Solving the linear moment problems for nonhomogeneous linear recursive sequences

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Abstract

The present paper aimed to explore the linear moment problem for the real sequences defined by the nonhomogeneous linear recursive relation. Various properties are provided, especially, those related to the Hankel matrices. Some considerations in connection with K-moment problem, for the nonhomogeneous recursive are discussed.

Keywords: Linear moment problem, $K-$moment problem, Hankel matrix, nonhomogeneous linear recursive sequences.

1 Introduction

In view of its fundamental role in various fields of mathematics and applied science, the linear moment problem has been extensively studied in the literature (see [4, 5, 9, 11–13]). Especially, it has been shown that this problem is useful for some topics in physics, such that the quantum dynamical systems, the resolvent $R_\varphi(\lambda)$ of a given Hamiltonian A, which can be written as an infinite series in terms of $1/\lambda$, whose coefficients are the moment $\mu_n = \langle\varphi|A^n|\varphi\rangle$ of order n of the operator A, where φ is a state vector of the given system (see [4,12] for example). Furthermore, the linear moment problem is also related to the Lanczos numerical method, which is an important technique for finding the positions of n particles such that the first $2n-1$ moments own given values (see [5, 13] for example).

Recently, the linear moment problem has been investigated in the literature, by various methods (see, for example, [4, 9, 11, 12]).

The linear moment problem is simple to formulate. Indeed, let $\mathcal{H}$ be a real separable Hilbert space, $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $\mathcal{H}$ and $\mathcal{S}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the subspace of self adjoint operators on $\mathcal{H}$. For a given operator $A \in \mathcal{L}(\mathcal{H})$ and non-vanishing $x \in \mathcal{H}$, the sequence $\Gamma = \{\alpha_n\}_{n\geq 0}$ defined by $\alpha_n = \langle A^n x|x\rangle$ for $n \geq 0$, is called the moment sequence of A on x, and α_n is the moment of order n of the operator A on x. The linear moment problem is the reciprocal of the previous situation. More precisely, let $\Gamma = \{\alpha_n\}_{0\leq n\leq p}$ ($p \leq +\infty$) be a sequence of real numbers, the linear moment problem associated with Γ consists to find a self-adjoint operator $A \in \mathcal{S}(\mathcal{H})$ and a non-vanishing vector $x \in \mathcal{H}$ such that,

$$\alpha_n = \langle A^n x|x\rangle, \quad \text{for } \ 0 \leq n \leq p. \tag{1}$$

The problem (1) is called the *full linear moment problem* when $p = +\infty$ and the *truncated linear moment problem* for $p < +\infty$ (see [7–9, 12], for example).

On the other hand, the linear moment problem (1) for the sequence Γ, is also related to the classical power K-moment problem (K is a closed set of $\mathbb{R}$), whose aim is to find a positive Borelean measure μ with $\operatorname{supp}(\mu) \subset K$ such that

$$\alpha_n = \int_K t^n d\mu(t), \quad \text{for } \ 0 \leq n \leq p, \tag{2}$$

where $p \leq +\infty$. The moment problem (2) is important in operator theory, particularly, it is related to the study of the shift of subnormal operators and subnormal extension (see [1, 3, 6–8]). Recently, the two preceding moment problems (1) and (2) have been studied in [3, 9–11], for some sequences defined by linear recursive relations. Moreover, it was established the closed connection between the full and the truncated moment problem for recursive sequences in [9, 11]. More precisely, let $\{u_n\}_{n\geq 0}$ be the sequence satisfying the following linear recursive relation of order r,

$$u_{n+1} = a_0 u_n + a_1 u_{n-1} + \cdots + a_{r-1} u_{n-r+1} \text{ for } n \geq r-1, \tag{3}$$

where $u_0, u_1, \ldots, u_{r-1}$ are the initial data, it was shown in [9–11] that, for the linear moment problems (1), the full one ($p = +\infty$) and the truncated one ($p < +\infty$) are closely related. Especially, it was shown in [9] that in the finite dimensional case ($\dim_{\mathbb{R}} \mathcal{H} < +\infty$), the two preceding linear moment problems (the full and the truncated) are identical. On the other side, it was shown in [9] that the full and truncated moment problem (2), for the recursive sequence (3), are equivalent.

The purpose of this paper is to study the linear moment problem (1), for a real non-homogeneous recursive sequence $\{v_n\}_{n\geq 0}$ of order r, defined by the following recursive relation,

$$v_{n+1} = a_0 v_n + a_1 v_{n-1} + \cdots + a_{r-1} v_{n-r+1} + c_{n+1} \text{ for } n \geq r-1, \tag{4}$$

where the coefficients $a_0, \ldots, a_{r-1}$ ($r \geq 2$, $a_{r-1} \neq 0$) are real numbers, $v_0 = \alpha_0, \ldots, v_{r-1} = \alpha_{r-1}$ are the initial values, and $\mathcal{C} = \{c_n\}_{n\geq r}$ is a (non trivial) real sequence. It seems to us that properties of the linear moment problem (1) for nonhomogeneous sequences (4), can be useful for the study of certain related perturbed physical systems. For the K-moment problem (2), it can be also, for studying the perturbed moment, of the shift of operators.

In this study, we characterize the solution of the linear moment problem (1) for sequences (4) in the general setting, especially, when the operator $A \in \mathcal{S}(\mathcal{H})$, namely, A is self-adjoint. When the real separable Hilbert space $\mathcal{H}$ is of finite dimension and the non-homogeneous sequence $\{v_n\}_{n\geq 0}$ is a moment sequence of an operator A, on a non-vanishing $x \in \mathcal{H}$, we establish that the sequence $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3). And when the real separable Hilbert space $\mathcal{H}$ is of infinite dimension and the non-homogeneous sequence $\{v_n\}_{n\geq 0}$ is a moment sequence of an operator A, on a non-vanishing $x \in \mathcal{H}$, then the general term of the sequence $\{c_n\}_{n\geq r}$, is expressed as a limit of $c_n = \lim_{s\to+\infty} c_n^{(s)}$, where $c_n^{(s)}$ is a linear recursive sequence of type (3). We establish the solution of the linear moment problem (1), using the properties of the Hankel matrices. The special case when $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3), is discussed. Moreover, the K-moment problem (2) for nonhomogeneous recursive sequences (4) is provided, using the spectral measures of self-adjoint operators. By the way, some other consequences are derived, especially, the Stieltjes and Hamburger moment problems (2), for the nonhomogeneous recursive sequences (4), are discussed through the spectral measures of self-adjoint operators. It should be noted that the study of these two problems for the sequences (4), is not common in the literature.

2 Linear moment problem and sequences (4)

Let improve the connections between solutions of (4) considered as a difference equation and the linear moment problem (1). Let $\{Q_n\}_{n\geq r}$ be the family of polynomials defined by $Q_n(z) = z^{n-r}P(z)$, where $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$, is the so-called characteristic polynomial of the homogeneous part of the sequence (4). Let $x \neq 0$ be an element of $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$. Suppose that $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$. Then, we have, $\langle A^{n+1}x|x\rangle = a_0\langle A^n x|x\rangle + \cdots + a_{r-1}\langle A^{n-r+1}x|x\rangle + c_{n+1}$, for every $n \geq r-1$. Therefore, we derive $c_{n+1} = \langle Q_{n+1}(A)x|x\rangle$, for every $n \geq r-1$. Consequently, we can state the following proposition.

Proposition 2.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Suppose that $\mathcal{T} = \{v_n\}_{n\geq 0}$, is a moment sequence of an operator $A \in \mathcal{S}(\mathcal{H})$, namely, $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$, where $x \neq 0$. Then, the sequence $\{c_n\}_{n\geq r}$ is given by $c_{n+1} = \langle Q_{n+1}(A)x|x\rangle$, for every $n \geq r-1$, where $Q_n(z) = z^{n-r}P(z)$.*

Therefore, the question of studying the converse of the preceding affirmation of Proposition 2.1 arises.

Theorem 2.2. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*, of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Let $A \in \mathcal{S}(\mathcal{H})$ and $x \neq 0 \in \mathcal{H}$. Then, we have $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$, if and only if, $v_n = \langle A^n x|x\rangle$ for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r}P(A)x|x\rangle$, for $n \geq r$.*

Proof. Suppose $v_n = \langle A^n x|x\rangle$ $(n \geq 0)$, for some $x \neq 0$ in $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$. Then, we have $c_k = v_k - \sum_{j=0}^{r-1} a_j v_{k-j-1} = \left\langle (A^k - \sum_{j=0}^{r-1} a_j A^{k-j-1})x|x \right\rangle = \langle A^{k-r}P(A)x|x\rangle$, for every $k \geq r$. Conversely, suppose that $v_n = \langle A^n x|x\rangle$, for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r}P(A)x|x\rangle$ for every $n \geq r$. Therefore, we have

$$v_r = \sum_{j=0}^{r-1} a_j \langle A^{r-j-1}x|x\rangle + \langle P(A)x|x\rangle = \langle A^r x|x\rangle.$$

And, by induction, we derive that $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$. □

As a consequence of Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Let $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$, then under the data of Theorem 2.2, the following statements are equivalent,*

(i) $v_n = \langle A^n x|x\rangle$*, for every* $n \geq 0$.

(ii) $v_n = \langle A^n x|x\rangle$*, for* $n = 0, 1, \ldots, 2r-1$*, and* $c_n = \sum_{j=0}^{r-1} a_j c_{n-j-1} + \langle A^{n-2r}z|z\rangle$ *for every* $n \geq 2r$*, where* $z = P(A)x$.

Proof. It suffices to establish the equivalence between (ii) and the second statement of Theorem 2.2. Let A be a self-adjoint operator, suppose that $v_n = \langle A^n x|x\rangle$ for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r}P(A)x|x\rangle$, for every $n \geq r$. Then, for $z = P(A)x$, we have, $\langle A^{n-2r}z|z\rangle = \langle A^{n-r}x|P(A)x\rangle - \sum_{j=0}^{r-1} a_j \langle A^{n-r-j-1}P(A)x|x\rangle = c_n - \sum_{j=0}^{r-1} a_j c_{n-j-1}$, for any $n \geq 2r$. Conversely, suppose that (ii) holds. A direct computation shows that $c_n = \langle A^{n-r}P(A)x|x\rangle$, for $n = r, r+1, \ldots, 2r-1$. On the other hand, by induction we prove that $c_n = \langle A^{n-r}P(A)x|x\rangle$, for every $n \geq 2r$. It follows that (i) and (ii) are equivalent. □

We conclude this section by the following observation. Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), whose characteristic polynomial is $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Suppose that there exist $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$ such that $v_n = \langle A^n x|x\rangle$. Then, we have, $c_{2k} - \sum_{j=0}^{r-1} a_j c_{2k-j-1} = \|A^{k-r}P(A)x\|^2$ for every $k \geq r$. Therefore, when $c_{2k} \neq 0$, for some $k \in \mathbb{N}$, we have $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$, for any $k \geq r$. This later inequality is a necessary condition for the existence of the solution of the linear moment problem (1), for the sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ defined by (4).

3 The linear moment problem (1) for sequences (4)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{R}$ ($m = \dim_{\mathbb{R}} \mathcal{H}$) and $\mathcal{T} = \{v_n\}_{n\geq 0}$ a sequence (4). A straightforward computation and by using Theorem 2.2, allows us to see that $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a moment sequences of a self-adjoint operator A on a non-vanishing vector x of $\mathcal{H}$ if and only if $v_n = \sum_{j=1}^{s} {\lambda_j}^n \|x_j\|^2$ for $n = 0, 1, \ldots, r-1$ and

$$c_n = \sum_{j=1}^{s} \frac{P(\lambda_j)}{\lambda_j^r} \|x_j\|^2 {\lambda_j}^n, \tag{5}$$

where $x_j = \Pi_j x \in \mathcal{H}_j$ $(0 \leq j \leq s)$, the subspace of the eigenvectors of A, corresponding to the eigenvalues λ_j $(0 \leq j \leq s)$. Expression (5) is nothing else but the analytic formula of the sequence $\{c_n\}_{n\geq r}$, viewed as a linear recursive sequence of type (3) of order s. More precisely, (5) implies that $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3), of characteristic polynomial $K(z) = \prod_{j=1}^{s} (z - \lambda_j)$. Thus, we can state the following proposition.

Proposition 3.1. *Let $\mathcal{T}$ be a sequence* (4). *Suppose that $\mathcal{T}$ is a moment sequences of a self-adjoint operator A on the finite dimensional Hilbert space $\mathcal{H}$. Then, the nonhomogeneous part $\mathcal{C}$ is a linear recursive sequence of type* (3) *of order s (with $s \leq \dim \mathcal{H}$). More precisely, the characteristic polynomial of $\mathcal{C}$ is $K(z) = \prod_{j=1}^{s} (z - \lambda_j)$, where the λ_j $(0 \leq j \leq s)$ are the eigenvalues of A.*

Suppose that $\mathcal{H}$ is a separable real Hilbert space (over $\mathbb{C}$) of infinite dimension. The simplest spectral theorem (after the algebraic case) concerns a compact self-adjoint and a compact normal operator A on $\mathcal{H}$, and asserts that $\mathcal{H}$ coincide with the closure of the orthogonal sum of the eigenspaces $\mathcal{H}_n$, corresponding to all possible eigenvalues $\{\lambda_n\}_{n\geq 0}$. With a view to generalization it is convenient to express it under the spectral resolution form $Ax = \sum_{n=0}^{+\infty} \lambda_n \Pi_n x$, where Π_n is an orthoprojection onto $\mathcal{H}_n$, the eigenspace corresponding to the eigenvalue λ_j, and $x = \sum_{n=0}^{+\infty} \Pi_n x$. We consider the class of operators satisfying the Spectral Theorem, which are called spectral operators or S-operators for short.

Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), with characteristic polynomial P. Suppose that $\mathcal{T}$ is a sequence of moments of an S-operator A of $\mathcal{L}(\mathcal{H})$, on a non-vanishing vector $x \in \mathcal{H}$, namely, $v_n = \langle A^n x | x\rangle$, for every $n \geq 0$, where A is an S-operator and $x = \sum_{n=0}^{+\infty} \Pi_n x \in \mathcal{H}$.

Let $s \geq 1$ and consider the sequence $\{v_n^{(s)}\}_{n\geq 0}$ defined as follows: $v_j^{(s)} = v_j$

for $i = 0, 1, \dots, r-1$, and

$$v_{n+1}^{(s)} = a_0 v_n^{(s)} + a_1 v_{n-1}^{(s)} + \cdots + a_{r-1} v_{n-r+1}^{(s)} + c_{n+1}^{(s)}, \tag{6}$$

for $n \geq r-1$, where $c_n^{(s)} = \sum_{p=0}^{s} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n$. It is easy to see that $c_n = \lim_{s\to+\infty} c_n^{(s)}$. For $n = r$, expression (6) shows that we have $v_r = \lim_{s\to+\infty} v_r^{(s)}$. By induction on n, we have $v_n = \lim_{s\to+\infty} v_n^{(s)}$, for every $n \geq r$. In conclusion, we have the following result.

Theorem 3.2. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*, with characteristic polynomial P. Suppose the Hilbert space $\mathcal{H}$ is of infinite dimension and that $\mathcal{T}$ is a moment sequences of an S-operator A on $\mathcal{H}$, on a non-vanishing vector $x = \sum_{n=0}^{+\infty} \Pi_n x$. Then, we have $v_n = \lim_{s\to+\infty} v_n^{(s)}$, for every $n \geq r$, where $\{v_n^{(s)}\}_{n\geq 0}$ is a sequence* (4)*, whose associate nonhomogeneous term is*

$$c_n^{(s)} = \sum_{p=0}^{s} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n, \tag{7}$$

where $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$ $(a_{r-1} \neq 0)$ is the characteristic polynomial of $\mathcal{T}$ and $x_p = \Pi_p x \in \mathcal{H}$. Moreover, expression (7) *stands for the analytic formula of the sequence $\{c_n^{(s)}\}_{n\geq 0}$, viewed as a linear recursive sequence of type* (3)*.*

From Theorem 3.2, we derive that

$$c_n = \sum_{p=0}^{+\infty} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n. \tag{8}$$

Remark 3.3. If there exists $s \geq 1$ such that $\lambda_p = 0$, for every $p \geq s+1$, we show that expressions (5) and (8) are identical. Suppose that for every $N > 0$ there exists $k \geq N$ such that $\lambda_k \neq 0$. Therefore, expression (8) doesn't represent a recursive sequence of finite order. Meanwhile, we can approximate this situation by a family of sequences (4), whose associated c_n is given by expression (7).

4 Hankel matrices and solution of the linear moment problem (1)

In this section, we present algebraic treatment of the Hankel matrix related to the sequences defined by (4), and its use for characterizing the existence of solutions for the linear moment problem (1).

Let H_k be the Hankel matrix of size $k+1$, whose entries are defined from the elements of the sequence $\mathcal{T} = \{v_i\}_{i\geq 0}$, in the sense that $H_k := (v_{i+j})_{0\leq i,j\leq k}$.

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The j^{th} column of H_k will be denoted by $\mathbf{V}_j := (v_{j+\ell})_{\ell=0}^{k}, 0 \leq j \leq k$, so that H_k can be briefly written as $H_k = (\mathbf{V}_0 \;\; \mathbf{V}_1 \;\; \cdots \;\; \mathbf{V}_k)$. Observe that we can verify that

$$\mathbf{V}_{r+k} = a_0\mathbf{V}_{r+k-1} + a_1\mathbf{V}_{r+k-2} + \cdots + a_{r-1}\mathbf{V}_k + \widehat{\mathbf{C}}_{r+k}, \tag{9}$$

where $\widehat{\mathbf{C}}_{r+k} := (c_{r+\ell})_{\ell=0}^{r+k-1}$.
With a vectorial representation, we can write the matrix H_{r+n} as follows

$$H_{r+n} = \begin{pmatrix} \mathbf{V}_0 & \mathbf{V}_1 & \cdots & \mathbf{V}_{r-1} & | & \mathbf{V}_r & \cdots & \mathbf{V}_{r+k} & \cdots & \mathbf{V}_{r+n-1} \end{pmatrix}.$$

Using expression (9) and some computational techniques emanated from determinant properties, we get,

$$\det H_{r+n} = \det \begin{pmatrix} \mathbf{V}_0 & \mathbf{V}_1 & \cdots & \mathbf{V}_{r-1} & | & \widehat{\mathbf{C}}_r & \cdots & \widehat{\mathbf{C}}_{r+k} & \cdots & \widehat{\mathbf{C}}_{r+n-1} \end{pmatrix}.$$

Repeating the same treatment on the matrix $S_k := (v_{i+j+1})_{0 \leq i,j \leq k}$, one gets out of it by the following result.

Proposition 4.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*,*

$$H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1} \text{ and } S_{r+n} = (v_{i+j+1})_{0\leq i,j\leq r+n-1}$$

be the Hankel matrices associated with $\mathcal{T}$. Then, we have

$$\det H_{r+n} = \begin{vmatrix} v_0 & \cdots & v_{r-1} & c_r & \cdots & c_{r+n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r-1} & \cdots & v_{2r-2} & c_{2r-1} & \cdots & c_{2r+n-2} \\ v_r & \cdots & v_{2r-1} & c_{2r} & \cdots & c_{2r+n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r+n-1} & \cdots & v_{2r+n-2} & c_{2r+n-1} & \cdots & c_{2r+2n-2} \end{vmatrix} \tag{10}$$

and

$$\det S_{r+n} = \begin{vmatrix} v_1 & \cdots & v_r & c_{r+1} & \cdots & c_{r+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_r & \cdots & v_{2r-1} & c_{2r} & \cdots & c_{2r+n-1} \\ v_{r+1} & \cdots & v_{2r} & c_{2r+1} & \cdots & c_{2r+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r+n} & \cdots & v_{2r+n-1} & c_{2r+n} & \cdots & c_{2r+2n-1} \end{vmatrix}. \tag{11}$$

Expression (10) shows that, for $n \geq 0$, it appears only the columns which depend on the entries of the sequence $\{c_n\}_{n\geq r}$ after the r-th column, in the determinant of the Hankel matrix H_{r+n}. A similar situation is observed for the matrix $S_k = (v_{i+j+1})_{0\leq i,j\leq k}$.

If the sequence $\mathcal{C} = \{c_n\}_{n\geq r}$ is also of type (3) of order s, then the $r+s-th$ column of the matrix H_{r+n} is a linear combination of the columns $r, r+1, \ldots, r+s-1$, and the $r+s+1-th$ column of the matrix S_{r+n} is a linear combination of the columns $r+1, r+2, \ldots, r+s$. Therefore, by Proposition 4.1, we get the following property.

Proposition 4.2. *If the sequence $\{c_n\}_{n\geq r}$ is also a linear recursive sequence of type* (3) *of order s, then we have,*

1. $\det H_{r+n} = 0$*, for $n \geq s$, if and only if, the $r+s+1$-column of the matrix H_{r+n} is a linear combination of the previous s columns, namely, the $r, r+1, \dots, r+s-1$ columns of the matrix H_{r+n}.*

2. $\det S_{r+n} = 0$*, for $n \geq s+1$, if and only if, the $s+1$-column of the matrix S_{r+n} is a linear combination of the previous s columns, namely, the $r+1, r+2, \dots, r+s$ columns of the matrix H_{r+n}.*

The two Hankel matrices $H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1}$ and $S_{r+n} = (v_{i+j+1})_{0\leq i,j\leq r+n-1}$ and their determinants (10)-(11), play a central role for solving the two moment problems (1)-(2) and their applications.
We recall that it was established in [12, Lemma 1.1] that a $N \times N$ Hermitean matrix A is strictly positive definite if and only if each sub-matrix $A_k = (a_{ij})_{1\leq i,j\leq k}$ has $\det(A_k) > 0$, for $k = 1, 2, \dots, N$. For a given Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$, we consider the family of sub-matrices $H_n = (m_{i+j})_{0\leq i,j\leq n}$. Then, [12, Proposition 1.2] shows that for a Hankel matrix the family of sesquilinear form $\mathcal{F} = \{\mathtt{H}_n\}_{n\geq 0}$, defined by $\mathtt{H}_n(\alpha, \beta) = \sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\beta}_k$, is (strictly) positive definite if and only if $\det(H_n) > 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$. Equivalently, we say that the Hankel matrix $H_n = (m_{i+j})_{0\leq i,j\leq n}$ is positive definite if and only if $\det(H_n) > 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$.

In order to establish the existence of solution of the linear moment problem (1), we will present a result of the closed relation between Hankel positive matrix, self-adjoint operator and measure. More precisely, we recall that from [6] the following theorem.

Theorem 4.3. *If $\{v_n\}_{n\geq 0}$ is a sequence of real numbers, the following statements are equivalent.*

(a) There is a self-adjoint operator A and a vector e such that $e \in \operatorname{dom} A^n$ for all n and $v_n = \langle A^n e, e\rangle$, for all $n \geq 0$.

(b) If $\alpha = (\alpha_0, \dots, \alpha_n)$, where $\alpha_j \in \mathbb{C}$, then we have $\displaystyle\sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\alpha}_k \geq 0$, for every $n \geq 0$.

(c) There is a positive regular Borelean measure μ on $\mathbb{R}$ such that $\displaystyle\int |t|^n d\mu(t) < \infty$ for all $n \geq 0$ and $v_n = \displaystyle\int t^n d\mu(t)$.

Therefore, for the Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$, the second assertion of Theorem 4.3, implies that the sesquilinear form defined by $\mathtt{H}_n(\alpha, \beta) = \sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\beta}_k$, is a (strictly) positive definite form if and only if the matrix $H_n = (m_{i+j})_{0\leq i,j\leq n}$ is (strictly) positive definite, for every $n \geq 0$. Equivalently, the second assertion of Theorem 4.3, shows that the Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$ is positive, or in an equivalent way, $\det H_n \geq 0$, for every $n \geq 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$.

Combining Proposition 4.1 and Theorem 4.3, we can formulate the following result.

Theorem 4.4. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4). *Then, the following assertions are equivalent,*

1. *The linear moment problem* (1) *for sequence* (4) *owns a solution.*
2. *The Hankel matrix $H = (v_{i+j})_{i,j\geq 0}$ is positive.*
3. $\det H_n \geq 0$*, for every $0 \leq n \leq r-1$ and* $\det H_{n+r} \geq 0$*, for every $n \geq 0$, where* $\det H_{n+r}$ *is given by* (10).

Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4) and suppose that the associated nonhomogeneous part $\mathcal{C} = \{c_n\}_{n\geq r}$ is a sequence of type (3) of order s, whose characteristic polynomial is $Q(z) = z^s - b_0z^{s-1} - b_1z^{s-2} - \cdots - b_{s-1}$. Let $R(z) = z^r - a_0z^{r-1} - a_1z^{r-2} - \cdots - a_{r-1}$ be the characteristic polynomial of the homogeneous part of (4). The linearization process of [2, Theorem 2.1 (Linearization Process)] applied to the sequence (4), allows us to show that $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a sequence of type (3) of order $r+s$, with initial data $v_0, v_1, \ldots, v_{r+s-1}$ and whose coefficients $c_0, c_1, \ldots, c_{r+s}$ are obtained from its characteristic polynomial given by $P(z) = Q(z)R(z)$. Therefore, following Proposition 4.2, we get the following property.

Proposition 4.5. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4) *and $H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1}$ its associated Hankel matrices of order $r+n$. Suppose that $\mathcal{C}$ is a sequence of type* (3) *of order s. Then, we have* $\det H_{r+n} = 0$*, for every $n \geq s$.*

On the other hand, let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ be a solution of the linear moment problem (1) on a vector on a non-vanishing $x \in \mathcal{H}$, associated with the sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ defined by (4). By the linear recursive relation (3), related to the linearized expression of (4), we have $\langle A^nP(A)x|x\rangle = \langle P(A)x|A^nx\rangle = 0$, for every $n \geq 0$, where $P(z) = Q(z)R(z)$ is the characteristic polynomial of the linearized sequence of (4). Therefore, we have $\langle A^nP(A)x|A^mP(A)x\rangle = 0$, for every $n \geq 0$, $m \geq 0$, especially $\|A^nP(A)x\| = 0$, for every $n \geq 0$. This implies that A^nx is a linear combination of x, Ax, $\ldots$, $A^{r+s-1}x$. Therefore, when the nonhomogeneous part $\mathcal{C}$ is an $s-GFS$, if the linear moment problem owns a solution A, a self-adjoint operator on a Hilbert space $\mathcal{H}$, then it has a solution A on some $r+s$-dimensional Hilbert space (for more details see [11, Proposition 2.2]). This allows us to suppose that the Hilbert space $\mathcal{H}$ is of finite dimension $(r+s)$. Therefore, we have the following result.

Proposition 4.6. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*, with positive definite associated Hankel matrix H_r, and let $P(z)$ the characteristic polynomial of its homogeneous part. Suppose that $\mathcal{C}$ is a linear recursive sequence of type* (3) *of order s, whose characteristic polynomial is $Q(z)$. Then, there exists a* $(\deg(P)+\deg(Q))$*-dimensional Hilbert space $\mathcal{H}_{(\mathcal{T})}$ and a self-adjoint operator A on $\mathcal{H}_{(\mathcal{T})}$, solution of the moment problem* (1).

Proposition 4.6 shows the main role of the recursiveness of the sequence $\{c_n\}_{n\geq 0}$, in reducing the study of the linear moment problem (1) to the finite dimensional Hilbert space $\mathcal{H}$.

5 Some considerations on the K-moment problems (2) for sequences (4)

The aim here is to apply results of the preceding sections for solving the K-moment problem (2) for nonhomogeneous recursive sequences (4), using results of the linear moments problems in Hilbert spaces $\mathcal{H}$. More precisely, the solution of K-moment problem (2) is obtained in terms of representing measure of the self-adjoint operator A and the vector $x \in \mathcal{H}$ solution of the linear moment problem (1), for the nonhomogeneous recursive sequences (4). The Stieltjes and Hamburger moment problems for the nonhomogeneous recursive sequences (4) are discussed.

5.1 $K-$moment problems associated with sequences (4)

Recall that the purpose of the $K-$moment problem associated with a given sequence $\mathcal{T} = \{v_n\}_{0\leq n\leq p}$, where K is a closed subset of $\mathbb{R}$, is to find a positive Borel measure μ such that Expression (2) is verified, namely,

$$v_n = \int_K t^n d\mu(t) \qquad \text{and} \qquad \operatorname{supp}(\mu) \subset K.$$

As mentioned above, the problem (2) has been studied in the literature, by various methods and techniques. It is called the *full moment problem* when $p = +\infty$ and the *truncated moment problem*, for $p < +\infty$ (see [7–9]). Using the spectral representation of the self-adjoint operators, we can show that the linear moment problem (1) and the moment problem (5.1) are equivalent (see for example [6]). Moreover, using Theorem 4.3 and Theorem 4.4, we get,

Theorem 5.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4). Suppose that the Hankel matrix $H = (v_{i+j})_{i,j\geq 0}$ is positive. Then, there exists a positive Borel measure μ such that*

$$v_n = \int_K t^n d\mu(t),$$

where $K = \operatorname{supp}(\mu)$. Namely, the there exists a positive Borel measure μ solution of the K-moment problem (2).

Now consider the moment problem (2) for a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ given by (4). Let μ be a positive Borel measure of support K. Then, following the proof of Theorem 2.2, we have $v_n = \int_K t^n d\mu(t)$ for every $n \geq 0$, if and only if, $v_n = \int_K t^n d\mu(t)$ for any $n = 0, \ldots, r-1$ and $c_n = \int_K t^{n-r} P(t) d\mu(t)$ for $n \geq r$, where

$K = \operatorname{supp}(\mu)$. Moreover, a direct computation allows us to get the following result.

Proposition 5.2. *Under the preceding data, the following assertions are equivalent.*

(i) $v_n = \int_K t^n d\mu(t)$, *for every* $n \geq 0$, *where* $K = \operatorname{supp}(\mu)$.

(ii) $v_n = \int_K t^n d\mu(t)$ *for* $n = 0, \ldots, 2r-1$ *and* $c_n - \sum\limits_{j=0}^{r-1} a_j c_{n-j-1} = \int_K t^{n-2r} P(t)^2 d\mu(t)$, *for every* $n \geq 2r$, *where* $K = \operatorname{supp}(\mu)$.

It is easy to show that the second assertion of the Proposition 5.2 implies that $c_{2k} - \sum\limits_{j=0}^{r-1} a_j c_{2k-j-1} = \int [t^{k-r} P(t)]^2 d\mu(t)$, for any $k \geq r$, and if there exists $k_0 \geq r$ such that $c_{2k_0} - \sum\limits_{j=0}^{r-1} a_j c_{2k_0-j-1} = 0$, then $\operatorname{supp}(\mu) \subset \mathcal{Z}(P) \cup \{0\}$ or equivalently the sequence $\mathcal{T}$ is an $r - GFS$, in which case the sequence $\mathcal{C}$ vanish. This allows us to give a necessary condition for a sequence (4) to be a moment sequences of some positive Borel measure. Thus, we recover Lemma 2.2 of [10], considered for the special case of the Hausdorff moment problem. Since the sequence $\mathcal{C}$ is a nontrivial, if a sequence (4) is a moment sequence of a positive Borel measure μ, we have $c_{2k} > \sum\limits_{j=0}^{r-1} a_j c_{2k-j-1}$, for $k \geq r$. Hence, we can obtain the following.

Proposition 5.3. *Let* $\mathcal{T} = \{v_n\}_{n\geq 0}$ *be a sequence* (4). *If* $\mathcal{T}$ *is a moment sequences of a positive Borel measure* μ, *then* $c_{2k} > \sum\limits_{j=0}^{r-1} a_j c_{2k-j-1}$ *for any* $k \geq r$.

Using Proposition 4.5, we can easily establish the following.

Proposition 5.4. *Let* $\mathcal{T} = \{v_n\}_{n\geq 0}$ *be a sequence* (4), μ *a positive Borel measure and* ρ *a measure given by* $t^r d\rho(t) = P(t)d\mu(t)$. *Then* μ *is a solution of the full moment problem* (2) *associated with* $\mathcal{T}$ *if and only if* μ *is a solution of the truncated moment problem* (2) *associated with* $\mathcal{T}_r = \{v_n\}_{0\leq n\leq r-1}$ *and* $\{c_{n+r}\}_{n\geq 0}$ *is a moment sequences of* ρ.

Particularly, when $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a sequence of type (3) of order r (i.e $c_n = 0$, for every $n \geq 0$), then the second assertion of the preceding proposition is equivalent to the fact that μ is a solution of the truncated moment problem (2) associated with $\mathcal{T}_r = \{v_n\}_{0\leq n\leq r-1}$ and $\int_K t^n d\rho(t) = \int_K t^{n-r} P(t) d\mu(t)$, for every $n \geq r$. The last statement is equivalent to $\operatorname{supp}(\mu) \subset \mathcal{Z}(P)$, and we obtain Lemma 2.2 of [10] in the particular case of the Hausdorff moment problem.

5.2 Moment problems (2) associated with sequences (4), with c_n satisfying (3)

Let consider the linear moment problem (1) for sequence sequences (4), where the sequence $\mathcal{C} = \{c_n\}_{n\geq r}$ satisfies the linear recursive relation (3). Then, by

Proposition 4.2, Theorem 4.4, Proposition 4.6 and Theorem 5.1, we get the following result concerning the Hamburger moment problem for sequences (4).

Theorem 5.5. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*. Suppose that $\mathcal{C} = \{c_n\}_{n\geq 0}$ is a sequence of type* (3) *of order s. Then, a necessary and sufficient condition that there exists a measure μ solution of the truncated Hamburger moment problem associated with a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ is that the Hankel matrix H_{r+s} is positive definite or equivalently* $\det H_n > 0$ *for $n = 0, 1, \ldots, r+s$.*

Similarly, we get the following result concerning the Stieltjes moment problem for sequences (4).

Theorem 5.6. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4)*. Suppose that $\mathcal{C} = \{c_n\}_{n\geq 0}$ is a sequence of type* (3) *of order s. Then, a necessary and sufficient condition that there exists a measure μ solution of the truncated Stieltjes moment problem associated with a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ is that the two matrices H_{r+s} and S_{r+s} are positive definite or equivalently* $\det H_n > 0$ *and* $\det S_n > 0$ *for $n = 0, 1, \ldots, r+s$.*

Note that a similar result can be established for the Hausdorff moment problem.

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References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, 2 Eds., New York: Hafner Publ. Co. 1965.

[2] R. Ben Taher, M. Mouline and M. Rachidi, Solving some general nonhomogeneous recurrence relations of order r by a linearization method and an application to polynomial and factorial polynomial cases, *Fibonacci Quart.* 40, No. 1, 79-84, 2002.

[3] R. Ben Taher, M. Rachidi, E. H. Zerouali, Recursive subnormal completion and the truncated moment problem, *Bull. Lond. Math. Soc.* 33 , No. 4, 425-432, 2001.

[4] D. Bessis, M. Vallini, Perturbative-variational approximations to the spectral properties of semibounded Hilbert space operator, based on the moment problem with finite or divergin moments. Application to quantum mechanical systems, *J. Math. Phys.* 16, No. 3, 462-474, 1975 .

[5] C. Brezinski, The methods of Vorobyev and Lanczos, *Linear Algebra Appl.* 234, 21-41, 1997.

[6] J. B. Conway, *A course in functional analysis*, 2 Eds., Springer-Verlag, 1990.

[7] R. Curto, L. Fialkow, Flat extensions of positive moment matrices : Recursively generated relations, *Mem. Amer. Math. Soc.* 136, No. 648, 1-54, 1998.

[8] R. Curto, L. Fialkow, Solution of the truncated complex moment problem for flat data, *Mem. Amer. Math. Soc.* 119, No. 568, 1-50, 1996.

[9] B. El Wahbi, M. Rachidi, r-Generalized Fibonacci sequences and the linear moment problem, *Fibonacci Quart.* 38, No. 5, 368-394, 2000.

[10] B. El Wahbi, M. Rachidi, On r-Generalized Fibonacci sequences and Hausdorff moment problem, *Fibonacci Quart.* 39, No. 1, 5-11, 2001.

[11] B. El Wahbi, M. Rachidi, E. H. Zerouali, Recursive relations, Jacobi matrices, moment problems and continued fractions, *Pacific J. Math.* 216, 39-50, 2004.

[12] B. Simon, The classical moment problem as a self-adjoint finite difference operator, *Adv. Math. (N. Y),* 137, No. 1, 82-203, 1998.

[13] R. R. Whitehead, Moment Methods and Lanczos Methods. In B.J. Dalton, S.M. Grimes, J.P. Vary, S.A. Williams (eds), Theory and Applications of Moment Methods in Many-Fermion Systems, Springer US, Boston, MA, pp. 235-255, 1980.