end{aligned}
$$

Where k is constant. Hence,

$$T(t) = K e^{\frac{t^\alpha}{\alpha}}, K = e^k. \qquad (14)$$

Again, by using the conditions $T(0) = T^{\alpha}(0) = 1$, we get

$$T(t) = e^{\frac{t^\alpha}{\alpha}}. \qquad (15)$$

Substitute in equation (4) to get

$$\begin{aligned} X(x) \otimes e^{\frac{t^\alpha}{\alpha}} &= (X^{2\beta}(x) + X^\beta(x)) \otimes e^{\frac{t^\alpha}{\alpha}} \\ X(x) &= X^{2\beta}(x) + X^\beta(x). \end{aligned} \tag{16}$$

Again, by using the result in [Khalil, 1985], and the conditions $X(0) = X^\beta(0) = 1$, we get

$$X(x) = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1+\sqrt{5}}{2}\frac{x^\beta}{\beta}} + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1-\sqrt{5}}{2}\frac{x^\beta}{\beta}} \tag{17}$$

From (15) and (17), we obtain the atomic solution of (4) as follows:

$$u(x,t) = \left(\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1+\sqrt{5}}{2}\frac{x^\beta}{\beta}} + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1-\sqrt{5}}{2}\frac{x^\beta}{\beta}}\right) e^{\frac{t^\alpha}{\alpha}} \tag{18}$$

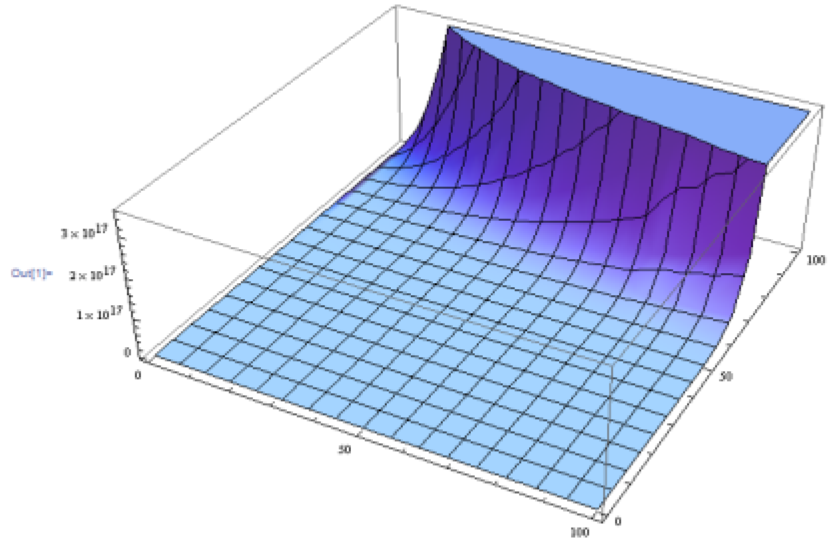
3.3 Example

Considering the following fractional wave equation

$$D_t^{0.5} D_t^{0.5} u = D_x^{0.2} D_x^{0.2} u + D_t^{0.5} D_x^{0.2} u. \tag{19}$$

The solution of (19) is

$$u(x,t) = e^{\frac{x^{0.2}}{0.2}} \left(\frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^{0.5}}{0.5}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{1-\sqrt{5}}{2})\frac{t^{0.5}}{0.5}} \right). \tag{20}$$



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