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ON HARMONIC MULTIVALENT FUNCTIONS DEFINED BY A NEW DERIVATIVE OPERATOR

ADRIANA CĂTAŞI1, ROXANA ŞENDRUŢIU2

Abstract. In the present paper, we define and investigate a new class of multivalent harmonic functions in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$, under certain conditions involving a new generalized differential operator. Coefficient inequalities, distortion bounds and a covering result are also obtained.

Keywords: differential operator, harmonic function, coefficient bounds.

2000 Mathematical Subject Classification: 30C45.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $D$ is said to be harmonic in $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. A necessary and sufficient condition for $f$ to be univalent and sense preserving in $D$ is that $|h'(z)| > |g'(z)|$, $z \in D$. (See [4] for more details.)

Denote by $S_{H}(p,n)$, $(p,n \in \mathbb{N})$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc $U$ for which $f(0) = \bar{z}(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_{H}(p,n)$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1.$$ (1.1)

Let $\tilde{S}_{H}(p,n,m)$, $(p,n \in \mathbb{N}, m \in \mathbb{N}_0 \cup \{0\})$ denote the family of functions $f_m = h + \bar{g}_m$ that are harmonic in $D$ with the normalization

$$h(z) = z^p - \sum_{k=p+n}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^m \sum_{k=p+n-1}^{\infty} |b_k| z^k, \quad |b_{p+n-1}| < 1.$$ (1.2)

2. Coefficient bounds for the new classes $AL_H(p,m,\delta,\alpha,\lambda,l)$ and $\tilde{A}L_H(p,m,\delta,\alpha,\lambda,l)$

We propose for the beginning a new generalized differential operator as follows.

Definition 2.1. Let $H(U)$ denote the class of analytic functions in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $A(p)$ be the subclass of the functions belonging...
Let \( H(U) \) of the form \( h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \). For \( m \in \mathbb{N}_0 \), \( \lambda \geq 0 \), \( \delta \in \mathbb{N}_0 \), \( l \geq 0 \) we define the generalized differential operator \( I_{\lambda,\delta}^m(p,l) \) on \( A(p) \)

\[
I_{\lambda,\delta}^m(p,l) h(z) = (p + l)^m z^p + \sum_{k=p+n}^{\infty} [p + \lambda(k - p) + l]^m C(\delta, k) a_k z^k,
\]  

\[
C(\delta, k) = \binom{k + \delta - 1}{\delta} = \frac{\Gamma(k + \delta)}{\Gamma(k) \Gamma(\delta + 1)}.
\]

**Remark 2.2.** When \( \lambda = 1 \), \( p = 1 \), \( l = 0 \), \( \delta = 0 \) we get Sălăgean differential operator \[10\]; \( p = 1 \), \( m = 0 \) gives Ruscheweyh operator \[9\]; \( p = 1 \), \( l = 0 \), \( \delta = 0 \) implies Al-Oboudi differential operator of order \( m \) (see \[11\]); \( \lambda = 1 \), \( p = 1 \), \( l = 0 \) operator \[2.1\] reduces to Al-Shaqsi and Darus differential operator \[2\] and when \( p = 1 \), \( l = 0 \) we reobtain the operator introduced by Darus and Ibrahim in \[3\].

**Definition 2.3.** Let \( f \in S_H(p,n) \), \( p \in \mathbb{N} \). Using the operator \[2.1\] for \( f = h + \tilde{g} \) given by \[1.1\] we define the differential operator of \( f \) as

\[
I_{\lambda,\delta}^m(p,l) f(z) = I_{\lambda,\delta}^m(p,l) h(z) + (-1)^m \tilde{I}_{\lambda,\delta}^m(p,l) \tilde{g}(z)
\]

\[
I_{\lambda,\delta}^m(p,l) h(z) = (p + l)^m z^p + \sum_{k=p+n}^{\infty} [p + \lambda(k - p) + l]^m C(\delta, k) a_k z^k
\]

\[
I_{\lambda,\delta}^m(p,l) \tilde{g}(z) = \sum_{k=p+n-1}^{\infty} [p + \lambda(k - p) + l]^m C(\delta, k) b_k z^k.
\]

**Remark 2.4.** When \( \lambda = 1 \), \( l = 0 \), \( \delta = 0 \) the operator \[2.3\] reduces to the operator introduced earlier in \[7\] by Jahangiri et al.

**Definition 2.5.** A function \( f \in S_H(p,n) \) belongs to the class \( AL_H(p,m,\delta,\alpha,\lambda,l) \) if

\[
\frac{1}{p + l} \text{Re} \left\{ \frac{I_{\lambda,\delta}^{m+1}(p,l)f(z)}{I_{\lambda,\delta}^{m}(p,l)f(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1,
\]

where \( I_{\lambda,\delta}^m f \) is defined by \[2.3\], for \( m \in \mathbb{N}_0 \). Finally, we define the subclass

\[
\tilde{AL}_H(p,m,\delta,\alpha,\lambda,l) \equiv AL_H(p,m,\delta,\alpha,\lambda,l) \cap \tilde{S}_H(p,n,m).
\]

**Remark 2.6.** The class \( AL_H(p,m,\delta,\alpha,\lambda,l) \) includes a variety of well-known subclasses of \( S_H(p,n) \). For example, letting \( n = 1 \) we get \( AL_H(1,1,0,\alpha,1,0) \equiv HK(\alpha) \) in \[6\], for \( n = 1 \), \( AL_H(1,1,0,\alpha,1,0) \equiv S_H(t,u,\alpha) \) in \[11\], \( AL_H(p,n+p,0,\alpha,1,0) \equiv SH_p(n,\alpha) \) in \[8\] and \( n = 1 \), \( AL_H(1,1,\delta,\alpha,1,0) \equiv M_H(\delta,\alpha) \) in \[3\].

**Theorem 2.7.** Let \( f = h + \tilde{g} \) be given by \[1.1\]. If

\[
\sum_{k=p+n}^{\infty} \frac{\{(p + l)(1 - \alpha) + \lambda(k - p)d_{p,k}(m,\lambda,l)C(\delta,k)}{(p + l)^{m+1}(1 - \alpha)} |a_k| +
\]
On harmonic multivalent functions defined by a new derivative operator

\[ + \sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} |b_k| \leq 1, \]

with \( \lambda n \geq \alpha(p+l) \), where

\[ d_{p,k}(m,\lambda,l) = [p+\lambda(k-p)+l]^m \]

then \( f \in AL_\mathcal{H}(p,m,\delta,\alpha,\lambda,l) \).

**Proof.** Using the fact that \( \frac{1}{p+m} \Re w \geq \alpha \) if and only if \( |(p+l)-(p+l)\alpha+w| \geq |(p+l)+(p+l)\alpha-w| \), it is sufficient to show that

\[ (2.10) \quad |(p+l)(1-\alpha)I_{\lambda,\delta}^{m}(p,l)f(z) + I_{\lambda,\delta}^{m+1}(p,l)f(z)| - \\
- |(p+l)(1+\alpha)I_{\lambda,\delta}^{m}(p,l)f(z) - I_{\lambda,\delta}^{m+1}(p,l)f(z)| \geq 0. \]

Substituting \( I_{\lambda,\delta}^{m}(p,l)f(z) \) and \( I_{\lambda,\delta}^{m+1}(p,l)f(z) \) in (2.10) yields by (2.8)

\[ (p+l)(1-\alpha)I_{\lambda,\delta}^{m}(p,l)f(z) + I_{\lambda,\delta}^{m+1}(p,l)f(z) - \\
- |(p+l)(1+\alpha)I_{\lambda,\delta}^{m}(p,l)f(z) - I_{\lambda,\delta}^{m+1}(p,l)f(z)| > \\
2(p+l)^{m+1}(1-\alpha) \left\{ 1 - \sum_{k=p+n}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} |a_k| - \\
- \sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} |b_k| \right\}. \]

The last expression is nonnegative by (2.8) and therefore the proof is complete. \( \square \)

**Remark 2.8.** The harmonic function

\[ f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} x_k z^k + \\
+ \sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} y_k z^k, \]

where \( \sum_{k=p+n}^{\infty} |x_k| + \sum_{k=p+n-1}^{\infty} |y_k| = 1, 0 \leq \alpha < 1, m \in \mathbb{N}_0, \lambda n \geq \alpha(p+l), \lambda \geq 0 \) and \( d_{p,k}(m,\lambda,l) \) is given in (2.9), show that the coefficient bound expressed by (2.8) is sharp.

**Theorem 2.9.** Let \( f_m = h + g_m \) be given by (1.2). Then \( f_m \in \tilde{AL}_\mathcal{H}(p,m,\delta,\alpha,\lambda,l) \) if and only if

\[ \sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} |a_k| + \\
+ \sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} |b_k| \leq 1, \]

where \( \lambda n \geq \alpha(p+l), 0 \leq \alpha < 1, m \in \mathbb{N}_0, \lambda \geq 0 \) and \( d_{p,k}(m,\lambda,l) \) is given in (2.9).
Proof. Since \( \overline{AL_H}(p, m, \delta, \alpha, \lambda, l) \subset AL_H(p, m, \delta, \alpha, \lambda, l) \), we only need to prove the "only if" part of the theorem. For this part we consider that \( f_m \in AL_H(p, m, \delta, \alpha, \lambda, l) \). Then

\[
\text{Re} \left\{ \frac{f_m+1(p, l)f(z)}{f_{\lambda,\delta}(p, l)f(z)} - \alpha(p + l) \right\} =
\]

\[
\frac{(p + l)^{m+1}(1 - \alpha)z^p - \sum_{k=p+n}^{\infty}[(p + l)(1 - \alpha) + \lambda(k - p)]\xi(m, \lambda, l; \delta, k)a_kz^k}{(p + l)^mz^p - \sum_{k=p+n}^{\infty}\xi(m, \lambda, l; \delta, k)a_kz^k + (1 - 2m)\sum_{k=p+n-1}^{\infty}\xi(m, \lambda, l; \delta, k)b_kz^k}
\]

\[
- \frac{(1 - 2m)\sum_{k=p+n-1}^{\infty}[(p + l)(1 + \alpha) + \lambda(k - p)]\xi(m, \lambda, l; \delta, k)b_kz^k}{(p + l)^mz^p - \sum_{k=p+n}^{\infty}\xi(m, \lambda, l; \delta, k)a_kz^k + (1 - 2m)\sum_{k=p+n-1}^{\infty}\xi(m, \lambda, l; \delta, k)b_kz^k}
\]

where \( \xi(m, \lambda, l; \delta, k) = d_{p,k}(m, \lambda, l)C(\delta, k) \).

The above required condition must hold for all values of \( z \) in \( U \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq |z| = r < 1 \), we must have

\[
(p + l)^{m+1}(1 - \alpha) - \sum_{k=p+n}^{\infty}[(p + l)(1 - \alpha) + \lambda(k - p)]\xi(m, \lambda, l; \delta, k)a_kr^{k-p}
\]

\[
- \frac{(p + l)^m - \sum_{k=p+n}^{\infty}\xi(m, \lambda, l; \delta, k)a_kr^{k-p} + \sum_{k=p+n-1}^{\infty}\xi(m, \lambda, l; \delta, k)b_kr^{k-p}}{(p + l)^m - \sum_{k=p+n}^{\infty}\xi(m, \lambda, l; \delta, k)a_kr^{k-p} + \sum_{k=p+n-1}^{\infty}\xi(m, \lambda, l; \delta, k)b_kr^{k-p}} \geq 0.
\]

If the condition (2.12) does not hold, then the numerator in (2.13) is negative for \( r \) sufficiently close to 1. Hence there exists a \( z_0 = r_0 \in (0, 1) \) for which the quotient in (2.13) is negative. This contradicts the required condition for \( f \in \overline{AL_H}(p, m, \delta, \alpha, \lambda, l) \) and so the proof is complete.

\[
\Box
\]

3. Distortion bounds

The following theorem gives the distortion bounds for functions in \( AL_H(p, m, \delta, \alpha, \lambda, l) \) which yields a covering result for this class.
On harmonic multivalent functions defined by a new derivative operator

**Theorem 3.1.** Let \( f \in \mathcal{A}_L(p, m, \delta, \alpha, \lambda, l) \), with \( 0 \leq \alpha < 1 \), \( \lambda n \geq \alpha(p + l) \), \( m \in \mathbb{N}_0 \), \( \lambda \geq 0 \). Then for \( |z| = r < 1 \) one obtains

\[
|f(z)| \leq (1 + |b_{p+n-1}|r^{n-1})r^p + \frac{(p + l)^{m+1}(1 - \alpha)}{[(p + l)(1 - \alpha) + \lambda n]d_{p,n+p}(m, \lambda, l)C(\delta, n + p)} \cdot \left\{ 1 - \frac{[(p + l)(1 + \alpha) + \lambda(n - 1)]d_{p,n+p+1}(m, \lambda, l)C(\delta, n + p - 1)}{(p + l)^{m+1}(1 - \alpha)}|b_{p+n-1}| \right\} r^{n+p}
\]

and

\[
|f(z)| \geq (1 - |b_{p+n-1}|r^{n-1})r^p - \frac{(p + l)^{m+1}(1 - \alpha)}{[(p + l)(1 - \alpha) + \lambda n]d_{p,n+p}(m, \lambda, l)C(\delta, n + p)} \cdot \left\{ 1 - \frac{[(p + l)(1 + \alpha) + \lambda(n - 1)]d_{p,n+p-1}(m, \lambda, l)C(\delta, n + p - 1)}{(p + l)^{m+1}(1 - \alpha)}|b_{p+n-1}| \right\} r^{n+p}.
\]

**Proof.** We only prove the left-hand inequality. The proof for the right-hand inequality is similar and will be omitted.

Let \( f \in \mathcal{A}_L(p, m, \delta, \alpha, \lambda, l) \). Taking the absolute value of \( f \) we have

\[
|f(z)| = |z^p - \sum_{k=p+n}^{\infty} a_k z^k| \geq (1 - |b_{p+n-1}|r^{n-1})r^p - \sum_{k=p+n}^{\infty} (|a_k| + |b_k|)r^{p+n} \geq (1 - |b_{p+n-1}|r^{n-1})r^p - \frac{(p + l)^{m+1}(1 - \alpha)}{[(p + l)(1 - \alpha) + \lambda n]d_{p,n+p}(m, \lambda, l)C(\delta, n + p)} \cdot \left\{ 1 - \frac{[(p + l)(1 + \alpha) + \lambda(n - 1)]d_{p,n+p+1}(m, \lambda, l)C(\delta, n + p - 1)}{(p + l)^{m+1}(1 - \alpha)}|b_{p+n-1}| \right\} r^{n+p}.
\]

The bounds given in Theorem 3.1 for the functions \( f \) of the form (1.1) also hold for the functions of the form (1.1) if the coefficient condition (2.8) is satisfied. The upper bound given for \( f \in \mathcal{A}_L(p, m, \delta, \alpha, \lambda, l) \) is sharp and the equality occurs for the function

\[
f(z) = z + |b_{p+n-1}|z^p + \frac{(p + l)^{m+1}(1 - \alpha)}{[(p + l)(1 - \alpha) + \lambda n]d_{p,n+p}(m, \lambda, l)C(\delta, n + p)} \cdot \left\{ 1 - \frac{[(p + l)(1 + \alpha) + \lambda(n - 1)]d_{p,n+p+1}(m, \lambda, l)C(\delta, n + p - 1)}{(p + l)^{m+1}(1 - \alpha)}|b_{p+n-1}| \right\} r^{n+p},
\]

where \( |b_{p+n-1}| \leq \frac{(p + l)^{m+1}(1 - \alpha)}{[(p + l)(1 + \alpha) + \lambda(n - 1)]d_{p,n+p+1}(m, \lambda, l)C(\delta, n + p - 1)} \).

The following covering result follows from the left-hand inequality in Theorem 3.1.
Corollary 3.2. If the function \( f \in \tilde{AL}_H(p,m,\delta,\alpha,\lambda,l) \), then
\[
\left\{ w; |w| < \frac{[(p + l)(1 - \alpha) + \lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p) - (p + l)^{m+1}(1 - \alpha)}{[(p + l)(1 - \alpha) + \lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} - \frac{[p(1 - \alpha) + \lambda n + l]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)}{E_{p,\delta}^\alpha(m,\lambda,l)} \cdot |b_{p+n-1}| \right\} \subset f(U)
\]
where \( E_{p,\delta}^\alpha(m,\lambda,l) = [(p + l)(1 + \alpha) + \lambda(n - 1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1) \).

References


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GOOD AND SPECIAL WEAKLY PICARD OPERATORS
PROPERTIES FOR A CLASS OF DISCRETE LINEAR
OPERATORS

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ABSTRACT. Based on the results of the weakly Picard operators theory, in this
paper we study the good and special convergence of the iterates of a general
class of positive linear operators of discrete type introduced by O.Agratini and
I.A. Rus ([1]).

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1. Introduction and Preliminaries

The study of the convergence of the sequence of successive approximations is
realized in metric spaces. That is, for \((X, d)\) metric space and \(A: X \to X\) an
operator, for any \(x \in X\) can be considered the sequence:

\[(A^m(x))_{m \in \mathbb{N}}, \quad x \in X\]

where \(A^0 = 1_X\) and \(A^m = A^{m-1} \circ A\) for \(m \in \mathbb{N}^*\).

Investigating the properties of sequence (1), L. d’Apuzzo introduced in 1976
(see [3]) the good and special convergence, giving necessary and sufficient condi-
tions for this kind of convergence (see [2]). In paper [3], she considers the good
and special convergence of type M, as a particular case, in which the sequence
\((d(A^m(x), A^\infty(x)))_{m \in \mathbb{N}}\) (respectively, \((d(A^m(x), A^{m-1}(x)))_{m \in \mathbb{N}}\) ) is strictly
decreasing for any \(x\). I.A. Rus introduced, in paper (see [8]), the good and special
weakly Picard operators.

In what follow, let \((X, d)\) be a metric space and \(A: X \to X\) an operator. In this
paper we will use the following notations:

\[P(X) := \{Y \subset X | Y \neq \emptyset\};\]
\[F_A := \{x \in X | A(x) = x\} - \text{the fixed point set of } A;\]
\[I(A) := \{Y \in P(X) | A(Y) \subset Y\} - \text{the family of the nonempty invariant subsets of } A.\]

Definition 1. (I.A. Rus - [6], [7], [8]) Let \((X, d)\) be a metric space.
1) An operator \(A: X \to X\) is weakly Picard operator (briefly WPO) if the sequence
of successive approximations \((A^m(x_0))_{m \in \mathbb{N}}\) converges for all \(x_0 \in X\) and the limit
(which may depend on \(x_0\)) is a fixed point of \(A\).
2) If the operator \(A: X \to X\) is WPO and \(F_A = \{x^*\}\), then by definition the
operator \(A\) is Picard operator (briefly PO).
3) If the operator \(A: X \to X\) is WPO, then can be considered the operator \(A^\infty\)
defined by \(A^\infty: X \to X, \quad A^\infty(x) := \lim_{m \to \infty} A^m(x).\)
The basic result in the WPO’s theory is the following:

**Theorem 1.** (Characterization theorem - [6], [7], [8]) An operator \( A : X \to X \) is WPO if and only if there exists a partition of \( X \), \( X = \bigcup_{\lambda \in \Lambda} X_\lambda \), such that:

(a) \( X_\lambda \in I(A) \), \( \forall \lambda \in \Lambda \);
(b) \( A|_{X_\lambda} : X_\lambda \to X_\lambda \) is PO, \( \forall \lambda \in \Lambda \).

**Definition 2.** Let \((X,d)\) be a metric space and \( A : X \to X \) a WPO.

1) \( A : X \to X \) is good WPO, if the series \( \sum_{m=1}^{\infty} d(A^{m-1}(x),A^m(x)) \) converges, for all \( x \in X \) (see [8]). In the case that the sequence \( \{d(A^{m-1}(x),A^m(x))\}_{m \in \mathbb{N}^*} \) is strictly decreasing for all \( x \in X \), the operator \( A \) is good WPO of type \( M \) (see [3]).

2) \( A : X \to X \) is special WPO, if the series \( \sum_{m=1}^{\infty} d(A^m(x),A^\infty(x)) \) converges, for all \( x \in X \) (see [8]). When the sequence \( \{d(A^m(x),A^\infty(x))\}_{m \in \mathbb{N}^*} \) is strictly decreasing for all \( x \in X \), \( A \) is special WPO of type \( M \) (see [3]).

In 2015, S. Mureșan and L.F. Iambor obtained the following result regarding to good and special weakly Picard operators.

**Theorem 2.** ([5]) Let \((X,d)\) be a metric space and \( A : X \to X \) a WPO. If \( A \) is special WPO then \( A \) is good WPO.

In the paper [4], A.Bica and L.F. Galea(Iambor) introduced the notions of uniform good and special weakly Picard operators like this:

**Definition 3.** (A.Bica, L.F. Galea - [4]) Let \((X,d)\) be a metric space and \( F \subset \{ A | A : X \to X \} \) a family of operators on \( X \). We say that \( F \) is a family of uniform special (good) WPO’s if for any \( A \in F \), \( A \) is special (good) WPO and there exist the functionals \( \varphi : X \to \mathbb{R}_+ \) and \( \psi, \psi' : F \to \mathbb{R}_+ \) such that \( \varphi \) is continuous and

\[
\sum_{m=1}^{\infty} d(A^m(x),A^\infty(x)) \leq \psi(A) \cdot \varphi(x), \quad \forall x \in X, \quad \forall A \in F
\]

(respectively, \( \sum_{m=1}^{\infty} d(A^m(x),A^{m-1}(x)) \leq \psi'(A) \cdot \varphi(x), \quad \forall x \in X, \quad \forall A \in F \)).

In what follow, we present the general class of linear positive operators of discrete type and some properties of these operators investigated by O. Agratini and I.A. Rus in [1].

At first they construct an approximation process of discrete type acting on the space \( C([a,b]) \) endowed with the Chebyshev norm \( \| \cdot \| \).

For each integer \( n \geq 1 \) they consider the following:

(i) A net on \([a,b]\) named \( \Delta_n \) is fixed \((a = x_{n,0} < x_{n,1} < \ldots < x_{n,n} = b)\).

(ii) A system \( \{\psi_{n,k}\}_{k=0}^{n} \) is given, where every \( \psi_{n,k} \) belongs to \( C([a,b]) \).

They assume that it is a blending system with a certain connection with \( \Delta_n \), more precisely the following conditions hold:

\[
\psi_{n,k} \geq 0, \quad (k = 0, \ldots, n), \quad \sum_{k=0}^{n} \psi_{n,k} = e_0, \quad \sum_{k=0}^{n} x_{n,k} \psi_{n,k} = e_1
\]

**Definition 4.** (O.Agratini, I.A. Rus - [1]) The operators \( L_n : C([a,b]) \to C([a,b]) \) defined by

\[
L_n(f) (x) = \sum_{k=0}^{n} \psi_{n,k} (x) f(x_{n,k})
\]
are called the operators of discrete type.

The operators of discrete type $L_n$, have the following properties:
1) $L_n$, $n \in \mathbb{N}$ are positive linear operators;
2) $L_n (e_0) = e_0$ and $L_n (e_1) = e_1$.

**Theorem 3.** (O.Agratini, I.A. Rus - [1]) Let $L_n$, $n \in \mathbb{N}$, such that $\psi_{n,0} (a) = \psi_{n,n} (b) = 1$. Let us denote $u_n := \min_{x \in [a,b]} [\Phi_{n,0} (x) + \Phi_{n,n} (x)]$.

If the $u_n > 0$ the iterates sequence $(L_n^n)_{m \geq 1}$ verifies
\[
\lim_{m \to \infty} (L_n^n f) (x) = f (a) + \frac{f(a) - f(b)}{b-a} (x - a), \ f \in C ([a,b])
\]
uniformly on $[a,b]$.

**Theorem 4.** (O.Agratini, I.A. Rus - [1]) Let $L_n$, $n \in \mathbb{N}$, such that $\psi_{n,0} (a) = \psi_{n,n} (b) = 1$. Then the operator $L_n$ is weakly Picard operator for every $n \in \mathbb{N}$ and
\[
L_n^n (f) = c_1 (f) e_1 + c_2 (f) \frac{f(a) - f(b)}{b-a} \ f \in C ([a,b])
\]
where $c_1 (f) = \frac{L_n^n (b) - f(a)}{b-a}$ and $c_2 (f) = \frac{f(a) - f(b)}{b-a}$.

The convergence exists on the space $(C [a,b], \| \cdot \|_\infty)$.

In the application of Characterization theorem of weakly Picard operator, it was considerate the partition of $C ([a,b])$
\[
C ([a,b]) := \bigcup_{\alpha, \beta \in \mathbb{R}} X_{\alpha, \beta}
\]
where $X_{\alpha, \beta} = \{ f \in C ([a,b]) : f (a) = \alpha, f (b) = \beta \}$, $\alpha, \beta \in \mathbb{R}$.

**Proposition 5.** (O. Agratini, I.A. Rus - [1]) The operators of discrete type satisfied the following contraction property relative to above partition:
\[
\| L_n (f) - L_n (g) \|_\infty \leq (1 - u_n) \| f - g \|_\infty, \ \forall f, g \in X_{\alpha, \beta}, \ \alpha, \beta \in \mathbb{R}
\]
where $u_n = \min_{x \in [a,b]} [\Phi_{n,0} (x) - \Phi_{n,n} (x)]$, $u_n > 0$.

2. Main results

In this section, we will investigate some properties of the iterates of discrete type of operators in sense of good and special convergence.

**Theorem 6.** The operators of discrete type $L_n$, $n \in \mathbb{N}$ are special WPO and good WPO of type M on $C ([a,b])$.

From Theorem 3, we have that $L_n$, $n \in \mathbb{N}$ is weakly Picard operator.

Let $f \in C ([a,b])$. Then $f \in X_{f(a),f(b)}$ and according to (1) we infer that $L_n$ is contraction on $X_{f(a),f(b)}$. So, the operator $L_n$, $n \in \mathbb{N}$ is special WPO of type M on $X_{f(a),f(b)}$. Finally, we get that $L_n$, $n \in \mathbb{N}$ is special WPO of type M on $C ([a,b])$.

From Theorem 2, any special WPO is good WPO. Then we have that $L_n$, $n \in \mathbb{N}$ is good WPO of type M on $C ([a,b])$.

**Theorem 7.** The family of the operators of discrete type $\{ L_n : n \in \mathbb{N}^* \}$ is family of uniform special and good WPO's on $C [a,b]$.

**Proof.** Using the inequality (1), we obtain the estimation:
\[
| L_n^1 (f) (x) - L_n^\infty (f) (x) | = | L_n^1 (f) (x) - L_n^1 (L_n^\infty (f)) (x) | \leq
\leq (1 - u_n) | f (x) - L_n^\infty (f) (x) | = (1 - u_n) | f (x) - c_1 (f) e_1 - c_2 (f) | \leq
\]
and (3). For instance, for the property of uniform special WPO we have:

$$\forall x \in [a, b],$$

where $C = \text{diam} (\text{Im } f) + 2 \max \{|f(a)|, |f(b)|\}$, with $\text{diam} (\text{Im } f) = \max \{|f(x) - f(y)| : x, y \in [a, b]\}$.

The constant $C$ was obtained using the following technique:

- If $x = a$ then:
  $$|f(x) - L_n^\infty (f) (x)| \leq |f(x) - \frac{f(b) - f(a)}{b-a} \cdot a - \frac{b(a) - a f(b)}{b-a}| =$$
  $$= |f(x) - \frac{f(a) - f(a)}{b-a}| = |f(x) - f(a)| \leq \text{diam} (\text{Im } f)$$

- If $x = b$ then:
  $$|f(x) - L_n^\infty (f) (x)| \leq |f(x) - \frac{f(b) - f(a)}{b-a} \cdot b - \frac{b(a) - a f(b)}{b-a}| =$$
  $$= |f(x) - \frac{f(b) - f(b)}{b-a}| = |f(x) - f(b)| \leq \text{diam} (\text{Im } f)$$

- If $x \in [a, b]$ then:
  $$|f(x) - L_n^\infty (f) (x)| \leq |f(x) - \frac{f(b) - f(a)}{b-a} \cdot x - \frac{f(a) - f(b)}{b-a}| =$$
  $$= |f(x) - \frac{x-a}{b-a} \cdot f(b) + \frac{x-b}{b-a} \cdot f(a)| \leq$$
  $$\leq |f(x) - f(b)| + |f(b)| \cdot |1 - \frac{x-a}{b-a}| + |\frac{x-b}{b-a}| \cdot |f(a)| \leq$$
  $$\leq \text{diam} (\text{Im } f) + 2 \max \{|f(a)|, |f(b)|\}.$$

By induction, for $m \in \mathbb{N}^*$, we have:

$$|L_n^m (f) (x) - L_n^\infty (f) (x)| \leq (1 - u_n)^m \cdot C, \forall x \in [a, b].$$

Then, $\sum_{m=1}^{\infty} |L_n^m (f) (x) - L_n^\infty (f) (x)| \leq$

$$\leq \lim_{m \to \infty} \left(1 - u_n\right)^m + \ldots + (1 - u_n)^m \right] =$$

$$= \lim_{m \to \infty} \left[ (1 - u_n) \cdot \frac{1 - (1 - u_n)^m}{1 - u_n} \right] \leq C \cdot \frac{1 - u_n}{u_n} \quad (2)$$

On the other hand, we have:

$$|L_n^1 (f) (x) - L_n^m (f) (x)| = \left| \sum_{n=0}^{n} \psi_{n,k} (x)f(x_{n,k}) - f(x) \right| =$$

$$= \left| \sum_{n=0}^{n} \psi_{n,k} (x)|f(x_{n,k}) - f(x)| \right| \leq C' \sum_{n=0}^{n} \psi_{n,k} (x) = C' e_0, \forall x \in [a, b],$$

where $C' = \text{diam} (\text{Im } f) = \max \{|f(x) - f(y)| : x, y \in [a, b]\}$.

By induction, for $m \in \mathbb{N}$, we have:

$$|L_n^m (f) (x) - L_n^{m-1} (f) (x)| = |L_n^1 (L_n^{m-1} (f)) (x) - L_n^1 (L_n^{m-2} (f)) (x)| \leq$$

$$\leq (1 - u_n)^{m-1} \cdot C e_0, \forall x \in [a, b]$$

Then, $\sum_{m=1}^{\infty} |L_n^m (f) (x) - L_n^{m-1} (f) (x)| \leq$

$$\leq \lim_{m \to \infty} C e_0 \left(1 + (1 - u_n)^2 + \ldots + (1 - u_n)^{m-1}\right) \leq$$

$$\leq C' e_0 \frac{1}{1 - u_n}, \forall f \in C [a, b] \quad (3).$$

Now, the property of uniform and special WPO follows from the estimations (2) and (3). For instance, for the property of uniform special WPO we have:

$$\varphi : C [a, b] \to \mathbb{R}_+,$$

$$\varphi (f) = \text{diam} (\text{Im } f) + 2 \max \{|f(a)|, |f(b)|\}$$

and for the property of uniform good WPO we have:
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\[ \varphi': C[a, b] \rightarrow \mathbb{R}_+, \varphi'(f) = \text{diam} (\text{Im} f) \]

and

\[ \psi': \{L_n : n \in \mathbb{N}^*\} \rightarrow \mathbb{R}_+, \psi'(L_n) = \frac{1}{1 - \beta^m} e_0, \ \forall \ n \in \mathbb{N}^*. \]

It is easy to prove that \( \varphi, \varphi': C[a, b] \rightarrow \mathbb{R}_+, \varphi(f) = \text{diam} (\text{Im} f) + 2 \max \{|f(a)|, |f(b)|\} \)
and \( \varphi'(f) = \text{diam} (\text{Im} f) \) are seminorms on \( C[a, b] \) and

\[ \varphi(f - g) \leq 2 \|f - g\|_C + 2 \|f\|_C, \varphi'(f - g) \leq 2 \|f - g\|_C \]

since \( |\varphi(f) - \varphi(g)| \leq \varphi(f - g), \ \forall \ f, g \in C[a, b] \) and

\[ |\varphi'(f) - \varphi'(g)| \leq \varphi'(f - g), \ \forall \ f, g \in C[a, b] \]

we infer the \( \varphi, \varphi' \) are continuous.

References


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General Iyengar type Inequalities

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Abstract

Here we present general Iyengar type inequalities with respect to $L_p$ norms, with $1 \leq p \leq \infty$. The method is based on the generalized Taylor’s formula.

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Key Words and Phrases: Iyengar inequality, Taylor formula.

1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [2].

**Theorem 1** Let $f$ be a differentiable function on $[a, b]$ and $|f'(x)| \leq M$. Then

$$ \left| \int_a^b f(x) \, dx - \frac{1}{2} (b - a) (f(a) + f(b)) \right| \leq \frac{M (b - a)^2}{4} - \frac{(f(b) - f(a))^2}{4M}. \quad (1) $$

2 Main Results

We present the following Iyengar type inequalities:

**Theorem 2** Let $n \in \mathbb{N}$, $f \in AC^n ([a, b])$ (i.e. $f^{(n-1)} \in AC ([a, b])$, absolutely continuous functions). We assume that $f^{(n)} \in L_\infty ([a, b])$. Then

$$ \left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k)}(a) (t - a)^{k+1} + (-1)^k f^{(k)}(b) (b - t)^{k+1} \right] \right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!} \left[ (t - a)^{n+1} + (b - t)^{n+1} \right], \quad (2) $$

$\forall \, t \in [a, b],$
ii) at $t = \frac{a+b}{2}$, the right hand side of (2) is minimized, and we get:

$$\left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n},$$

(3)

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n - 1$, we obtain

$$\left| \int_a^b f(x) \, dx \right| \leq \frac{\|f^{(n)}\|_{\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n},$$

(4)

which is a sharp inequality.

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( \frac{b-a}{N} \right)^{k+1} \left[ j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n} \left( \frac{N}{b-a} \right)^n,$$

(5)

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n - 1$, from (5) we get:

$$\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{N} \right) \left[ j f(a) + (N-j) f(b) \right] \right| \leq \frac{\|f^{(n)}\|_{\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n} \left( \frac{N}{b-a} \right)^n \left[ j^{n+1} + (N-j)^{n+1} \right],$$

(6)

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

vi) when $N = 2$ and $j = 1$, (6) turns to

$$\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f^{(n)}\|_{\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n},$$

(7)

vii) when $n = 1$ (without any boundary conditions), we get from (7) that

$$\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f'\|_{\infty([a,b])}}{4} \frac{(b-a)^2}{4},$$

(8)

a similar to Iyengar inequality (1).
Proof. Here \( n \in \mathbb{N} \) and \( f^{(n-1)} \) is absolutely continuous on \([a, b]\). We assumed that
\[
\|f^{(n)}\|_{\infty, [a, b]} := \|f^{(n)}\|_{L^\infty([a, b])} < +\infty.
\]
By [1], we have the following generalized Taylor’s formulae:
\[
f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t) \, dt \quad (9)
\]
and
\[
f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t) \, dt, \quad (10)
\]
\( \forall \ x \in [a, b] \).
Then we get
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\|f^{(n)}\|_{\infty, [a, b]}}{n!} (x-a)^n, \quad (11)
\]
\( \forall \ x \in [a, b] \),
and
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| = \frac{1}{(n-1)!} \left| \int_a^b (x-t)^{n-1} f^{(n)}(t) \, dt \right| =
\]
\[
\frac{1}{(n-1)!} \left| \int_x^b (t-x)^{n-1} f^{(n)}(t) \, dt \right| \leq \frac{1}{(n-1)!} \left| \int_x^b (t-x)^{n-1} f^{(n)}(t) \, dt \right| \leq \frac{\|f^{(n)}\|_{\infty, [a, b]}}{n!} (b-x)^n,
\]
that is
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\|f^{(n)}\|_{\infty, [a, b]}}{n!} (b-x)^n, \quad (12)
\]
\( \forall \ x \in [a, b] \).
We call
\[
\delta := \frac{\|f^{(n)}\|_{\infty, [a, b]}}{n!}.
\]
So we have
\[
-\delta (x-a)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \leq \delta (x-a)^n \quad (14)
\]
\[ -\delta (b - x)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x - b)^k \leq \delta (b - x)^n, \quad (15) \]

\forall x \in [a, b].

Therefore it holds
\[ \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x - a)^k - \delta (x - a)^n \leq f(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x - a)^k + \delta (x - a)^n \quad (16) \]

and
\[ \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x - b)^k - \delta (b - x)^n \leq f(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x - b)^k + \delta (b - x)^n, \quad (17) \]

\forall x \in [a, b].

Let any \( t \in [a, b] \), then
\[ \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k + 1)!} (t - a)^{k+1} - \frac{\delta}{(n + 1)} (t - a)^{n+1} \leq \int_a^t f(x) \, dx \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k + 1)!} (t - a)^{k+1} + \frac{\delta}{(n + 1)} (t - a)^{n+1}, \quad (18) \]

and
\[ \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k + 1)!} (t - b)^{k+1} - \frac{\delta}{(n + 1)} (b - t)^{n+1} \leq \int_t^b f(x) \, dx \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k + 1)!} (t - b)^{k+1} + \frac{\delta}{(n + 1)} (b - t)^{n+1}. \quad (19) \]

Adding (18) and (19), we obtain:
\[
\sum_{k=0}^{n-1} \frac{1}{(k + 1)!} \left[ f^{(k)}(a) (t - a)^{k+1} - f^{(k)}(b) (t - b)^{k+1} \right] - \frac{\delta}{(n + 1)} \left[ (t - a)^{n+1} + (b - t)^{n+1} \right] \leq \int_a^b f(x) \, dx \leq \sum_{k=0}^{n-1} \frac{1}{(k + 1)!} \left[ f^{(k)}(a) (t - a)^{k+1} - f^{(k)}(b) (t - b)^{k+1} \right] + \frac{\delta}{(n + 1)} \left[ (t - a)^{n+1} + (b - t)^{n+1} \right], \quad (20) \]

\forall t \in [a, b].
We notice that
\[
\int_{a}^{b} f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \leq \frac{\delta}{(n+1)} \left[ (t-a)^{n+1} + (b-t)^{n+1} \right],
\]
\[\forall \, t \in [a, b].\]

Hence
\[g(t) := (t-a)^{n+1} + (b-t)^{n+1}, \quad \forall \, t \in [a, b].\]

Let us consider
\[g'(t) = (n+1) [(t-a)^n - (b-t)^n] = 0,
\]
giving
\[(t-a)^n = (b-t)^n\]
and
\[t = a + \frac{b-t}{2},\]
that is
\[t = \frac{a+b}{2}\]
the only critical number here.

We have
\[g(a) = g(b) = (b-a)^{n+1},\]
and
\[g \left( \frac{a+b}{2} \right) = \frac{(b-a)^{n+1}}{2^n},\]
which is the minimum of \(g\) over \([a, b]\).

Consequently the right hand side of (21) is minimized when
\[t = \frac{a+b}{2},\]
with value
\[\frac{\|f^{(n)}\|_{\infty, [a,b]} (b-a)^{n+1}}{(n+1)!} \frac{2^n}{2^n}.\]

Assuming
\[f^{(k)}(a) = f^{(k)}(b) = 0,\]
for \(k = 0, 1, \ldots, n-1\), then we obtain that
\[\int_{a}^{b} f(x) \, dx \leq \frac{\|f^{(n)}\|_{\infty, [a,b]} (b-a)^{n+1}}{(n+1)!} \frac{2^n}{2^n},\]
which is a sharp inequality.

When
\[t = \frac{a+b}{2},\]
then (21) becomes
\[\int_{a}^{b} f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \leq \frac{\|f^{(n)}\|_{\infty, [a,b]} (b-a)^{n+1}}{(n+1)!} \frac{2^n}{2^n}.\]

Next let
\[N \in \mathbb{N}, j = 0, 1, 2, \ldots, N\]
and
\[t_j = a + j \left( \frac{b-a}{N} \right),\]
that is \(t_0 = a, t_1 = a + \frac{b-a}{N}, \ldots, t_N = b\).

Hence it holds
\[t_j - a = j \left( \frac{b-a}{N} \right), \quad (b-t_j) = (N-j) \left( \frac{b-a}{N} \right), \quad j = 0, 1, 2, \ldots, N.\]

We notice that
\[(t_j - a)^{n+1} + (b-t_j)^{n+1} = \left( \frac{b-a}{N} \right)^{n+1} \left[ j^{n+1} + (N-j)^{n+1} \right],\]
\[ j = 0, 1, 2, \ldots, N, \]
\[ \text{and} \quad (k = 0, 1, \ldots, n-1) \]
\[ \left[ f^{(k)}(a) (t_j - a)^{k+1} + (-1)^k f^{(k)}(b) (b - t_j)^{k+1} \right] = \]
\[ \left[ f^{(k)}(a) j^{k+1} \left( \frac{b-a}{N} \right)^{k+1} + (-1)^k f^{(k)}(b) (N - j)^{k+1} \left( \frac{b-a}{N} \right)^{k+1} \right] = \quad (27) \]
\[ \left( \frac{b-a}{N} \right)^{k+1} \left[ f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b) (N - j)^{k+1} \right], \]
\[ j = 0, 1, 2, \ldots, N. \]

By (21) we get
\[ \int_a^b f(x) dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( \frac{b-a}{N} \right)^{k+1} \left[ f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b) (N - j)^{k+1} \right] \leq \]
\[ \| f^{(n)} \|_{\infty, [a,b]} \left( \frac{b-a}{N} \right)^{n+1} \left[ j^{n+1} + (N - j)^{n+1} \right], \quad (28) \]
\[ j = 0, 1, 2, \ldots, N. \]

If \( f^{(k)}(a) = f^{(k)}(b) = 0, k = 1, \ldots, n-1 \), then (28) becomes
\[ \left| \int_a^b f(x) dx - \left( \frac{b-a}{N} \right)^{n+1} \left[ j f(a) + (N - j) f(b) \right] \right| \leq \]
\[ \| f^{(n)} \|_{\infty, [a,b]} \left( \frac{b-a}{N} \right)^{n+1} \left[ j^{n+1} + (N - j)^{n+1} \right], \quad (29) \]
for \( j = 0, 1, 2, \ldots, N. \)

When \( N = 2 \) and \( j = 1 \), then (29) becomes
\[ \left| \int_a^b f(x) dx - \left( \frac{b-a}{2} \right) [f(a) + f(b)] \right| \leq \]
\[ \| f^{(n)} \|_{\infty, [a,b]} \left( \frac{b-a}{2} \right)^{n+1} 2 = \| f^{(n)} \|_{\infty, [a,b]} \left( \frac{b-a}{2} \right)^{n+1} \frac{2^n}{(n+1)!}. \quad (30) \]

And, if \( n = 1 \) (without any boundary conditions), we get from (30) that
\[ \left| \int_a^b f(x) dx - \left( \frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \| f' \|_{\infty, [a,b]} \frac{(b-a)^2}{4}, \quad (31) \]
which a similar inequality to Iyengar inequality (1). \( \blacksquare \)

We give
Theorem 3 Let \( f \in AC^n ([a, b]), n \in \mathbb{N} \). Then

i) \[
\left| \int_a^b f(x) \, dx - \frac{1}{n!} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ \frac{f^{(k)}(a)}{N} \right]^{k+1} \left[ j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} [(t-a)^n + (b-t)^n],
\]
\( \forall t \in [a,b] \),

ii) at \( t = \frac{a+b}{2} \), the right hand side of (32) is minimized, and we get:

\[
\left| \int_a^b f(x) \, dx - \frac{1}{n!} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ \frac{b-a}{2^{k+1}} \right] \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}},
\]

iii) if \( f^{(k)}(a) = f^{(k)}(b) = 0 \), for all \( k = 0, 1, ..., n-1 \), we obtain

\[
\left| \int_a^b f(x) \, dx \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}},
\]

which is a sharp inequality,

iv) more generally, for \( j = 0, 1, 2, ..., N \in \mathbb{N} \), it holds

\[
\left| \int_a^b f(x) \, dx - \frac{1}{n!} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ \frac{b-a}{N} \right]^{k+1} \left[ j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \left( \frac{b-a}{N} \right)^n \left[ j^n + (N-j)^n \right],
\]

v) if \( f^{(k)}(a) = f^{(k)}(b) = 0 \), \( k = 1, ..., n-1 \), from (35) we get:

\[
\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{N} \right) j f(a) + (N-j) f(b) \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \left( \frac{b-a}{N} \right)^n \left[ j^n + (N-j)^n \right],
\]

vi) when \( N = 2 \) and \( j = 1 \), (36) turns to

\[
\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{2} \right) \left( f(a) + f(b) \right) \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \left( \frac{b-a}{2} \right)^n \left[ j^n + (N-j)^n \right].
\]
\[
\frac{\|f^{(n)}\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}}, \quad (37)
\]

vii) when \( n = 1 \) (without any boundary conditions), we get from (37) that
\[
\left| \int_a^b f(x)\, dx - \left( \frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \|f'\|_{L_1([a,b])} (b-a). \quad (38)
\]

**Proof.** Here \( n \in \mathbb{N} \) and \( f^{(n-1)} \) is absolutely continuous on \([a,b]\). Hence \( f^{(n)} \) exists almost everywhere and \( f^{(n)} \in L_1([a,b]) \). By (9) we get
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| = \frac{1}{(n-1)!} \left| \int_a^x (x-t)^{n-1} f^{(n)}(t) \, dt \right| \leq \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \left| f^{(n)}(t) \right| \, dt
\]
\[
= \frac{(x-a)^{n-1}}{(n-1)!} \left\| f^{(n)} \right\|_{L_1([a,b])}.
\]
That is
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{(n-1)!} (x-a)^{n-1}, \quad (40)
\]
\( \forall x \in [a,b]. \)

By (10) we get
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| = \frac{1}{(n-1)!} \left| \int_b^x (t-x)^{n-1} f^{(n)}(t) \, dt \right|
\]
\[
= \frac{1}{(n-1)!} \int_b^x (t-x)^{n-1} \left| f^{(n)}(t) \right| \, dt \leq \frac{1}{(n-1)!} \int_b^x (t-x)^{n-1} \left| f^{(n)}(t) \right| \, dt 
\]
\[
= \frac{(b-x)^{n-1}}{(n-1)!} \int_a^b \left| f^{(n)}(t) \right| \, dt = \frac{(b-x)^{n-1}}{(n-1)!} \left\| f^{(n)} \right\|_{L_1([a,b])}.
\]
That is
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{(n-1)!} (b-x)^{n-1}, \quad (42)
\]
\( \forall x \in [a,b]. \)

Set
\[
\rho := \frac{\|f^{(n)}\|_{L_1([a,b])}}{(n-1)!}.
\]
Hence
\[ f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \leq \rho(x-a)^{n-1}, \tag{43} \]
and
\[ f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \leq \rho(b-x)^{n-1}, \tag{44} \]
\[ \forall x \in [a, b]. \]

As in the proof of Theorem 2 we get:
\[ \left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \]
\[ \frac{\rho}{n} \left[ (t-a)^n + (b-t)^n \right], \tag{45} \]
\[ \forall t \in [a, b]. \]

The rest of the proof is similar to the proof of Theorem 2. \( \blacksquare \)

We continue with

**Theorem 4** Let \( f \in AC^n([a, b]), \, n \in \mathbb{N}; \, p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \) and \( f^{(n)} \in L_q([a, b]). \) Then
\[ i) \]
\[ \left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k)}(a) (t-a)^{k+1} + (-1)^k f^{(k)}(b) (b-t)^{k+1} \right] \right| \leq \]
\[ \frac{\|f^{(n)}\|_{L_q([a, b])}}{(n-1)! \left( n + \frac{1}{p} \right) \left( p(n-1) + 1 \right)^\frac{1}{p}} \left[ (t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right], \tag{46} \]
\[ \forall t \in [a, b], \]
\[ ii) \text{at } t = \frac{a+b}{2}, \text{ the right hand side of (46) is minimized, and we get:} \]
\[ \left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq \]
\[ \frac{\|f^{(n)}\|_{L_q([a, b])}}{(n-1)! \left( n + \frac{1}{p} \right) \left( p(n-1) + 1 \right)^\frac{1}{p}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n+\frac{1}{2}}}, \tag{47} \]
\[ iii) \text{if } f^{(k)}(a) = f^{(k)}(b) = 0, \text{ for all } k = 0, 1, ..., n-1, \text{ we obtain} \]
\[ \left| \int_a^b f(x) \, dx \right| \leq \frac{\|f^{(n)}\|_{L_q([a, b])}}{(n-1)! \left( n + \frac{1}{p} \right) \left( p(n-1) + 1 \right)^\frac{1}{p}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n+\frac{1}{2}}}, \tag{48} \]
which is a sharp inequality.

iv) more generally, for \( j = 0, 1, 2, \ldots, N \in \mathbb{N} \), it holds

\[
\left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( \frac{b-a}{N} \right)^{k+1} \left[ j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left( n + \frac{1}{p} \right) \left( p(n-1) + 1 \right)^{\frac{1}{p}}} \left( \frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[ j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right],
\]

(49)

v) if \( f^{(k)}(a) = f^{(k)}(b) = 0, k = 1, \ldots, n-1 \), from (49) we get:

\[
\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{N} \right) [jf(a) + (N-j)f(b)] \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left( n + \frac{1}{p} \right) \left( p(n-1) + 1 \right)^{\frac{1}{p}}} \left( \frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[ j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right],
\]

(50)

for \( j = 0, 1, 2, \ldots, N \in \mathbb{N} \).

vi) when \( N = 2 \) and \( j = 1 \), (50) turns to

\[
\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! \left( n + \frac{1}{p} \right) \left( p(n-1) + 1 \right)^{\frac{1}{p}}} \left( \frac{b-a}{2} \right)^{n+\frac{1}{p}} \frac{2^{n+\frac{1}{p}}}{2^{n+\frac{1}{p}}},
\]

(51)

vii) when \( n = 1 \) (without any boundary conditions), we get from (51) that

\[
\left| \int_a^b f(x) \, dx - \left( \frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(1 + \frac{1}{p})} \left( \frac{b-a}{2} \right)^{1+\frac{1}{p}}.
\]

(52)

**Proof.** Here \( f^{(n)} \in L_q([a,b]) \), where \( p, q > 1 \), such that \( \frac{1}{p} + \frac{1}{q} = 1 \). By (9) we get

\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| = \frac{1}{(n-1)!} \left| \int_a^x (x-t)^{n-1} f^{(n)}(t) \, dt \right| \leq \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} |f^{(n)}(t)| \, dt \leq \frac{1}{(n-1)!} \left( \int_a^x (x-t)^{p(n-1)} \, dt \right)^{\frac{1}{p}} \left( \int_a^x |f^{(n)}(t)|^q \, dt \right)^{\frac{1}{q}} \leq
\]

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\[
\frac{(x-a)^{p(n-1)+1}}{(n-1)! (p(n-1)+1)^\frac{1}{p}} \| f^{(n)} \|_{L_q([a,b])}.
\] (53)

That is
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\| f^{(n)} \|_{L_q([a,b])}}{(n-1)! (p(n-1)+1)^\frac{1}{p}} \frac{1}{(x-a)^{n-\frac{1}{q}}}, \quad (54)
\]
\forall x \in [a,b].

By (10) we get
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| = \frac{1}{(n-1)!} \left| \int_x^b (t-x)^{n-1} f^{(n)}(t) \, dt \right| \leq \frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} \left| f^{(n)}(t) \right| \, dt \leq \frac{1}{(n-1)!} \left( \int_x^b (t-x)^{p(n-1)} \, dt \right)^\frac{1}{p} \left( \int_x^b \left| f^{(n)}(t) \right|^q \, dt \right)^\frac{1}{q} \leq \frac{(b-x)^{p(n-1)+1}}{(n-1)! (p(n-1)+1)^\frac{1}{p}} \| f^{(n)} \|_{L_q([a,b])}. \] (55)

That is
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\| f^{(n)} \|_{L_q([a,b])}}{(n-1)! (p(n-1)+1)^\frac{1}{p}} \frac{1}{(b-x)^{n-\frac{1}{q}}}, \quad (56)
\]
\forall x \in [a,b].

Set
\[
\gamma := \frac{\| f^{(n)} \|_{L_q([a,b])}}{(n-1)! (p(n-1)+1)^\frac{1}{p}}, \quad (57)
\]
and
\[
m := n - \frac{1}{q} > 0. \quad (58)
\]

So, we can write
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \gamma (x-a)^m, \quad (59)
\]
and
\[
\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \gamma (b-x)^m, \quad (60)
\]
\forall x \in [a,b].
As in the proof of Theorem 2 we obtain:

\[
\left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq \gamma \frac{(m+1)}{(n+1)!} \left[ (t-a)^{m+1} + (b-t)^{m+1} \right] = (61)
\]

\[
\left\| f^{(n)} \right\|_{L_a(a,b)} \frac{1}{(n+1)! (p(n-1)+1)^{\frac{1}{p}}} \left[ (t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right], \quad (62)
\]

\( \forall t \in [a,b]. \)

The rest of the proof is similar to the proof of Theorem 2. ■

References


A NOTE ON THE APPROXIMATE SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

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Abstract. By using the Caratheodory approximation method, the current article presents the analysis of exact and approximate solutions for stochastic differential equations (SDEs) in the framework of G-Brownian motion. In view of the non-linear growth and non-Lipschitz conditions, the boundedness of the Caratheodory approximate solutions $Y_q(t)$, $q \geq 1$ in the space $M^G_2([t_0, T]; \mathbb{R}^n)$ has been determined. Estimate for the difference between the exact solution $Y(t)$ and the Caratheodory approximate solutions $Y_q(t)$ has been derived.

Keywords: G-Brownian motion, non-linear growth and non-Lipschitz conditions, Caratheodory approximation procedure, bounded solutions, stochastic differential equations

MSC: 60H20, 60H10, 60H35, 62L20.

1. Introduction

Stochastic differential equations (SDEs) are employed by several and diverse scientific disciplines such as chemistry, statistical physics, biology and engineering. In finance and economics, they are utilized to find out the risk measures and stochastic volatility problems. SDEs describe heavy traffic behavior of communication networks and control systems [16]. Mathematics use the concept of SDEs to incorporate random fluctuations in the model when one investigates the evolution of the number of cells in an organism infected by a virus. The weather and climate can be modeled by these equations. The clarification of fluid through porous structures and water catchment can be modeled by SDEs [17]. They are used to describe the motion of wildlife [4]. SDEs play an important role to study the animal’s swarm, such as schooling of fish, flocking of birds or herding of mammals, to find resource of food in noisy and obstacle environment [30]. In physics, SDEs are used to study and model the effect of random variations on distinct physical processes. A large literature is available on the applications of SDEs in numerous fields of engineering such as computer engineering [16, 22], mechanical engineering [26, 28, 29], random vibrations [3, 24], stability theory [25] and wave processes [27]. In general, one can not find the explicit solutions for non-linear SDEs, so we have to present and study the analysis for the solutions of these equations. Moreover, the developments of computational techniques are very important for solving several demanding problems, for instance to find the optimal construction of a design and to determine input data from fundamental principles. Therefore it is valuable to know computational accuracy, which leads us to convergence results and estimates for the difference between exact and approximate solutions.

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The aim of the current article is to investigate estimates for the difference between exact and approximate solutions for SDEs driven by G-Brownian motion with Carathéodory approximation procedure. In view of growth and Lipschitz conditions, the existence-uniqueness results for G-SDEs was studied by Peng [20, 21] and Gao [15]. Later, Bai and Lin [1] established the existence theory for G-SDEs with integral Lipschitz coefficients. Subject to some discontinuous coefficients, the said theory was generalized by Faizullah [11]. Let \( 0 \leq t_0 \leq t \leq T < \infty \). Consider the following SDE in the framework of G-Brownian motion

\[
dY(t) = \kappa(t, Y(t))dt + \lambda(t, Y(t))d\langle W, W \rangle(t) + \mu(t, Y(t))dW(t),
\]

with initial value \( Y(t_0) \in \mathbb{R}^n \). The given coefficients \( g(., x), h(., x) \) and \( w(., x) \) belong to space \( M^2_G([t_0, T]; \mathbb{R}^n) \), for all \( x \in \mathbb{R}^n \). SDE (1.1) in the integral form is expressed as the following

\[
Y(t) = Y(t_0) + \int_{t_0}^{t} \kappa(s, Y(s))ds + \int_{t_0}^{t} \lambda(s, Y(s))d\langle W, W \rangle(s) + \int_{t_0}^{t} \mu(s, Y(s))dW(s),
\]

on \( t \in [t_0, T] \). Its solution is a process \( Y \in M^2_G([t_0, T]; \mathbb{R}^n) \) and satisfying SDE (1.2). The rest of the current paper contains three sections. Building on the previous notions of G-expectation, section 2 presents the fundamental definitions and results of G-Brownian motion, sub-expectation, Growthwall's inequality, Doobs martingale inequality, G-Itô's integral and Hölder’s inequality etc. Section 3 reveals the Carathéodory approximate solutions procedure for SDEs driven by G-Brownian motion. This section give an important result, which shows that the Carathéodory approximate solutions are bounded. Section 4 derives estimates for the difference between approximate and exact solutions to SDEs driven by G-Brownian motion.

2. Preliminaries

We present some basic results and notions required for the subsequent sections of the current article. We don’t give detailed literature on basic notions of G-expectation, so readers are suggested to consult the more depth oriented papers [9, 13, 18, 20, 21]. Let \( \Omega \) be a given basic non-empty set. Assume \( \mathcal{H} \) be a space of linear real functions defined on \( \Omega \) so that (i) \( 1 \in \mathcal{H} \) (ii) for every \( n \geq 1, Y_1, Y_2, ..., Y_n \in \mathcal{H} \) and \( \varphi \in C_{b, Lp} (\mathbb{R}^n) \) it satisfies \( \varphi(Y_1, Y_2, ..., Y_n) \in \mathcal{H} \) i.e., subject to Lipschitz bounded functions, \( \mathcal{H} \) is stable. Then \( (\Omega, \mathcal{H}, E) \) is a sub-expectation space, where \( E \) is a sub-expectation defined as follows.

**Definition 2.1.** A functional \( E : \mathcal{H} \rightarrow \mathbb{R} \) satisfying the below four features is known as a sub-expectation. Let \( X, Y \in \mathcal{H} \), then

1. **Monotonicity:** \( E(X) \leq E(Y) \) if \( X \leq Y \).
2. **Constant preservation:** \( E(M_1) = M_1 \), for all \( M_1 \in \mathbb{R} \).
3. **Positive homogeneity:** \( E(N_1 Y) = N_1 E(Y) \), for all \( N_1 \in \mathbb{R}^+ \).
4. **Sub-additivity:** \( E(X) + E(Y) \geq E(X + Y) \).
Moreover, let \( \Omega \) be the space of all \( \mathbb{R}^n \)-valued continuous paths \((w_t)_{t \geq 0}\) starting from zero. In addition, assume that subject to the below distance, \( \Omega \) is a metric space

\[
\rho(w^1, w^2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \max_{t \in [0,k]} |w^1_t - w^2_t| \wedge 1.
\]

Fix \( T \geq 0 \) and set

\[
L_{ip}^0(\Omega_T) = \{ \phi(W_{t_1}, W_{t_2}, ..., W_{t_m}) : m \geq 1, t_1, t_2, ..., t_m \in [0, T], \phi \in C_b, Lip(\mathbb{R}^{m \times n}) \},
\]

where \( W \) is the canonical process, \( L_{ip}^0(\Omega_t) \subseteq L_{ip}^0(\Omega_T) \) for \( t \leq T \) and \( L_{ip}^0(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}^0(\Omega_n) \). The completion of \( L_{ip}^0(\Omega) \) under the Banach norm \( E[|\cdot|^p]^{1/p} \), \( p \geq 1 \) is denoted by \( L_p^G(\Omega) \), where \( L_p^G(\Omega_t) \subseteq L_p^G(\Omega_T) \subseteq L_p^G(\Omega) \) for \( 0 \leq t \leq T < \infty \). Generated by the canonical process \( \{W(t)\}_{t \geq 0} \), the filtration is represented as \( \mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\} \), \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \). Suppose \( \pi_T = \{t_0, t_1, ..., t_N\} \), \( 0 \leq t_0 \leq t_1 \leq ... \leq t_N \leq \infty \) be a division of \([0, T]\). For \( p \geq 1 \), \( M^p_G(0, T) \) denotes a set of the processes given by

\[
\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w)I_{[t_i, t_{i+1}]}(t),
\]

where \( \xi_i \in L_p^G(\Omega_{t_i}) \), \( i = 0, 1, ..., N - 1 \). Furthermore, the completion of \( M^p_G(0, T) \) with the below given norm is indicated by \( M^p_G(0, T) \), \( p \geq 1 \)

\[
||\eta|| = \left\{ \int_0^T E[|\eta_s|^p]ds \right\}^{1/p}.
\]

**Definition 2.2.** An \( n \)-dimensional stochastic process \( \{W(t)\}_{t \geq 0} \) is called a G-Brownian motion if

1. \( W(0) = 0 \).
2. For any \( t, m \geq 0 \), \( W_{t+m} - W_t \) is G-normally distributed and independent from \( W_{t_1}, W_{t_2}, ..., W_{t_n} \), for \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq t_2 \leq ..., \leq t_n \leq t \),

**Definition 2.3.** Let \( \eta_t \in M^2_G(0, T) \) having the form (2.1). Then the G-quadratic variation process \( \{\langle W \rangle_t\}_{t \geq 0} \) and G-Itô’s integral \( I(\eta) \) are respectively defined by

\[
\langle W \rangle_t = W_t^2 - 2 \int_0^t W_s dW(s),
\]

\[
I(\eta) = \int_0^T \eta_s dW(s) = \sum_{i=0}^{N-1} \xi_i(W_{t_{i+1}} - W_{t_i}).
\]

The following two lemmas can be found in the book [19]. They are known as Hölder’s and Gronwall’s inequalities respectively.

**Lemma 2.4.** Assume \( m, n > 1 \) such that \( \frac{1}{m} + \frac{1}{n} = 1 \) and \( \beta \in L^2 \) then \( \alpha \beta \in L^1 \) and

\[
\int_a^b \alpha \beta \leq \left( \int_a^b |\alpha|^m \right)^{\frac{1}{m}} \left( \int_a^b |\beta|^n \right)^{\frac{1}{n}}.
\]

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Lemma 2.5. Let \( \alpha(t) \geq 0 \) and \( \beta(t) \) be continuous real functions defined on \([a, b]\). If for all \( t \in [a, b] \),
\[
\beta(t) \leq K + \int_a^b \alpha(s) \beta(s) ds,
\]
where \( K \geq 0 \), then
\[
\beta(t) \leq K e^{\int_a^t \alpha(s) ds},
\]
for all \( t \in [a, b] \).

The following lemma, known as Doob’s martingale inequality, is borrowed from [15].

Lemma 2.6. Assume \([c, d]\) be a bounded interval of \( \mathbb{R}_+ \). Consider an \( \mathbb{R}^n \)-valued \( G \)-martingale \( \{X(t) : t \geq 0\} \). Then we have
\[
E(\sup_{c \leq t \leq d} |Y(t)|^p) \leq (\frac{p}{p-1})^p E(|Y(d)|^p),
\]
where \( p > 1 \) and \( Y(t) \in L^p_G(\Omega, \mathbb{R}^d) \). In particular, if \( p = 2 \) then \( E(\sup_{c \leq t \leq d} |Y(t)|^2) \leq 4E(|Y(d)|^2) \).

3. Caratheodory approximate solutions

We now present the Caratheodory approximation procedure for equation (1.2). Let \( q \geq 1 \) be any positive integer. For \( t \in [t_0 - 1, t_0] \), we set \( Y^q(t) = Y_0 \) and for \( t \in [t_0, T] \),
\[
Y^q(t) = Y_0 + \int_{t_0}^t \kappa(s, Y^q(s - \frac{1}{q})) ds + \int_{t_0}^t \lambda(s, Y^q(s - \frac{1}{q})) d\langle W, W \rangle(s) + \int_{t_0}^t \mu(s, Y^q(s - \frac{1}{q})) dW(s).
\]
(3.1)

The approximate solutions \( Z^q(.) \) can be determined step-by-step on the intervals \([t_0, t_0 + \frac{1}{q}]\), \([t_0 + \frac{1}{q}, t_0 + \frac{2}{q}]\) and so on with the following procedure. For \( t \in [t_0, t_0 + \frac{1}{q}] \), we have
\[
Y^q(t) = Y_0 + \int_{t_0}^t \kappa(s, Y_0) ds + \int_{t_0}^t \lambda(s, Y_0) d\langle W, W \rangle(s) + \int_{t_0}^t \mu(s, Y_0) dW(s),
\]
and for \( t \in (t_0 + \frac{1}{q}, t_0 + \frac{2}{q}] \),
\[
Y^q(t) = Y^q(t_0 + \frac{1}{q}) + \int_{t_0 + \frac{1}{q}}^t \kappa(s, Y^q(s - \frac{1}{q})) ds + \int_{t_0 + \frac{1}{q}}^t \lambda(s, Y^q(s - \frac{1}{q})) d\langle W, W \rangle(s) + \int_{t_0 + \frac{1}{q}}^t \mu(s, Y^q(s - \frac{1}{q})) dW(s),
\]

etc. All through this article, we assume two conditions, described as follows. Let \( M \) be a positive constant. For any \( t \in [t_0, T] \) and \( \kappa(t, 0), \lambda(t, 0), \mu(t, 0) \in L^2 \),
\[
|\kappa(t, 0)|^2 + |\lambda(t, 0)|^2 + |\mu(t, 0)|^2 \leq M,
\]
(3.2)
which is weakened linear growth condition. Let \( t \in [t_0, T] \). For every \( u, v \in \mathbb{R}^n \), there exists a concave non-decreasing function \( \Psi(.) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \Psi(0) = 0 \) and for \( s > 0 \), \( \Psi(s) > 0 \) such that

\[
\left| \kappa(t, u) - \kappa(t, v) \right|^2 + \left| \lambda(t, u) - \lambda(t, v) \right|^2 + \left| \mu(t, u) - \mu(t, v) \right|^2 \leq \Psi(|u - v|^2),
\]

where \( \int_{0^+} \frac{ds}{\Psi(s)} = \infty \) and for all \( s \geq 0 \), \( C, D > 0 \), \( \Psi(s) \leq C + Ds \). Assumption (3.3) is a non-uniform Lipschitz condition. Subject to conditions (3.2) and (3.3), we assume that problem (1.1) has a unique solution \( Y(t) \in \mathcal{M}_2^\infty([t_0, T]; \mathbb{R}^n) \) [1].

**Lemma 3.1.** Let assumptions (3.2) and (3.3) are satisfied. For every \( q \geq 1 \) and any \( T > 0 \),

\[
(3.4) \quad \sup_{t_0 \leq t \leq T} E(|Y^q(t)|^2) \leq N_1,
\]

where \( N_1 = H_1 e^{H_2(T-t_0)} \), \( H_1 = 4E|Y_0|^2 + 8T(T+2)(2M+C) \), \( H_2 = 8(T+2)D \) and \( M, C, D \) are arbitrary positive constants.

**Proof.** In view of the inequality \( \sum_{i=1}^4 c_i^2 \leq 7 \sum_{i=1}^4 |c_i|^2 \), from (3.1) we derive

\[
|Y^q(t)|^2 \leq 4|Y_0|^2 + 4 \int_{t_0}^t \kappa(s, Y^q(s) - \frac{1}{q})|ds|^2 + 4 \int_{t_0}^t \lambda(s, Y^q(s) - \frac{1}{q})|d\langle W \rangle(s)|^2 + 4 \int_{t_0}^t \mu(s, Y^q(s) - \frac{1}{q})|dW|^2.
\]

Apply G-subexpectation on both sides. Then by virtue of the Doob’s martingale, Holder’s and BDG [5] inequalities we have

\[
E(\sup_{t_0 \leq s \leq t} |Y^q(s)|^2) \leq 4E(|Y_0|^2) + 4T \int_{t_0}^t E|\kappa(s, Y^q(s) - \frac{1}{q})|^2|ds + 4T \int_{t_0}^t E|\lambda(s, Y^q(s) - \frac{1}{q})|^2|ds
\]

\[
+ 16 \int_{t_0}^t E|\mu(s, Y^q(s) - \frac{1}{q})|^2|ds
\]

\[
\leq 4E(|Y_0|^2) + 4T \int_{t_0}^t E|\kappa(s, Y^q(s) - \frac{1}{q}) - \kappa(s, 0)|^2 + |\kappa(s, 0)|^2|ds
\]

\[
+ 8T \int_{t_0}^t E|\lambda(s, Y^q(s) - \frac{1}{q}) - \lambda(s, 0)|^2 + |\lambda(s, 0)|^2|ds
\]

\[
+ 32 \int_{t_0}^t E|\mu(s, Y^q(s) - \frac{1}{q}) - \mu(s, 0)|^2 + |\mu(s, 0)|^2|ds.
\]

Using (3.2) and (3.3), we derive

\[
E(\sup_{t_0 \leq s \leq t} |Y^q(s)|^2) \leq 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2) \int_{t_0}^t E[|Y^q(s) - \frac{1}{q}|^2]|ds
\]

\[
\leq 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2) \int_{t_0}^t [C + DE|^Y^q(s) - \frac{1}{q}|^2]|ds
\]

\[
\leq 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2)C + 8(T+2)D \int_{t_0}^t E[\sup_{t_0 \leq r \leq s} |Y^q(r)|^2]|ds.
\]
By virtue of the Gronwall’s inequality, we derive
\[
E\left( \sup_{t_0 \leq s \leq t} |Y^q(s)|^2 \right) \leq H_1 e^{H_2 (t-t_0)},
\]
where \( H_1 = 4E|Y_0|^2 + 8T(T + 2)(2M + C) \) and \( H_2 = 8(T + 2)D \). Consequently, supposing \( t = T \), we obtain
\[
E\left( \sup_{t_0 \leq s \leq T} |Y^q(s)|^2 \right) \leq H_1 e^{H_2 (T-t_0)} = N_1.
\]
The proof stands completed. \( \square \)

In a similar way as lemma 3.1, we can prove the following result.

**Lemma 3.2.** Subject to the growth condition (3.2), for any \( T > 0 \),

\[
\sup_{t_0 \leq t \leq T} E(|Y(t)|^2) \leq N_1,
\]
where \( N_1 \) is a positive constant.

4. Estimates for the difference between exact and Caratheodory approximate solutions

We first give an important result. Then in view of weakened growth and non-uniform Lipschitz conditions, we derive an estimate for the difference between the approximate and exact solutions to problem (1.1).

**Lemma 4.1.** Let \( 0 \leq r < t \leq T \). Suppose that the assumptions of lemma 3.1 are satisfied. For all \( q \geq 1 \)

\[
E[|Z^q(t) - Z^q(u)|^2] \leq G_1(t-u),
\]
where \( G_1 = 12(T+2)(M+C+DN_1) \), \( M, C, D \) and \( N_1 \) are positive constants.

**Proof.** In view of the fundamental inequality \( |\sum_{i=1}^3 c_i|^2 \leq 7 \sum_{i=1}^3 |c_i|^2 \), for any \( q \geq 1 \) and \( 0 \leq r < t \leq T \), from (3.1) we derive
\[
|Y^q(t) - Y^q(u)|^2 \leq 3\int_{u}^{t} \kappa(s, Y^q(s - \frac{1}{q}))ds|^2 + 3\int_{u}^{t} \lambda(s, Y^q(s - \frac{1}{q})))d\langle W, W \rangle(s)|^2
\[
+ 3\int_{u}^{t} \mu(s, Y^q(s - \frac{1}{q})))dW(s)|^2.
\]
Apply G-subexpectation on both sides. Then by virtue of the Doob’s martingale, Holder’s and BDG [5] inequalities we have

\[
|Y^q(t) - Y^q(r)|^2 \leq 3T \int_r^t E|\kappa(s, Y^q(s - \frac{1}{q}))|^2 ds + 3T \int_t^r E|\lambda(s, Y^q(s - \frac{1}{q}))|^2 ds \\
+ 12 \int_r^t |\mu(s, Y^q(s - \frac{1}{q}))|^2 ds \\
\leq 6T \int_r^t E|\kappa(s, Y^q(s - \frac{1}{q})) - \kappa(s, 0)|^2 + |\kappa(s, 0)|^2| ds \\
+ 6T \int_r^t E|\lambda(s, Y^q(s - \frac{1}{q})) - \lambda(s, 0)|^2 + |\lambda(s, 0)|^2| ds \\
+ 24 \int_r^t E|\mu(s, Y^q(s - \frac{1}{q})) - \mu(s, 0)|^2 + |\mu(s, 0)|^2| ds.
\]

Using (3.2) (3.3), we derive

\[
|Y^q(t) - Y^q(u)|^2 \leq 6TM(t - u) + 6TM(t - u) + 24M(t - u) + 12(T + 2) \int_u^t E|\Psi(|Y^q(s - \frac{1}{q})|)| ds \\
\leq 12TM(t - u) + 24M(t - u) + 12C(T + 2)(t - u) + 12D(T + 2) \int_u^t E|\Psi(|Y^q(s - \frac{1}{q})|)| ds \\
\leq 12TM(t - u) + 24M(t - u) + 12C(T + 2)(t - u) \\
+ 12D(T + 2) \int_u^t E|\sup_{t_0 \leq r \leq s} |Y^q(r)||^2| ds
\]

In view of lemma 3.1, we have

\[
|Y^q(t) - Y^q(u)|^2 \leq 12TM(t - u) + 24M(t - u) + 12C(T + 2)(t - u) \\
+ 12D(T + 2)N_1(t - u)
\]

Consequently,

\[
|Y^q(t) - Y^q(u)|^2 \leq G_1(t - u),
\]

where \(G_1 = 12(T + 2)(M + C + DN_1)\). The proof is complete. \(\square\)

Next lemma can be proved by using similar arguments as used in lemma 4.1.

**Lemma 4.2.** Let \(0 \leq r < t \leq T\). Subject to conditions (3.2) and (3.3),

\[
E|Z(t) - Z(t)|^2 \leq G_1(t - u),
\]

where \(G_1\) is a positive constant.

**Theorem 4.3.** Assume (3.2) and (3.3) holds. Then

\[
E(\sup_{t_0 \leq s \leq T} |Y(s) - Y^q(s)|^2) \leq 6T(T + 2)[C + \frac{2D}{q}e^{12(T + 2)D(T - t_0)}],
\]

where \(C\) and \(D\) are positive constants.
Proof. By using the inequality \(|\sum_{i=1}^{3} c_i|^2 \leq 7 \sum_{i=1}^{3} |c_i|^2|, from (1.2) and (3.1) we obtain

\[
|Y(t) - Y^q(t)|^2 \leq 3\int_{t_0}^{t} [\kappa(s, Y(s)) - \kappa(s, Y^q(s - \frac{1}{q}))]ds + 3\int_{t_0}^{t} [\lambda(s, Y(s)) - \lambda(s, Y^q(s - \frac{1}{q}))]|d(W, W')(s)|^2
\]

\[
+ 3\int_{t_0}^{t} [\mu(s, Y(s)) - \mu(s, Y^q(s - \frac{1}{q}))]|dW(s)|^2.
\]

Apply G-subexpectation on both sides. Then by virtue of the Doob’s martingale, Holder’s and BDG [5] inequalities we derive

\[
E(\sup_{t_0 \leq s \leq t} |Y(s) - Y^q(s)|^2) \leq 3T \int_{t_0}^{t} E[|\kappa(s, Y(s)) - \kappa(s, Y^q(s - \frac{1}{q}))|^2]ds
\]

\[
+ 3T \int_{t_0}^{t} E[|\lambda(s, Y(s)) - \lambda(s, Y^q(s - \frac{1}{q}))|^2]ds
\]

\[
+ 12 \int_{t_0}^{t} E[|\mu(s, Y(s)) - \mu(s, Y^q(s - \frac{1}{q}))|^2]ds.
\]

Using the non-uniform Lipschitz condition we get

\[
E(\sup_{t_0 \leq s \leq t} |Y(s) - Y^q(s)|^2) \leq 6(T + 2) \int_{t_0}^{t} E[\Psi(|Y(s) - Y^q(s - \frac{1}{q})|^2)]ds
\]

\[
\leq 6T(T + 2)C + 6(T + 2)D \int_{t_0}^{t} E[|Y(s) - Y^q(s - \frac{1}{q})|^2]ds
\]

\[
= 6T(T + 2)C + 6(T + 2)D \int_{t_0}^{t} E[|Y(s) - Y^q(s)^q + Y^q(s) - Y^q(s - \frac{1}{q})|^2]ds
\]

\[
\leq 6T(T + 2)C + 12(T + 2)D \int_{t_0}^{t} E[|Y(s) - Y^q(s)|^2]ds
\]

\[
+ 12(T + 2)D \int_{t_0}^{t} E[|Y^q(s) - Y^q(s - \frac{1}{q})|^2]ds
\]

Utilizing lemma 4.1, we determine

\[
E(\sup_{t_0 \leq s \leq t} |Y(s) - Y^q(s)|^2) \leq 6T(T + 2)C + 12(T + 2)D \int_{t_0}^{t} E(\sup_{t_0 \leq r \leq s} |Y(r) - Y^q(r)|^2)ds + 12T(T + 2)D \frac{1}{q}.
\]

Finally, the Gronwall’s inequality gives

\[
E(\sup_{t_0 \leq s \leq t} |Y(s) - Y^q(s)|^2) \leq [6T(T + 2)C + 12T(T + 2)D \frac{1}{q}]e^{12(T+2)D(t-t_0)}.
\]

Consequently, by letting \(t = T\), we get

\[
E(\sup_{t_0 \leq s \leq T} |Y(s) - Y^q(s)|^2) \leq 6T(T + 2)[C + \frac{2D}{q}]e^{12(T+2)D(T-t_0)}.
\]

The proof stands completed. \(\square\)
5. Conclusion

This paper opens several new research directions with arising the following open problems. What will be the estimates for the difference between exact and Caratheodory approximate solutions to G-SFDEs under non-linear growth and non-Lipschitz conditions? How can one solve the stated problem for G-NSFDEs? Can one give estimates for the difference between exact and Caratheodory approximate solutions to backward stochastic differential equations in the framework of G-Brownian motion? Under what conditions, can we develop the mentioned theory for stochastic pantograph equations [2, 12, 31, 32]? We hope the current paper will play an essential role to establish a foundation for the concepts briefly discussed.

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References


Behavior of a system of higher-order difference equations

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Abstract

We study the local stability about equilibria, periodicity nature of positive solutions and existence of unbounded solutions of higher-order system of rational difference equations. The results presented here are considerably extended and improve some existing results in the literature. Finally theoretical results are verified numerically.

Keywords: difference equations; local stability; periodicity; unbounded solutions

AMS subject classifications: 39A10, 40A05

1 Introduction

In [1], Bajo and Liz have investigated the global behavior of the difference equation: $x_{n+1} = \frac{x_{n-1}}{a+bx_{n-1}x_n}$, where $a, b, x_0, x_{-1} \in \mathbb{R}^2_+$. Aloqeili [2] has investigated the stability and semi-cycle analysis of the difference equation: $x_{n+1} = \frac{x_{n-1}}{a-x_{n-1}x_n}$, $n = 0, 1, \cdots$, where $a, x_0, x_{-1} \in \mathbb{R}^2_+$. For systemic study of difference equations and systems of difference equations, we refer the reader [3–7] and references cited therein. Motivated by the above studies, our aim in this paper is to investigate the local stability about equilibria, periodicity nature of the positive solutions and existence of unbounded solutions of the following higher-order system of difference equations:

$$
\begin{align*}
  x_{n+1} &= \frac{\alpha_1 x_{n-k}}{\beta_1 - \gamma_1 \prod_{i=0}^{k} y_{n-i}}, \\
  y_{n+1} &= \frac{\alpha_2 y_{n-k}}{\beta_2 - \gamma_2 \prod_{i=0}^{k} x_{n-i}},
\end{align*}
$$

where $\alpha_i, \beta_i, \gamma_i$ for $i = 1, 2$ and $x_{-j}, y_{-j}$ for $j = 0, 1, \cdots, k$ are belong to $\mathbb{R}^2_+$. The rest of the paper is organized as follows: Existence of equilibria and local stability are studied in Section 2. Section 3 deals with the study of periodicity nature and existence of unbounded solutions of system (1). In Section 4, numerical simulations are presented to verify theoretical discussion. A brief conclusion is given in last Section.

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2 Existence of equilibria and local stability

In this section, we will study the existence of equilibria and local stability of system (1). The results about the existence of equilibria are summarized into following Lemma:

Lemma 1. System (1) has two equilibria in the interior of $\mathbb{R}_+^2$. More precisely

(i) For parametric values, system (1) has a unique boundary equilibrium point $O(0, 0)$;

(ii) If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, then $A\left(\frac{\beta_2 - \alpha_2}{\gamma_2}, \frac{\beta_1 - \alpha_1}{\gamma_1}\right)$ is the unique positive equilibrium point of system (1).

Hereafter we will study the local stability of system (1) about boundary equilibrium $(0, 0)$ and the unique positive equilibrium point $A\left(\frac{\beta_2 - \alpha_2}{\gamma_2}, \frac{\beta_1 - \alpha_1}{\gamma_1}\right)$ of system (1).

Lemma 2. For local dynamics about $O(0, 0)$ and $A\left(\frac{\beta_2 - \alpha_2}{\gamma_2}, \frac{\beta_1 - \alpha_1}{\gamma_1}\right)$, the following statements hold:

(i) For equilibrium $O(0, 0)$, the following holds:

(i.1) $O(0, 0)$ is locally asymptotically stable if $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$;

(i.2) $O(0, 0)$ is unstable if $\alpha_1 > \beta_1$ or $\alpha_2 < \beta_2$.

(ii) $A\left(\frac{\beta_2 - \alpha_2}{\gamma_2}, \frac{\beta_1 - \alpha_1}{\gamma_1}\right)$ is unstable.

Proof. (i.1) The linearized system of (1) about $(0, 0)$ becomes: $X_{n+1} = J_{(0,0)}X_n$ where

$$
\begin{pmatrix}
  x_n \\
  x_{n-1} \\
  \vdots \\
  x_{n-k} \\
  y_n \\
  y_{n-1} \\
  \vdots \\
  y_{n-k}
\end{pmatrix}, \quad J_{(0,0)} =

\begin{pmatrix}
  0 & 0 & \cdots & 0 & \frac{\alpha_2}{\beta_1} & 0 & \cdots & 0 & 0 \\
  1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_2}{\beta_2} \\
  0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
$$

The characteristic equation of $J_{(0,0)}$ about $(0, 0)$ is

$$
\lambda^{2k+2} - \left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}\right)\lambda^{k+1} + \frac{\alpha_1\alpha_2}{\beta_1\beta_2} = 0. \tag{2}
$$

If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ then all roots of (2) lie inside unit disk. So $O(0, 0)$ of system (1) is locally asymptotically stable.

(i.2) It is easy to show that if $\alpha_1 > \beta_1$ or $\alpha_2 > \beta_2$ then $O(0, 0)$ is unstable.

(ii). The linearized system of (1) about $A\left(\frac{\beta_2 - \alpha_2}{\gamma_2}, \frac{\beta_1 - \alpha_1}{\gamma_1}\right)$ becomes: $X_{n+1} = J_A X_n$ where
\[
J_A = E_{(2k+2) \times (2k+2)} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 & \frac{\gamma_1 \bar{x} \bar{y}^k}{\alpha_1} & \frac{\gamma_1 \bar{x} \bar{y}^k}{\alpha_1} & \ldots & \frac{\gamma_1 \bar{x} \bar{y}^k}{\alpha_1} & \frac{\gamma_1 \bar{x} \bar{y}^k}{\alpha_1} \\
1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\frac{\gamma_2 \bar{x} \bar{y}^k}{\alpha_2} & \frac{\gamma_2 \bar{x} \bar{y}^k}{\alpha_2} & \ldots & \frac{\gamma_2 \bar{x} \bar{y}^k}{\alpha_2} & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 
\end{pmatrix}.
\]

Let \(\lambda_1, \lambda_2, \ldots, \lambda_{2k+2}\) denote the \(2k+2\) eigenvalues of matrix \(E\). Let \(D = \text{diag}(d_1, d_2, \ldots, d_{2k+2})\) be a diagonal matrix, where \(d_1 = d_{k+2} = 1\), \(d_i = d_{k+1+i} = 1 - i\epsilon\), \(i = 2, 3, \ldots, k+1\) for \(0 < \epsilon < 1\). Clearly, \(D\) is invertible. In computing \(DED^{-1}\), we obtain that

\[
DED^{-1} = \begin{pmatrix}
0 & 0 & \ldots & 0 & d_1d_{k+1}^{-1} \\
d_2d_1^{-1} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & d_{k+2}^{-1} & 0 \\
\frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{k+2}^{-1}}{\alpha_2} & \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{k+3}^{-1}}{\alpha_2} & \ldots & \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{2k+1}^{-1}}{\alpha_2} & \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{2k+2}^{-1}}{\alpha_2} \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & d_{k+2}^{-1} \\
d_{k+3}d_{k+2}^{-1} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & d_{2k+2}^{-1} & 0 
\end{pmatrix}.
\]

From \(d_1 > d_2 > \cdots > d_{k+1} > 0\) and \(d_{k+2} > d_{k+3} > \cdots > d_{2k+2} > 0\) it implies that \(d_2 d_{k+1}^{-1} < 1, d_3 d_{k+2}^{-1} < 1, \ldots, d_{k+1} d_{k-1}^{-1} < 1\) and \(d_{k+3} d_{k+2}^{-1} < 1, d_{k+4} d_{k+3}^{-1} < 1, \ldots, d_{2k+2} d_{2k+1}^{-1} < 1\). Furthermore,

\[
d_1 d_{k+1}^{-1} + \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{k+2}^{-1}}{\alpha_1} + \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{k+3}^{-1}}{\alpha_1} + \cdots + \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{2k+1}^{-1}}{\alpha_1} + \frac{\gamma_1 \bar{x} \bar{y}^k d_1 d_{2k+2}^{-1}}{\alpha_1} = \frac{1}{1 - (k+1)\epsilon} + \frac{\gamma_1 \bar{x} \bar{y}^k}{\alpha_1} \left(1 + \frac{1}{1 - 2\epsilon} + \cdots + \frac{1}{1 - k\epsilon} + \frac{1}{1 - (k+1)\epsilon}\right) > 1.
\]

Also

\[
\frac{\gamma_2 \bar{x} \bar{y}^k d_{k+2} d_{k+1}^{-1}}{\alpha_2} + \frac{\gamma_2 \bar{x} \bar{y}^k d_{k+3} d_{k+2}^{-1}}{\alpha_2} + \cdots + \frac{\gamma_2 \bar{x} \bar{y}^k d_{k+2} d_{k+3}^{-1}}{\alpha_2} + \frac{\gamma_2 \bar{x} \bar{y}^k d_{k+2} d_{k+4}^{-1}}{\alpha_2} + \frac{\gamma_2 \bar{x} \bar{y}^k d_{k+2} d_{k+5}^{-1}}{\alpha_2} + \cdots + \frac{\gamma_2 \bar{x} \bar{y}^k d_{k+2} d_{2k+2}^{-1}}{\alpha_2} = \frac{\gamma_2 \bar{x} \bar{y}^k}{\alpha_2} \left(1 + \frac{1}{1 - 2\epsilon} + \cdots + \frac{1}{1 - k\epsilon} + \frac{1}{1 - (k+1)\epsilon}\right) + \frac{1}{1 - (k+1)\epsilon} > 1.
\]
It is well-known fact that $E$ has the same eigenvalues as $DED^{-1}$. Hence, we obtain
\[
\max_{1 \leq m \leq 2k+2} |\lambda_m| \leq \|DED^{-1}\|_\infty = \max\{d_2d_1^{-1}, \ldots, d_{k+1}d_k^{-1}, d_{k+3}d_{k+2}^{-1}, \ldots, d_{2k+2}d_{2k+1}^{-1}, \frac{1}{1 - (k+1)\epsilon} + \gamma_1 \bar{y} \alpha_1  \\
\cdot \left( 1 + \frac{1}{1 - 2\epsilon} + \cdots + \frac{1}{1 - k\epsilon} + \frac{1}{1 - (k+1)\epsilon} \right) \cdot \frac{1}{1 - (k+1)\epsilon} \}
\]
This implies that $A\left( \left( \frac{\beta_2 - \alpha_2}{\gamma_2} \right)^{\frac{1}{k+1}}, \left( \frac{\beta_1 - \alpha_1}{\gamma_1} \right)^{\frac{1}{k+1}} \right)$ of system (1) is unstable. $\square$

3 Periodicity nature and existence of unbounded solutions

In this section, we will study the periodicity nature and existence of unbounded solutions of system (1). Let us denote $a_1 = \gamma_1 y_{-k}y_{-k} \cdots y_0$, $a_2 = \gamma_2 x_{-k}x_{-k} \cdots x_0$ to study the periodicity nature of positive solution of system (1).

**Theorem 1.** If $a_1 = \beta_1 - \alpha_1$ and $a_2 = \beta_2 - \alpha_2$, then system (1) has prime period-(k+1) solutions.

**Proof.** From system (1) and $a_1 = \beta_1 - \alpha_1$, $a_2 = \beta_2 - \alpha_2$, we have
\[
x_1 = \frac{\alpha_1 x_{-k}}{k} = \frac{\alpha_1 x_{-k}}{\beta_1 - \gamma_1 \sum_{i=0} y_{-i}} = x_{-k}, \quad y_1 = \frac{\alpha_2 y_{-k}}{k} = \frac{\alpha_2 y_{-k}}{\beta_2 - \gamma_2 \sum_{i=0} x_{-i}} = y_{-k}.
\]
\[
x_2 = \frac{\alpha_1 x_{-k}}{k} = \frac{\alpha_1 x_{-k}}{\beta_1 - \gamma_1 \sum_{i=0} y_{1-i}} = x_{1-k}, \quad y_2 = \frac{\alpha_2 y_{-k}}{k} = \frac{\alpha_2 y_{-k}}{\beta_2 - \gamma_2 \sum_{i=0} x_{1-i}} = y_{1-k}.
\]
By induction, one has
\[
x_{k+2} = \frac{\alpha_1 x_1}{k} = \frac{\alpha_1 x_1}{\beta_1 - \gamma_1 \sum_{i=0} y_{k+1-i}} = x_1, \quad y_{k+2} = \frac{\alpha_2 y_1}{k} = \frac{\alpha_2 y_1}{\beta_2 - \gamma_2 \sum_{i=0} x_{k+1-i}} = y_1.
\]

**Theorem 2.** Assume that $\beta_1 < \alpha_1$, $\beta_2 < \alpha_2$. Then, every positive solution $\{(x_n, y_n)\}$ of system (1) tends to $\infty$ as $n \to \infty$. $\square$
Example 1. In this section we will present numerical simulations to verify theoretical results.

Example 2. Moreover, in Fig. 1 the plot of $x_n$, $x_{n-1}$, $x_{n-2}$, $x_{n-3}$ is shown in Fig. 1a, the plot of $y_n$, $y_{n-1}$, $y_{n-2}$, $y_{n-3}$ is shown in Fig. 1b and global attractor of system (5) is shown in Fig. 1c.

4 Numerical simulations

In this section we will present numerical simulations to verify theoretical results.

Example 1. If $\alpha_1 = 50, \beta_1 = 63, \gamma_1 = 4, \alpha_2 = 90, \beta_2 = 122, \gamma_2 = 2$ then system (1) with $x_5 = 3.9, x_4 = 1.5, x_3 = 12.4, x_2 = 11.9, x_1 = 1.6, x_0 = 2.9, y_5 = 2.6, y_4 = 3.8, y_3 = 5.8, y_2 = 3.5, y_1 = 3.1, y_0 = 0.9$ can be written as:

$$x_{n+1} = \frac{50x_{n-5}}{63 - 4y_n y_{n-1} y_{n-2} y_{n-3} y_{n-4} y_{n-5}}, \quad y_{n+1} = \frac{90y_{n-5}}{122 - 2x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4} x_{n-5}}.$$  \hspace{1cm} (5)

Moreover, in Fig. 1 the plot of $x_n$ is shown in Fig. 1a, the plot of $y_n$ is shown in Fig. 1b and global attractor of system (5) is shown in Fig. 1c.

Example 2. If $\alpha_1 = 15.5, \beta_1 = 17, \gamma_1 = 27, \alpha_2 = 11.2, \beta_2 = 12, \gamma_2 = 23$, then system (1) with $x_8 = 1.9, x_7 = 1.7, x_6 = 2.5, x_5 = 0.9, x_4 = 1.5, x_3 = 10.4, x_2 = 6.9, x_1 = 0.6, x_0 = 2.9, y_8 = 2.8, y_7 = 1.6, y_6 = 1.8, y_5 = 2.6, y_4 = 2.8, y_3 = 2.8, y_2 = 3.5, y_1 = 2.1, y_0 = 1.6$ can be written as:

$$x_{n+1} = \frac{15.5x_{n-8}}{17 - 27y_n y_{n-1} y_{n-2} y_{n-3} y_{n-4} y_{n-5} y_{n-6} y_{n-7} y_{n-8}}, \quad y_{n+1} = \frac{11.2y_{n-8}}{12 - 23x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4} x_{n-5} x_{n-6} x_{n-7} x_{n-8}}.$$  \hspace{1cm} (6)

Moreover, in Fig. 2 the plot of $x_n$ is shown in Fig. 2a, the plot of $y_n$ is shown in Fig. 2b and global attractor of system (6) is shown in Fig. 2c.
5 Conclusion and future work

This work is related to the qualitative behavior of a system of higher-order rational difference equations. We have proved that under some restrictions to parameters, system (6) has a boundary equilibrium \( O(0, 0) \) and the unique positive equilibrium point \( A \left( \left( \frac{\beta_2 - \alpha_2}{\gamma_2} \right)^{1/k}, \left( \frac{\beta_1 - \alpha_1}{\gamma_1} \right)^{1/k} \right) \) in the closed first quadrant \( \mathbb{R}_+^2 \).

We have analyzed the local stability about equilibria, periodicity nature of positive solutions and existence of unbounded solutions of system (6). Finally, theoretical results are verified numerically. Besides the local properties, the global stability of under consideration system (6), which is our further aim to study.

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References


ON APPROXIMATING THE GENERALIZED EULER-MASCHERONI CONSTANT∗

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ABSTRACT. In the article, we provide several sharp bounds for the the generalized Euler-Mascheroni constant, which are the generalizations of the previously results on the Euler-Mascheroni constant.

1. INTRODUCTION

It is well known that the sequence
\begin{equation}
\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n
\end{equation}
is convergent towards the Euler-Mascheroni constant
\begin{equation}
\gamma = 0.57721566490115328\ldots.
\end{equation}

The Euler-Mascheroni constant has been involved in a variety of mathematical formulas and results [1-6], many special functions are closely related to the Euler-Mascheroni constant [7-63]. Recently, the bounds for \(\gamma_n - \gamma\) have attracted the attention of many researchers.

Alzer [64] proved that the double inequality
\begin{equation}
\frac{1}{2n+1} \leq \gamma_n - \gamma \leq \frac{1}{2n}
\end{equation}
holds for \(n \geq 1\).

In [65], Tóth proved that the two-sided inequality
\begin{equation}
\frac{1}{2n + \frac{2}{3}} \leq \gamma_n - \gamma \leq \frac{1}{2n + \frac{1}{3}}
\end{equation}
takes place for \(n \geq 1\).

Chen [66] proved that \(\alpha = (2\gamma - 1)/(1 - \gamma)\) and \(\beta = 1/3\) are the best possible constants such that the double inequality
\begin{equation}
\frac{1}{2n + \alpha} \leq \gamma_n - \gamma \leq \frac{1}{2n + \beta}
\end{equation}
holds for \(n \geq 1\).

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In [67], Qiu and Vuorinen proved that the double inequality
\begin{equation}
\frac{1}{2n} - \frac{\lambda}{n^2} < \gamma_n - \gamma \leq \frac{1}{2n} - \frac{\mu}{n^2}
\end{equation}
holds for \( n \geq 1 \) if and only if \( \lambda \geq 1/12 \) and \( \mu \leq \gamma - 1/2 \).

Let \( a > 0 \). Then the generalized Euler-Mascheroni constant \( \gamma(a) \) is defined by
\begin{equation}
\gamma(a) = \lim_{n \to \infty} \left( \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log \frac{a+n-1}{a} \right),
\end{equation}
which was introduced by Knopp [68]. We clearly see that \( \gamma(1) = \gamma \). Recently, the generalized Euler-Mascheroni constant \( \gamma(a) \) has been the subject of intensive research [69-71].

In [70], Štimărian introduced the sequences
\begin{equation}
x_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log \frac{a+n-1}{a},
\end{equation}
\begin{equation}
y_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \log \frac{a+n-1}{a},
\end{equation}
and proved that the double inequalities
\begin{equation}
\frac{1}{2(n+a)} \leq \gamma(a) - x_n \leq \frac{1}{2(n+a-1)},
\end{equation}
\begin{equation}
\frac{1}{2(n+a)} \leq y_n - \gamma(a) \leq \frac{1}{2(n+a-1)}
\end{equation}
hold for \( n \geq 1 \).

In [71], Berinde and Mortici established Theorems 1.1 and 1.2 as follows.

**Theorem 1.1.** The double inequalities
\begin{equation}
\frac{1}{2(n+a)} - \frac{1}{4} < \gamma(a) - x_n < \frac{1}{2(n+a) - \frac{1}{4}},
\end{equation}
\begin{equation}
\frac{1}{2(n+a)} - \frac{1}{3} < y_n - \gamma(a) < \frac{1}{2(n+a) - \frac{1}{3}}
\end{equation}
hold for \( a > 0 \) and \( n \geq 2 \).

**Theorem 1.2.** (a) The inequality
\begin{equation}
\frac{1}{2(n+a)} - \frac{1}{3} + \frac{1}{18n} \leq \gamma(a) - x_n
\end{equation}
holds for \( a \geq 13/30 \) and any integer \( n \geq 1 \).

(b) The inequality
\begin{equation}
\frac{1}{2(n+a)} - \frac{5}{3} + \frac{1}{18n} \leq y_n - \gamma(a)
\end{equation}
holds for \( a \geq 17/30 \) and \( n \geq 1 \).

The main purpose of this article is to generalize inequalities (1.4) and (1.5) to the generalized Euler-Mascheroni constant \( \gamma(a) \). Our main results are the following Theorems 1.3 and 1.4.
Theorem 1.3. Let $a > 0$, $n \geq 1$. Then one has

(1) the double inequality

\[ \frac{1}{2(n+a) - \alpha_1} \leq \gamma(a) - x_n < \frac{1}{2(n+a) - \beta_1} \]

holds with the best possible constants

\[ \alpha_1 = 2(1 + a) - \frac{1}{\psi(1 + a) - \log(1 + a)}, \quad \beta_1 = \frac{1}{3}, \]

(2) the two-sided inequality

\[ \frac{1}{2(n+a) - \alpha_2} \leq y_n - \gamma(a) < \frac{1}{2(n+a) - \beta_2} \]

is valid with the best possible constants

\[ \alpha_2 = 2(1 - d), \quad \beta_2 = \frac{5}{3}, \]

where

\[ d = \max\{\tilde{f}_2(a), \tilde{f}_2(1+a), \tilde{f}_2(2+a)\}, \quad \tilde{f}_2(x) = \frac{1}{2(\psi(x+1) - \log(x)) - x}. \]

Theorem 1.4. Let $a > 0$, $n \geq 1$. Then the double inequalities

\[ \frac{1}{2(n+a) + \frac{\alpha_3}{(n+a)^2}} \leq \gamma(a) - x_n < \frac{1}{2(n+a) + \frac{\beta_3}{(n+a)^2}}, \]

\[ \frac{1}{2(n+a-1) + \frac{\alpha_4}{(n+a-1)^2}} < y_n - \gamma(a) \leq \frac{1}{2(n+a-1) + \frac{\beta_4}{(n+a-1)^2}} \]

hold with the best possible constants

\[ \alpha_3 = (1 + a)^2[\log(1 + a) - \psi(1 + a)] - \frac{1+a}{2}, \quad \beta_3 = \frac{1}{12}, \]

\[ \alpha_4 = -\frac{1}{12}, \quad \beta_4 = a^2[\psi(a) - \log(a)] + \frac{a}{2}. \]

2. Lemmas

In order to prove our main results, we need the following formulas and lemmas.

For $x > 0$, the classical gamma function $\Gamma(x)$ and psi function $\psi(x)$ [72-84] are defined as

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \]

respectively.

The psi function $\psi(x)$ has the following recurrence and asymptotic formulas [85]

\[ \psi(n + x) = \frac{1}{(n-1) + x} + \frac{1}{(n-2) + x} + \cdots + \frac{1}{2 + x} + \frac{1}{1 + x} + \frac{1}{x} + \psi(x), \]

\[ \psi(x) \sim \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty) \]
According to (2.1) and the definitions of \( x_n \) and \( y_n \) given in (1.7) and (1.8), we clearly see that \( x_n \) and \( y_n \) can be rewritten as

\[
\begin{align*}
\text{(2.3)} & \quad x_n = \psi(n + a) - \psi(a) - \log \frac{n + a}{a}, \\
\text{(2.4)} & \quad y_n = \psi(n + a) - \psi(a) - \log \frac{n + a - 1}{a}.
\end{align*}
\]

It follows from (1.6) and (2.2) that

\[
\begin{align*}
\text{(2.5)} & \quad \gamma(a) = \lim_{n \to \infty} y_n \\
& \quad = \lim_{n \to \infty} (\psi(n + a) - \log(n + a - 1) + \log(a) - \psi(a)) = \log(a) - \psi(a).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\text{(2.6)} & \quad \gamma(a) - x_n = \log(n + a) - \psi(n + a), \\
\text{(2.7)} & \quad y_n - \gamma(a) = \psi(n + a) - \log(n + a - 1).
\end{align*}
\]

**Lemma 2.1.** The function

\[
\text{(2.8)} \quad f_1(x) = \frac{1}{\log(x) - \psi(x)} - 2x
\]

is strictly decreasing from \((1, \infty)\) onto \((-1/3, 1/\gamma - 2)\).

The function

\[
\text{(2.9)} \quad f_2(x) = \frac{1}{\psi(x + 1) - \log(x)} - 2x
\]

is strictly decreasing from \([2, \infty)\) onto \((1/3, f_2(2)]\).

**Proof.** Differentiating \( f_1(x) \) gives

\[
(\log(x) - \psi(x))^2 f_1'(x) = \psi'(x) - \frac{1}{x} - 2(\log(x) - \psi(x))^2.
\]

It follows from the inequalities

\[
\begin{align*}
\psi'(x) - \frac{1}{x} < & \quad \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \\
\log(x) - \psi(x) > & \quad \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4}
\end{align*}
\]

given in [86] that

\[
(\log(x) - \psi(x))^2 f_1'(x) < \frac{1}{50400x^5} F_1(x),
\]

where

\[
F_1(x) = -207 - 3840(x - 1) - 6580(x - 1)^2 - 3640(x - 1)^3 - 700(x - 1)^4 < 0
\]

for \( x \in (1, \infty) \).

Therefore, the monotonicity of \( f_1(x) \) follows easily from (2.10) and (2.11).

Clearly, \( f_1(1) = 1/\gamma - 2 \). The limiting value \( \lim_{x \to \infty} f_1(x) = -1/3 \) follows from the asymptotic formula (2.2).

Differentiating \( f_2(x) \) leads to

\[
2(\psi(x + 1) - \log(x))^2 f_2'(x) = \frac{1}{x} + \frac{1}{x^2} - \psi'(x) - 2(\psi(x) + \frac{1}{x} - \log(x))^2.
\]
It follows from the inequalities
\[
\frac{1}{x} + \frac{1}{x^2} - \psi'(x) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5},
\]
\[
\psi(x) + \frac{1}{x} - \log(x) > \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}
\]
for \( x > 0 \) given in [86] that
\[
2(\psi(x + 1) - \log(x)) f_2'(x) < -\frac{F_2(x)}{3175200x^{12}},
\]
where
\[
F_2(x) = 3217636 + 17887632(x - 2) + 39443124(x - 2)^2 + 47009928(x - 2)^3 + 39443124(x - 2)^4 + 47009928(x - 2)^5 + 4189500(x - 2)^6 + 652680(x - 2)^7 + 44100(x - 2)^8 > 0
\]
for \( x \geq 2 \).

Therefore, \( f_2(x) \) is a strictly decreasing function on \([2, \infty)\). The limit \( \lim_{x \to \infty} f_2(x) = 1/6 \) follows from the asymptotic formula (2.2).

**Remark 1.** Qi et al. [87] proved that the function \( f_2(x) \) defined by (2.9) is strictly decreasing on \((12/5, \infty)\).

The following Lemma 2.2 can be found in [88, 89].

**Lemma 2.2.** The function

\[
f_3(x) = x^2(\psi(x) - \log(x)) + \frac{x}{2}
\]
is strictly decreasing from \((0, \infty)\) onto \((-1/12, 0)\) and completely monotonic on \((0, \infty)\).

---

### 3. Proof of Theorems 1.3 and 1.4

**Proof of Theorem 1.3.** From (2.6) we clearly see that inequality (1.15) can be rewritten as
\[
-\beta < \frac{1}{\log(n + a) - \psi(n + a)} - 2(n + a) < -\alpha.
\]
It follows from Lemma 2.1 that the sequence
\[
f_1(n + a) = \frac{1}{\log(n + a) - \psi(n + a)} - 2(n + a)
\]
is strictly decreasing, which leads to the conclusion that
\[
-\frac{1}{3} = \lim_{n \to \infty} f_1(n) < f_1(n) \leq f_1(1) = \frac{1}{\log(1 + a) - \psi(1 + a)} - 2(1 + a).
\]
Therefore,
\[
\alpha_1 = 2(1 + a) - \frac{1}{\psi(1 + a) - \log(1 + a)}, \quad \beta_1 = \frac{1}{3}
\]
are the best possible constants such that inequality (1.15) holds.

From (2.7) we clearly see that inequality (1.17) is equivalent to
\[
1 - \frac{\beta}{2} < \frac{1}{2(\psi(n + a) - \log(n + a - 1))} - (n + a - 1) \leq 1 - \frac{\alpha}{2}.
\]
It follows from Lemma 2.1 that the sequence
\[ \tilde{f}_2(n + a - 1) = \frac{1}{2(\psi(n + a) - \log(n + a - 1))} - (n + a - 1) \]
is strictly decreasing for \( n \geq 2 \), which leads to the conclusion that
\[ \frac{1}{6} = \lim_{n \to \infty} \tilde{f}_2(n) < \tilde{f}_2(n) \leq \max \{ \tilde{f}_2(a), \tilde{f}_2(1 + a), \tilde{f}_2(2 + a) \} = d. \]
Therefore,
\[ (3.1) \quad \alpha_2 = 2(1 - d), \quad \beta_2 = \frac{5}{3} \]
are the best possible constants such that inequality (1.17) holds.

**Proof of Theorem 1.4.** From (2.6) and (2.7) we know that inequalities (1.19) and (1.20) can be rewritten as
\[ \alpha_3 \leq (n + a)^2 (\log(n + a) - \psi(n + a)) - \frac{(n + a)}{2} < \beta_3, \]
\[ \alpha_4 < (n + a - 1)^2 (\psi(n + a - 1) - \log(n + a - 1)) + \frac{(n + a - 1)}{2} \leq \beta_4, \]
respectively.

It follows from Lemma 2.2 that the sequence
\[ \tilde{f}_3(n + a - 1) = (n + a - 1)^2 (\psi(n + a - 1) - \log(n + a - 1)) + \frac{(n + a - 1)}{2} \]
is strictly decreasing for \( n \in \mathbb{N} \).

Note that
\[ \lim_{n \to \infty} f_3(n) = -\frac{1}{12}. \]
Therefore,
\[ \alpha_3 = (1 + a)^2 [\log(1 + a) - \psi(1 + a)] - \frac{1 + a}{2}, \quad \beta_3 = \frac{1}{12}, \]
\[ \alpha_4 = -\frac{1}{12}, \quad \beta_4 = (a)^2 [\psi(a) - \log(a)] + \frac{a}{2} \]
are the best possible constants such that inequalities (1.19) and (1.20) hold.

**Remark 2.** (1) Let \( a = 1 \). Then Theorem 1.3(2) leads to inequality (1.4) with the best possible constants \( \alpha = (2\gamma - 1)/(1 - \gamma) \) and \( \beta = 1/3 \).

(2) Let \( a = 1 \). Then inequality (1.20) becomes inequality (1.5) with the best possible constants \( \alpha = 1/12 \) and \( \beta = \gamma - 1/2 \).

(3) From Theorem 1.3 we know that both the upper bounds \( 1/[2(n + a) - 1/3] \) for \( \gamma(a) - x_n \) and \( 1/[2(n + a) - 5/3] \) for \( y_n - \gamma(a) \) given in (1.11) and (1.12) are sharp for any \( a > 0 \).
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General study on Volterra integral equations of the second kind in space with weight function

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Abstract

This paper is devoted to present a new and simple algorithm to prove that the function $\varphi_n(x)$ is a good approximation to the solution $\varphi(x)$ for Volterra integral equations (VIEs) of the second kind in the space $L^2_{p(x)}[0, 2\pi]$ with weight function $p(x)$. This approximation is discussed in details with help of the Vallée-Poussin’s and Fêjer’s, operators. Special attention is given to study the convergence analysis and estimation of an upper bound for the error of the approximated solution.

Key-Words: Volterra integral equations; Vallée-Poussin’s and Fêjer’s operators; Convergence analysis;

1. Introduction

In this paper, we present the approximate solution for Volterra integral equations (VIEs) of the second kind in the space $L^2_{p(x)}[0, 2\pi]$ with weight function $p(x) \geq 1$ where $p(x)$ is a summable function on $[0, 2\pi]$ $\varphi(x) = f(x) + \lambda \int_0^x k(x, y)\varphi(y)dy, \quad 0 \leq x, y \leq 2\pi,$ (1)
where the functions $f(x), k(x, y)$ belong to $L^2_{p(x)}[0, 2\pi]$ and are 2$\pi$-periodic functions, $\frac{1}{\chi}$ is a regular value of the kernel $k(x, y)$ and the kernel $k(x, y)$ satisfies the following conditions

1. $\{\int_0^\pi p(y)|k(x, y)|^2dy\}^{\frac{1}{2}} = \chi(x) \in L^2_{p(x)}[0, 2\pi]$;

2. $|\lambda||k(x, y)||_{L^2_p} < 1$,

where $\|k(x, y)||_{L^2_p} = \|k(x, y)||_{L^2_{p(x)}}[0, 2\pi] = \left[\int_0^{2\pi} \int_0^{2\pi} p(x)p(y)|k(x, y)|^2dydx\right]^{\frac{1}{2}}$.

The simplicity of finding a solution for Fredholm integral equations (FIEs) of the second kind with degenerate kernel naturally leads one to think of replacing the given equation (1) by FIE with degenerate kernel, see [1, 2, 8, 9]. The solution of the new equation is taken as an approximate solution of the original equation. The study employs Dzyadyk’s method which is based on the linear polynomial operator ((3)-[5]).

Eq.(1) can be written in the new form $\varphi(x) = f(x) + \lambda \int_0^x k(x, y)\varphi(y)dy,$ (2)
where

\[ \tilde{k}(x, y) = e(x, y)k(x, y), \quad e(x, y) = \begin{cases} 1, & \text{for } y \leq x, \\ 0, & \text{for } y > x. \end{cases} \]

(3)

From (3), it is found that the kernel \( \tilde{k}(x, y) \) in (2) satisfies the following conditions (A*)

1. \( \int_{0}^{2\pi} p(y)|\tilde{k}(x, y)|^2dy \frac{1}{2} = \rho(x) \in L_{p(x)}^2[0, 2\pi] \);

2. \( |\lambda||\tilde{k}(x, y)||_{L_{p}^{2}} < 1 \),

where

\[ ||\tilde{k}(x, y)||_{L_{p}^{2}[0,2\pi]} = \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)|\tilde{k}(x, y)|^2dydx \right]^{\frac{1}{2}}. \]

Now, instead of Eqs.(1) and (2), let us solve the following equations

\[ \varphi_n(x) = U_n(f; x) + \lambda \int_{0}^{2\pi} U_n[\tilde{k}(., y); x]\varphi_n(y)dy, \quad 0 \leq x, y \leq 2\pi, \]

(4)

The notation \( U_n[\tilde{k}(., y); x] \) will mean that the operator \( U_n \) acts on \( \tilde{k}(x, y) \) as a function of \( x \) and at the same time, the variable \( y \) plays the role of the parameter.

Now, since the functions \( U_n(f; x) \) and \( U_n[\tilde{k}(., y); x] \) are both trigonometric polynomials of order \( n \) with respect to \( x \), the solution \( \varphi_n(x) \) of the Eq.(4) will also be trigonometric polynomial of order \( n \) in \( x \). It is well known that the problem of determination of the solution of Fredholm integral equation of the second kind with degenerate kernel is reduced to the solution of corresponding system of algebraic equations [11]. In this study, it will be proved that the function \( \varphi_n(x) \) is a good approximation to the solution \( \varphi(x) \) of Eq.(1) on the space \( L_{p(x)}^2[0, 2\pi] \). This approximation is discussed in details for Vallée-Poussin’s and Féjer’s operators.

2. Preliminaries

Starting from the known linear polynomial operators \( U_n(g; x) \) which are good approximation to the function \( g(x) \) in the space \( L_{p(x)}^2 \), and have the form:

\[ U_n(g; x) = \frac{1}{\pi} \int_{0}^{2\pi} g(t)U_n(x - t)dt = \frac{1}{\pi} \int_{0}^{2\pi} g(x - t)U_n(t)dt, \]

(5)

where

\[ U_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos(kx), \]

(6)

\( \lambda_k^{(n)} \) are constants which define the method of approximation.

**Theorem 1.** [6]

For \( k(x, y) \) belongs to \( L_{p(x)}^2[0, 2\pi] \), such that \( |\lambda||\tilde{k}(x, y)||_{L_{p}^{2}} < 1 \), and \( f(x) \) belongs to \( L_{p(x)}^2 \), then the integral equation

\[ \varphi(x) = f(x) + \lambda \int_{0}^{2\pi} k(x, y)\varphi(y)dy, \]

has an unique solution \( \varphi(x) \) in \( L_{p(x)}^2[0, 2\pi] \).
Now, with the help of the following theorem we will find the condition by which the equation (4) has an unique solution.

Theorem 2. [6]

If $A$ and $B$ are two bounded linear operators in Banach space $E$, while $A$ has an inverse and $\|B\|_E \|A^{-1}\|_E < 1$, then the operator $(A + B)$ has also an inverse and
\[
\|(A + B)^{-1}\|_E \leq \|A^{-1}\|_E (1 - \|B\|_E \|A^{-1}\|_E)^{-1}.
\]
To find this condition, we write both of Eqs.(2) and (4) in the operator form
\[
(I - \lambda \tilde{K}) \varphi = f, \quad (I - \lambda U_n(\tilde{K})) \varphi_n = f_n,
\]
where
\[
\tilde{K} \varphi = \int_{-\pi}^{\pi} \tilde{k}(x, y) \varphi(y) dy, \quad U_n(\tilde{K}) \varphi_n = \int_{-\pi}^{\pi} U_n[\tilde{k}(\cdot, y); x] \varphi_n(y) dy.
\]
It is obvious that $I - \lambda \tilde{K} = A$, $\lambda(\tilde{K} - U_n(\tilde{K})) = B$, are two bounded linear operators in the space $L^2_{p(x)}$.
It is well-known that the operator $I - \lambda \tilde{K}$ has an inverse for each $\lambda$ such that $\frac{1}{\lambda}$ is a regular value of $\tilde{K}$ [6]. So Eq.(2) has an unique solution and we can write
\[
\varphi = (I + \lambda R) f = f + \lambda R f,
\]
where $(I - \lambda \tilde{K})^{-1} = (I + \lambda R)$ and $R$ is the resolvent of the operator $\tilde{K}$. From theorem 2 if $|\lambda| \|(I - \lambda \tilde{K})^{-1}\|_E \|\tilde{K} - U_n(\tilde{K})\|_E < 1$, then $(I - \lambda U_n(\tilde{K}))$ has also an inverse, thereby Eq.(4) has an unique solution and can be written in the form
\[
\varphi_n = (I + \lambda R_n) f_n = f_n + \lambda R_n f_n,
\]
where $(I - \lambda U_n(\tilde{K}))^{-1} = I + \lambda R_n$ and $R_n$ is the resolvent of the operator $U_n(\tilde{K})$.

Now, we return to the functional representation of resolvents $R(x, y; \lambda); R_n(x, y; \lambda)$ and equations (2) and (4). Knowing the resolvent $R(x, y; \lambda)$, we at once obtain the solution of the original equation (2) with an arbitrary right hand side $f(x)$ in the following form
\[
\varphi(x) = f(x) + \lambda \int_{0}^{2\pi} R(x, y; \lambda) f(y) dy.
\]
Also, the solution of Eq.(4) can be represented through the resolvent as follows
\[
\varphi_n(x) = f_n(x) + \lambda \int_{0}^{2\pi} R_n(x, y; \lambda) f_n(y) dy.
\]
Theorem 3.

For any kernel \( k(x, y) \in L^2_p[0, 2\pi] \), if the linear polynomial operator \( U_n \) of order \( n \) is defined in \( L^2_{p(x)} \) and if the function \( f(x) \in L^2_{p(x)} \), then

\[
U_n \left[ \int_a^b k(., y)f(y)dy; x \right] = \int_a^b U_n[k(., y); x]f(y)dy.
\]

The proof of this theorem is very similar to the proof of a theorem in [4].

3. Auxiliary definitions and theorems

Definition 1.

The averaged-modulus of continuity of the kernel \( k(x, y) \in L^2_p[0, 2\pi] \) is defined as follows

\[
w_{L^2_p}(k; t) = w_{L^2_p}(t) = \frac{1}{2\pi} \sup_{|s| \leq t} \left[ \int_0^{2\pi} \int_0^{2\pi} p(x)p(y)[k(x-s, y) - k(x, y)]^2 dx dy \right]^\frac{1}{2}.
\]

(7)

Lemma 1.

The function \( w_{L^2_p}(t) \) has the following properties:

1. \( w_{L^2_p}(t) \to 0 \) for \( t \to 0 \);
2. \( w_{L^2_p}(t) \) is positive and monotonic increasing;
3. \( w_{L^2_p}(t_1 + 2) \leq w_{L^2_p}(t_1) + w_{L^2_p}(t_2) \);
4. \( w_{L^2_p}(t) \) is continuous;
5. for any positive real number \( \eta \), the following inequality holds \( w_{L^2_p}(\eta t) \leq (1 + \eta)w_{L^2_p}(t) \).

Also, by the averaged-modulus of continuity with respect to \( x \) and \( y \) of a function \( \tilde{k}(x, y) = e(x, y)k(x, y) \) defined in \([0, 2\pi]\), we mean the following function \( \Omega_{L^2_p}(t) \)

\[
\Omega_{L^2_p}(k; t) = \Omega_{L^2_p}(t) = \frac{1}{2\pi} \sup_{|s| \leq t} \left[ \int_0^{2\pi} \int_0^{2\pi} p(x)p(y)[\tilde{k}(x-s, y) - \tilde{k}(x, y)]^2 dx dy \right]^\frac{1}{2}.
\]

(8)

It is evident that the function \( \Omega_{L^2_p}(t) \) satisfies the above properties of the modulus of continuity (1-5).

Definition 2.

The value of the following norm

\[
\delta_n(\tilde{k}) = \delta(\tilde{k}; U_n) = \|U_n(\tilde{k}(., y); x) - \tilde{k}(x, y)\|_{L^2_p} = \left[ \int_0^{2\pi} \int_0^{2\pi} p(x)p(y)[U_n(\tilde{k}(., y); x) - \tilde{k}(x, y)]^2 dx dy \right]^\frac{1}{2},
\]

(9)

will play an important role in estimating the error arising from replacement of Eq.(1) by Eq.(4). The following theorem provides an estimate of \( \delta(\tilde{k}, U_n) \).
Theorem 4.
For any kernel \( \tilde{k}(x, y) \in L_p^2[0, 2\pi] \), and for any linear polynomial operator \( U_n(g; x) \), we always have the following inequality

\[
\delta_n(\tilde{k}) \leq 2 \left[ w_{L_p^2} \left( \frac{1}{n} \right) + \Omega_{L_p^2} \left( \frac{1}{n} \right) \right] \int_{-\pi}^{\pi} |n|t + 1)|U_n(t)|dt. \tag{10}
\]

Proof. Using Minkowski inequality and equalities (5) and (7), we obtain

\[
\delta_n(\tilde{k}) = \|U_n(\tilde{k}(., y); x) - \tilde{k}(x, y)\|_{L_p^2} = \\
= \frac{1}{\pi} \left\| \int_{-\pi}^{\pi} [\tilde{k}(x - t, y) - \tilde{k}(x, y)]U_n(t)dt \right\|_{L_p^2} \\
= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} p(x)p(y) \left[ \int_{-\pi}^{\pi} U_n(t)(\tilde{k}(x - t, y) - \tilde{k}(x, y)) \right]^2 dy \right] dx \right]^\frac{1}{2} \\
\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |U_n(t)| \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)[\tilde{k}(x - t, y) - \tilde{k}(x, y)]^2 dy \right] dx \right]^\frac{1}{2} dt \\
\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |U_n(t)| \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)[e(x - t, y)k(x - t, y) - e(x, y)k(x, y)]^2 dy dx \right] dx \right]^\frac{1}{2} dt \\
\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |U_n(t)| \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)k(x, y)[e(x - t, y) - e(x, y)]^2 dy dx \right] dx \right]^\frac{1}{2} dt \\
+ \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)e(x - t, y)[k(x - t, y) - k(x, y)]^2 dy dx \right] dx \right]^\frac{1}{2} dt \\
\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |U_n(t)| \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)k(x, y)[e(x - t, y) - e(x, y)]^2 dy dx \right] dx \right]^\frac{1}{2} dt \\
+ \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} p(x)p(y)e(x - t, y)[k(x - t, y) - k(x, y)]^2 dy dx \right] dx \right]^\frac{1}{2} dt \\
\leq 2 \int_{-\pi}^{\pi} |U_n|[w_{L_p^2}(t) + \Omega_{L_p^2}(t)]dt \leq \\
\leq 2 \left[ w_{L_p^2} \left( \frac{1}{n} \right) + \Omega_{L_p^2} \left( \frac{1}{n} \right) \right] \int_{-\pi}^{\pi} |n|t + 1)|U_n(t)|dt.
\]

Definition 3.
We define the error of approximation of \( \tilde{k}(x, y) \) as follows

\[
E_{n,m}^*(\tilde{k})_{L_p^2} = \|\tilde{k}(x, y) - T_{n,m}(x, y)\|_{L_p^2} \\
= \inf_{T_{n,m}(x, y)} \left[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p(x)p(y)[\tilde{k}(x, y) - T_{n,m}(x, y)]^2 dxdy \right]^\frac{1}{2},
\]

\[
E_{n,\infty}^*(\tilde{k})_{L_p^2} = \|\tilde{k}(x, y) - T_{n,\infty}(x, y)\|_{L_p^2} \\
= \inf_{T_{n,\infty}(x, y)} \left[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p(x)p(y)[\tilde{k}(x, y) - T_{n,\infty}(x, y)]^2 dxdy \right]^\frac{1}{2},
\]

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\[ E_{\infty,m}^*(\tilde{k})_{L_p^2} = \| \tilde{k}(x,y) - T_{\infty,m}^*(x,y) \|_{L_p^2} \]

where \( T_{n,m}^*(x,y) \) denotes the trigonometric polynomial in \( x \) of order \( n \) and in \( y \) of order \( m \) of best approximation of \( \tilde{k}(x,y) \) in the metric \( L_p^2[0,2\pi] \), \( T_{n,\infty}^*(x,y) \) denotes the trigonometric polynomial in \( x \) of order \( n \) of best approximation of \( \tilde{k}(x,y) \) in the metric \( L_p^2[0,2\pi] \), \( T_{\infty,m}^*(x,y) \) denotes the trigonometric polynomial in \( y \) of order \( m \) of best approximation of \( \tilde{k}(x,y) \) in the metric \( L_p^2[0,2\pi] \). The estimates of how rapidly the quantities \( E_{n,m}^*(\tilde{k})_{L_p^2} \), \( E_{n,\infty}^*(\tilde{k})_{L_p^2} \) and \( E_{\infty,m}^*(\tilde{k})_{L_p^2} \) tend to zero as \( n \to \infty, m \to \infty \) are given in [10], where

\[
E_{n,m}^*(\tilde{k})_{L_p^2} \to 0, \quad n, m \to \infty, \\
E_{n,m}^*(\tilde{k})_{L_p^2} \geq E_{n,\infty}^*(\tilde{k})_{L_p^2}, \quad E_{n,m}^*(\tilde{k})_{L_p^2} \geq E_{\infty,m}^*(\tilde{k})_{L_p^2}
\]

then

\[
E_{n,\infty}^*(\tilde{k})_{L_p^2} \to 0, \quad as \quad n \to \infty, \\
E_{\infty,m}^*(\tilde{k})_{L_p^2} \to 0, \quad as \quad m \to \infty.
\]

Now, we will mention the bounds of the norm (9) for various linear polynomial operators \( U_n \) as the following cases:

**Case 1: Vallee-Poussin’s method [5]:**

\( U_n = V_n \), we have

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} |V_n(t)| dt \leq \frac{1}{3} + \frac{2\sqrt{3}}{\pi},
\]

from Eq.(10) and definition 3, we get

\[
E_{n,\infty}^*(\tilde{k})_{L_p^2} \leq 12\pi \left[ w_{L_p^2}\left(\frac{1}{n}\right) + \Omega_{L_p^2}\left(\frac{1}{n}\right) \right].
\]

By using the inequality (13) and considering that the method of Vallee-Poussin’s \( V_n \) leaves trigonometric polynomial of order \( n \) invariant, then

\[
\delta_n(\tilde{k}; V_n) = \| \tilde{k}(x,y) - V_n(\tilde{k}(., y); x) \|_{L_p^2}
\]

\[
= \| \tilde{k}(x,y) - T_{n,\infty}^*(x,y) - V_n[\tilde{k}(., y) - T_{n,\infty}^*(., y); x] \|_{L_p^2}
\]

\[
\leq E_{n,\infty}^*(\tilde{k})_{L_p^2} + \frac{1}{\pi} \int_{-\pi}^{\pi} |V_n(t)| \left[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p(x)p(y)[\tilde{k}(x-y, t) - T_{n,\infty}^*(x-t, y)]^2 dy dx \right]^{\frac{1}{2}} dt
\]

\[
\leq \left[ 1 + \frac{1}{\pi} \int_{-\pi}^{\pi} |V_n(t)| dt \right] E_{n,\infty}^*(\tilde{k})_{L_p^2} \approx 2.436 E_{n,\infty}^*(\tilde{k})_{L_p^2},
\]

and from (14) we get

\[
\delta_n(\tilde{k}; V_n) \leq 29.232\pi \left[ w_{L_p^2}\left(\frac{1}{n}\right) + \Omega_{L_p^2}\left(\frac{1}{n}\right) \right].
\]
Case 2: Féjer’s method [5]:

\[ U_n = F_n, \]

we have

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} |F_n(t)| dt = 1, \]  

(17)

\[ \int_{-\pi}^{\pi} (1 + n|t|) |F_n(t)| dt < 6(1 + \ln n), \quad \forall n \geq 3. \]  

(18)

We let \( n' = \frac{\sqrt{n}}{2}, a_i(y), b_i(y), a_i^*(y) \) and \( b_i^*(y) \) denote the corresponding coefficients of Fourier series in the variable \( x \) of the functions \( \tilde{k}(x, y) \) and \( V_{n'}(\tilde{k}(., y); x) \). Then,

\[
\| V_{n'}(\tilde{k}(., y); x) - F_n[V_{n'}(\tilde{k}(., y); x)] \|_{L_p^2} \\
= \left\| \sum_{i=1}^{2n'} \frac{1}{i} \left[ a_i^*(y) \cos ix + b_i^*(y) \sin ix \right] \right\|_{L_p^2} \\
\leq \left\| \sum_{i=1}^{2n'} \left( \frac{i}{n} \right)^2 \left[ \sum_{i=1}^{2n'} \left( a_i^*(y) \cos ix + b_i^*(y) \sin ix \right)^2 \right]^{\frac{1}{2}} \right\|_{L_p^2} \\
\leq \left[ \sum_{i=1}^{2n'} \left( \frac{i}{n} \right)^2 \right]^{\frac{1}{2}} \left( \sum_{i=1}^{2n'} \left[ (a_i^2(y) + b_i^2(y)) \right] \right)^{\frac{1}{2}} \left\| \tilde{k}(x, y) \right\|_{L_p^2} \\
\leq \frac{1}{\sqrt{\pi n}} \left( \sum_{i=1}^{2n'} i^2 \right)^{\frac{1}{2}} \left\| \tilde{k}(x, y) \right\|_{L_p^2} \leq \frac{1}{\sqrt{\pi n}} (2n')^{\frac{3}{2}} \left\| \tilde{k}(x, y) \right\|_{L_p^2} \\
\leq \frac{1}{\sqrt{\pi n^4}} \left\| \tilde{k}(x, y) \right\|_{L_p^2}.
\]

Thereby

\[
\delta(\tilde{k}; F_n) = \left\| \tilde{k}(x, y) - F_n(\tilde{k}(., y); x) \right\|_{L_p^2} \\
= \left\| \tilde{k}(x, y) - V_{n'}(\tilde{k}(., y); x) + V_{n'}(\tilde{k}(., y); x) - F_n[V_{n'}(\tilde{k}(., y); x)] + F_n(\tilde{k}(., y); x) \right\|_{L_p^2} \\
\leq \left\| \tilde{k}(x, y) - V_{n'}(\tilde{k}(., y); x) \right\|_{L_p^2} + \left\| F_n[V_{n'}(\tilde{k}(., y); x)] - F_n(\tilde{k}(., y); x) \right\|_{L_p^2} \\
\leq \left( 1 + \frac{1}{\pi} \int_{-\pi}^{\pi} |F_n(t)| dt \right) (2.5) E_{n', \infty}^r(\tilde{k})_{L_p^2} + \frac{1}{\sqrt{\pi n^4}} \left\| \tilde{k}(x, y) \right\|_{L_p^2},
\]

(19)

from Eqs.(17) and (19), we get

\[
\delta(\tilde{k}; F_n) \leq 5E_{n', \infty}^r(\tilde{k})_{L_p^2} + \frac{1}{\sqrt{\pi n^4}} \left\| \tilde{k}(x, y) \right\|_{L_p^2}.
\]

(20)

Also, from Eqs.(18) and (10), we have

\[
\delta(\tilde{k}; F_n) \leq 12(1 + \ln n) [w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})].
\]

(21)

Now from (16), (20) and (21) it is clear that \( \delta_n(\tilde{k}) \to 0 \) as \( n \to \infty \) for Valleé-Poussin’s and Féjer’s methods for every periodic function \( \tilde{k}(x, y) \in L_p^2[0, 2\pi], w_{L_p^2}(\frac{1}{n}) = o(1/\ln n) \) and \( \Omega_{L_p^2}(\frac{1}{n}) = o(1/\ln n) \).
Definition 4.

The following quantities will play an important role in estimating the error of our approximation

\[ \xi(\tilde{k}; U_n; \varphi) = \xi_n = \left\| \int_0^{2\pi} \tilde{k}(x, y)[\varphi(y) - U_n(\varphi; y)]dy \right\|_{L_p^2}, \]  
(22)

\[ \gamma_m = \gamma_m(U_n; \varphi) = \sum_{i=1}^{m} |1 - \lambda_i^{(n)}| E_{i-1}(\varphi)_{L_p^2}, \]  
(23)

where

\[ E_n(\varphi)_{L_p^2} = \inf_{T_n} \| \varphi(x) - T_n(x) \|_{L_p^2}, \]

\( T_n(x) \) is a trigonometric polynomial of order \( n \) in \( x, m \leq n. \)

Theorem 5.

For any kernel \( \tilde{k}(x, y) \in L_p^2[0, 2\pi] \) and for linear polynomial operator \( U_n(g; x) \) the following inequality holds

\[ \xi_n(\tilde{k}) = \xi_n(\tilde{k}; U_n; \varphi) = \left\| \int_0^{2\pi} \tilde{k}(x, y)[\varphi(y) - U_n(\varphi; y)]dy \right\|_{L_p^2} \]

\[ \leq E_{\infty,m}^*(\tilde{k})_{L_p^2} \| \varphi(y) - U_n(\varphi; y) \|_{L_p^2} + \sqrt{\frac{2}{\pi}} \gamma_m(U_n; \varphi) \left[ \int_0^{2\pi} p(x)dx \right]^\frac{1}{2} \left[ \| \tilde{k}(x, y) \|_{L_p^2} + E_{\infty,m}^*(\tilde{k})_{L_p^2} \right], \]  
(24)

for any positive integer \( m \leq n. \)

Proof. For any function \( \varphi(x) \in L_p^2 \) with Fourier coefficients \( c_i \) and \( d_i \) in view of Bunyakovskii inequality and \( p(x) \geq 1 \), we obtain

\[ |c_i \cos ix + d_i \sin ix| = \inf_{T_{i-1}(t) \pi} \frac{1}{\pi} \left| \int_{-\pi}^{\pi} [\varphi(t) - T_{i-1}(t)] \cos[(i(x - t))dt] \right| \]

\[ \leq \frac{1}{\pi} \inf_{T_{i-1}(t) \pi} \left[ \int_{-\pi}^{\pi} p(t)|\varphi(t) - T_{i-1}(t)|dt \right]^\frac{1}{2} \left[ \frac{\int_{-\pi}^{\pi} \cos[(i(x - t))]^2}{p(t)}dt \right]^\frac{1}{2} \]

\[ \leq \sqrt{\frac{2}{\pi}} \inf_{T_{i-1}(t) \pi} \left[ \int_{-\pi}^{\pi} p(t)|\varphi(t) - T_{i-1}(t)|^2dt \right]^\frac{1}{2} \]

\[ \leq \sqrt{\frac{2}{\pi}} E_{i-1}^*(\varphi)_{L_p^2}, \]

therefore

\[ \| c_i \cos ix + d_i \sin ix \|_{L_p^2} \leq \sqrt{\frac{2}{\pi}} E_{i-1}^*(\varphi)_{L_p^2} \left( \int_{-\pi}^{\pi} p(x)dx \right)^\frac{1}{2}. \]

Letting

\[ T_{\infty,m}(x, y) = \sum_{i=0}^{m} a_i(x) \cos iy + b_i(x) \sin iy, \]
and taking into consideration (23) and using Bunyakovskii inequality, we obtain

\[
\delta_n = \delta_{n}(\tilde{k}; U_n; \varphi) = \left\| \int_{0}^{2\pi} \tilde{k}(x, y)[\varphi(y) - U_n(\varphi; y)]dy \right\|_{L^2_p}
\]

\[
= \left[ \int_{0}^{2\pi} p(x) \left[ \int_{0}^{2\pi} \tilde{k}(x, y)[\varphi(y) - U_n(\varphi; y)]dy \right] dx \right]^{\frac{1}{2}}
\]

\[
\leq \left[ \int_{0}^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \left[ \int_{0}^{2\pi} |\tilde{k}(x, y) - T_{\infty,m}(x, y)||\varphi(y) - U_n(\varphi; y)|dy \right] dx \right]^{\frac{1}{2}}
\]

\[
+ \left[ \int_{0}^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \left[ \int_{0}^{2\pi} (\tilde{k}(x, y) + T_{\infty,m}(x, y) - \tilde{k}(x, y))(\varphi(y) - U_n(\varphi; y))dy \right] dx \right]^{\frac{1}{2}}
\]

\[
\leq E^*_{\infty,m}(\tilde{k})L^2_p \|\varphi(y) - U_n(\varphi; y)\|_{L^2_p}
\]

\[
+ \sqrt{\frac{2}{\pi}} \gamma_m(U_n; \varphi) \left[ \int_{0}^{2\pi} p(x)dx \right]^{\frac{1}{2}} \left[ \|\tilde{k}(x, y)\|_{L^2_p} + E^*_{\infty,m}(\tilde{k})L^2_p \right].
\]

\[\square\]

4. The approximate solution and its error bounds

The following theorem shows that for sufficiently good linear methods $U_n(g; x)$, the difference between the polynomials $\varphi_n(x)$ and the original solution $\varphi(x)$ is sufficiently small.

Theorem 6.

If the kernel $\tilde{k}(x, y)$ in Eq.(2) satisfies the assumptions ($A^*$), all functions appearing in (2) are $2\pi$—periodic in $x$ and $y$, then any linear polynomial operator $U_n(g; x)$, if $|\lambda|R\delta(\tilde{k}; U_n) < 1$ and if Eq.(1) is replaced by Eq.(4), the following inequality holds

\[
\|\varphi(x) - \varphi_n(x)\|_{L^2_p} \leq (1 + \alpha_n(\tilde{k}))\|\varphi(x) - U_n(\varphi; x)\|_{L^2_p},
\]

in which

\[
\alpha_n(\tilde{k}) = |\lambda| R \left[ \delta(\tilde{k}; U_n) + \frac{\xi(\tilde{k}; U_n; \varphi)}{\|\varphi(x) - U_n(\varphi; x)\|_{L^2_p}} \right] /[1 - |\lambda|R\delta(\tilde{k}; U_n)],
\]

where $\delta(\tilde{k}; U_n)$ and $\xi(\tilde{k}; U_n; \varphi)$ are defined in (9) and (22), respectively, and $R = 1 + |\lambda||R(x, y)||_{L^2_p}$, where $R(x, y)$ denotes the resolvent of the kernel $\tilde{k}(x, y)$.
Proof. Using theorem 3, and Eq.(2), we represent the solution \( \varphi_n(x) \) of Eq.(4) in the form

\[
\varphi_n(x) = U_n(f; x) + \lambda U_n \left[ \int_0^{2\pi} \tilde{k}(., y) \varphi_n(y) dy \right]
\]

\[
= U_n(f; x) + \lambda U_n \left[ \int_0^{2\pi} \tilde{k}(., y)(\varphi_n(y) - \varphi(y)) dy + \int_0^{2\pi} \tilde{k}(., y) \varphi_n(y) dy \right]
\]

\[
= \lambda \int_0^{2\pi} U_n[\tilde{k}(., y); x] [\varphi_n(y) - \varphi(y)] dy + U_n \left[ f(\cdot) + \lambda \int_0^{2\pi} \tilde{k}(., y) \varphi(y) dy \right] x
\]

\[
= \lambda \int_0^{2\pi} U_n[\tilde{k}(., y); x] [\varphi_n(y) - \varphi(y)] dy + U_n(\varphi; x),
\]

it follows that

\[
\varphi_n(x) - U_n(\varphi; x) = \lambda \int_0^{2\pi} \tilde{k}(x, y) [\varphi_n(y) - U_n(\varphi; y)] dy + g_n(x),
\]

where

\[
g_n(x) = \lambda \int_0^{2\pi} [U_n(\tilde{k}(., y); x) - \tilde{k}(x, y)][\varphi_n(y) - \varphi(y)] dy + \lambda \int_0^{2\pi} \tilde{k}(x, y)[U_n(\varphi; y) - \varphi(y)] dy.
\]

Thus, by Eqs.(9), (10) and (22) we get the estimate

\[
\|g_n(x)\|_{L^2_p} \leq |\lambda| \left\| \int_0^{2\pi} [U_n(\tilde{k}(., y); x) - \tilde{k}(x, y)][\varphi_n(y) - \varphi(y)] dy \right\|_{L^2_p}
\]

\[
+ |\lambda| \left\| \int_0^{2\pi} \tilde{k}(x, y)[U_n(\varphi; y) - \varphi(y)] dy \right\|_{L^2_p}
\]

\[
\leq |\lambda| \delta(\tilde{k}; U_n) \left[ \|\varphi_n(x) - U_n(\varphi; x)\|_{L^2_p} + \|U_n(\varphi; x) - \varphi(x)\|_{L^2_p} \right] + |\lambda| \xi(\tilde{k}; U_n; \varphi).
\]

In view of \( |\lambda||\tilde{k}(x, y)||_{L^2_p} < 1 \), Eq.(28) has an unique solution given by

\[
\varphi_n(x) - U_n(\varphi; x) = g_n(x) + \lambda \int_0^{2\pi} R(x, y) g_n(y) dy.
\]

Therefore

\[
\|\varphi_n(x) - U_n(\varphi; x)\|_{L^2_p} \leq \|g_n(x)\|_{L^2_p} \left[ 1 + \|\lambda||R(x, y)||_{L^2_p} \right] = R\|g_n(x)\|_{L^2_p}
\]

\[
\leq R|\lambda| \left[ \delta(\tilde{k}; U_n) \|\varphi(x) - U_n(\varphi; x)\|_{L^2_p} + \|\varphi_n(x) - U_n(\varphi; x)\|_{L^2_p} + \xi(\tilde{k}; U_n; \varphi) \right].
\]

Taking into consideration \( |\lambda| R\delta(\tilde{k}; U_n) < 1 \), we obtain

\[
\|\varphi_n(x) - U_n(\varphi; x)\|_{L^2_p} \leq \frac{|\lambda|R[\delta(\tilde{k}; U_n)]\|U_n(\varphi; x) - \varphi(x)\|_{L^2_p} + \xi(\tilde{k}; U_n; \varphi)}{1 - |\lambda| R\delta(\tilde{k}; U_n)}
\]

Therefore

\[
\|\varphi(x) - \varphi_n(x)\|_{L^2_p} \leq \|\varphi(x) - U_n(\varphi; x)\|_{L^2_p} + \|\varphi_n(x) - U_n(\varphi; x)\|_{L^2_p}
\]

\[
\leq \|\varphi(x) - U_n(\varphi; x)\|_{L^2_p} + \frac{|\lambda|R[\delta(\tilde{k}; U_n)]\|U_n(\varphi; x) - \varphi(x)\|_{L^2_p} + \xi(\tilde{k}; U_n; \varphi)}{1 - |\lambda| R\delta(\tilde{k}; U_n)}
\]

\[
\leq (1 + \alpha_n(\tilde{k}))\|\varphi(x) - U_n(\varphi; x)\|_{L^2_p},
\]

where \( \alpha_n \) is given by (26). Thus, the inequality (25) is proved. \( \square \)
5. The results

It is well-known that in [7], one cannot achieve an error less than the corresponding to the best approximation. The error estimate in (25) with rate of convergence $\alpha_n(\tilde{k})$, means that, the rate of convergence of $\phi_n(x)$ to $\phi(x)$ is comparable with the rate of convergence of the best approximation, which means that the error estimate (25) is optimal. Applying theorem 6, and also the corresponding results from section 3, we obtain the following results:

In the case of the application of Vallee-Poussin’s method:

From [10] and (25) we obtain
\[
\|\phi(x) - \phi_n(x)\|_{L^p} \leq (1 + \alpha_n(\tilde{k}))(\frac{4}{3} + \frac{2\sqrt{3}}{\pi})E_n^*(\phi)_{L^p} \leq (1 + \alpha_n(\tilde{k}))(2.5)E_n^*(\phi)_{L^p},
\]
where by (15) we have
\[
\alpha_n(\tilde{k}) \leq |\lambda|R\frac{2.5E_n^{*,\infty}(\tilde{k})_{L^p} + E_n^{*,\infty}(\tilde{k})_{L^p}}{1 - \lambda R(2.5)E_n^{*,\infty}(\tilde{k})_{L^p}},
\]
then $\alpha_n(\tilde{k}) \to 0$ as $n \to \infty$ for all $\phi(x) \in L^2_{p(x)}$, $\tilde{k}(x,y) \in L^2_{p(x)}[0,2\pi]$.

In the case of the application of Féjer’s method:

The quantity $\alpha_n(\tilde{k})$ in the relation (25) will not tend to zero for any solution $\phi(x)$, but will tend to zero only under the condition that "the solution $\phi(x)$ belongs to some subclasses of integrable functions". Restricting ourselves to the Holder classes $W^{(r)}H^\beta(L^p_{n})$ where $r$ is a non-negative integer and $0 < \beta \leq 1$, we obtain the following case:

In order that $\alpha_n(\tilde{k}) \to 0$ as $n \to \infty$ considering (20), (21) and [10], it is sufficient that the following conditioned be satisfied
\[
\phi(x) \in W^{(0)}H^\beta(L^2_{p}), \quad \text{i.e.} \quad r = 0, \quad 0 < \beta \leq 1, \quad w(\frac{1}{n})_{L^p} = o(1/\ln n), \quad \Omega(\frac{1}{n})_{L^p} = o(1/\ln n).
\]

6. Conclusion and remarks

In this article, we presented the approximate solutions of the Volterra integral equations of the second kind in the space $L^2_{p(x)}[0,2\pi]$ with weight function $p(x)$ with the help of the Vallee-Poussin’s and Féjer’s operators. In the same time, we proved that the function $\phi_n(x)$ is a good approximation to the exact solution $\phi(x)$ for the Volterra integral equations. From the obtained approximate solutions using ADM, we can conclude that the proposed approach is easy to implement and computationally very attractive. A good agreement between the theoretical study with the obtained approximate solutions have been obtained.
References


A Modified SSDP Method for Nonlinear Semidefinite Programming*

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Abstract In this paper, we investigate nonlinear semidefinite programming and propose a modified sequential semidefinite programming (SSDP for short) algorithm without a penalty function or a filter. At each iteration, the search direction is yielded by solving a linear semidefinite programming subproblem and a quadratic semidefinite programming subproblem. The nonmonotone line search ensures that the objective function or constraint violation function is sufficiently reduced. Under some appropriate conditions, the global convergence of the proposed algorithm is shown. Some preliminary numerical results are reported.

Key words nonlinear semidefinite programming; sequential semidefinite programming; nonmonotone line search; global convergence

1 Introduction

Consider the following nonlinear semidefinite programming (NLSDP):

\[
\min f(x) \\
\text{s.t. } G(x) \preceq 0, \tag{1.1}
\]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is assumed to be a smooth and real value function, \( G : \mathbb{R}^n \rightarrow \mathbb{S}^m \) is a smooth and matrix value function. \( \mathbb{S}^m \) represents the set of all real symmetric matrices. The symbol \( A \preceq B \) means that \( A - B \) is a negative semidefinite matrix.

Nonlinear semidefinite programming has many real-world applications, such as engineering design, optimal structure design, optimal robust control and robust feedback control design (see [1]-[4]). In recent years, the investigation of NLSDP has attracted much attention. The main solution methods for NLSDP are augmented Lagrange method [5]-[10], interior point method [11]-[15], SSDP method [16]-[21]. In this paper, our focus is on SSDP method. Correa and Ramirez in [16] proposed an SSDP algorithm. At each iteration, the search direction is generated by solving a traditional quadratic semidefinite programming (QSDP for short) subproblem. A subdifferentiable penalty function is used as a merit function to design line search. Under some conditions, the algorithm is globally convergent. However, it is not easy for the choice of an appropriate penalty parameter. Gomez in [17] proposed a filter-type SSDP algorithm for nonlinear semidefinite programming problem. For each iteration point, by solving a trust-region type QSDP subproblem

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to get search direction. When objective function value or the constraint violation function is improved, the trial point is accepted by filter. Chen in [21] proposed a trust region SSDP method without a penalty function or a filter. The search direction is obtained by solving trust region QS-DP subproblem. Whether the trial point is accepted or not depends on the decline of the objective function or constraint violation function.

In all above SSDP algorithms, the traditional QSDP subproblem, which generated the search direction, may be incompatible. Motivated by the idea of modified SQP methods for nonlinear programming, in this paper, we proposed a modified SSDP algorithm for NLSDP (1.1). At each iteration, the search direction is yielded by solving a linear semidefinite programming (LSDP for short) subproblem and a modified QSDP subproblem. Nonmonotone line search technique is used to determine step size.

The paper is organized as follows. In the next section, the algorithm is described in detail. The global convergence is shown in Section 3. Some preliminary numerical results are reported in Section 4 and some concluding remarks are given in the final section.

2 Description of Algorithm

In this section, we first restate some concepts and notations about nonlinear semidefinite programming, and then describe the proposed algorithm.

Let \( G(x) : \mathbb{R}^n \to \mathbb{S}^m \) be a matrix value function, we use the notation

\[
DG(x) = \left( \frac{\partial G(x)}{\partial x_1}, \ldots, \frac{\partial G(x)}{\partial x_n} \right)^T
\]

for its differential operator evaluated at \( x \). For any \( d = (d_1, \ldots, n) \in \mathbb{R}^n \), \( DG(x)d \) is defined by

\[
DG(x)d = \sum_{i=1}^{n} d_i \frac{\partial G(x)}{\partial x_i}.
\]

The adjoint operator \( DG(x)^* \) of the linear operator \( DG(x) \) satisfies

\[
DG(x)^*Y = \left( < \frac{\partial G(x)}{\partial x_1}, Y >, < \frac{\partial G(x)}{\partial x_2}, Y >, \ldots, < \frac{\partial G(x)}{\partial x_n}, Y > \right)^T, \quad \forall Y \in \mathbb{S}^m.
\]

where \(< A, B >\) means the inner product of the matrix \( A \) and \( B \).

**Definition 2.1** [16] Let \( \bar{x} \in \mathbb{R}^n \) be a feasible point of NLSDP (1.1), if there exists \( \bar{Y} \in \mathbb{S}^m \) satisfying the following KKT conditions

\[
\nabla_x L(\bar{x}, \bar{Y}) = \nabla f(\bar{x}) + DG(\bar{x})^*\bar{Y} = 0,
\]

\[
\bar{Y} \succeq 0, \quad < G(\bar{x}), \bar{Y} > = 0,
\]

where \( L : \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R} \) is the Lagrangian function of NLSDP (1.1), that is,

\[
L(x, \lambda, Y) = f(x) + < Y, G(x) >,
\]

then \( \bar{x} \) is called a KKT point of NLSDP (1.1), the matrix \( \bar{Y} \) is called a Lagrangian multiplier associated with \( \bar{x} \).
Let \( x^k \in \mathbb{R}^n \) be the current iterate point. In order to generate search directions, we borrow the ideas in [22] and construct the following linear semidefinite programming (LSDP \((x^k)\) for short):

\[
\begin{align*}
\min & \quad z \\
\text{s.t.} & \quad G(x^k) + DG(x^k)d \preceq zI_m, \\
& \quad z \geq 0,
\end{align*}
\] (2.6)

where \( I_m \) is the \( m \) order identity. Obviously, the feasible set of LSDP\((x^k)\) (2.6) is not empty, so there exists an optimal solution of (2.6). Let \((\hat{d}^k, z_k)^T\) be an optimal solution of (2.6), then we construct a quadratic semidefinite programming (QSDP \((x^k, H_k)\) for short) as follows:

\[
\begin{align*}
\min_{d \in \mathbb{R}^n} & \quad \nabla f(x^k)^T d + \frac{1}{2} d^T H_k d \\
\text{s.t.} & \quad G(x^k) + DG(x^k)d \preceq z_k I_m.
\end{align*}
\] (2.7)

If \( H_k \) is a symmetric positive definite matrix, then the solution of QSDP\((x^k, H_k)\) (2.7) is unique.

To measure the degree of feasibility at the iterate point, we define the degree of constraint violation as follows:

\[
h(x) = (\lambda_1(G(x)))_+, \] (2.8)

where \( \lambda_1(\cdot) \) is the largest eigenvalue of a matrix, \((\alpha)_+ = \max\{0, \alpha\}\). Obviously, \( h(x) = 0 \) is equivalent with that \( x \) is a feasible point of NLSDP (1.1).

Let \( d^k \) be the solution of QSDP\((x^k, H_k)\) (2.7). Similar to the idea of filter method, we hope that the search direction \( d^k \) can improve the feasibility of the iterate point or the value of the objective function. In other words, if \( d^k \) satisfies

\[
\nabla f(x^k)^T d^k \leq -\frac{1}{2} (d^k)^T H_k d^k, \] (2.9)

and \( t \) satisfies

\[
\begin{align*}
f(x^k + td^k) & \leq \max_{0 \leq j \leq m(k)} \{f(x^{k-j})\} - t\alpha (d^k)^T H_k d^k, \\
h(x^k + td^k) & \leq \beta \max_{0 \leq j \leq m(k)} \{h(x^{k-j})\},
\end{align*}
\] (2.10)

where \( \alpha \in (0, \frac{1}{2}) \), \( m(0) = 0 \), \( m(k) = \min\{m(k - 1) + 1, M\} \), \( M \) is a positive integer, then the corresponding trial step \( x^k + td^k \) is accepted.

If \( d^k \) does not satisfy (2.9), that is,

\[
\nabla f(x^k)^T d^k > -\frac{1}{2} (d^k)^T H_k d^k, \] (2.12)

then let \( t = 1 \). If the following inequality

\[
h(x^k + d^k) \leq \beta \max_{0 \leq j \leq m(k)} \{h(x^{k-j})\} \] (2.13)

hold, then the corresponding trial step \( x^k + d^k \) is accepted.

Based on the above strategy, we now present the new algorithm in detail.
Algorithm A

\textbf{S0.} Given $x^0 \in \mathbb{R}^n$, $H_0 = I_m$, $\alpha \in (0, \frac{1}{2})$, $\sigma \in (0, 1)$, $\beta \in (\frac{1}{2}, 1)$, $m(0) = 0$, a positive integer $M$. Let $k := 0$.

\textbf{S1.} Solve LSDP($x^k$) (2.6) to get a solution $(\hat{d}^k, z^k)^T$. If $\hat{d}^k = 0$ and $z^k \neq 0$, stop.

\textbf{S2.} Solve QSDP ($x^k, H_k$) (2.7) to get the solution $d^k$. If $d^k = 0$, stop.

\textbf{S3.} If $d^k$ satisfies (2.9), then let $t_k$ be the first number in the sequence of $\{1, \sigma, \sigma^2, \cdots\}$ satisfying the following inequality

$$f(x^k + td^k) \leq \max_{0 \leq j \leq m(k)} \{f(x^{k-j})\} - t\alpha(d^k)^TH_kd^k,$$

and go to S4; otherwise, let $t_k = 1$ and go to S4.

\textbf{S4.} Let $x^{k+1} = x^k + t_kd^k$. If the following inequality

$$h(x^{k+1}) \leq \beta \max_{0 \leq j \leq m(k)} \{h(x^{k-j})\},$$

holds, then set $m(k+1) = \min\{m(k) + 1, M\}$. Update $H_k$ such that $H_{k+1}$ is a positive definite matrix. Let $k = k + 1$ and go to S1; otherwise, go into the restoration phase to obtain a new point $x^{k+1}$. Let $k = k + 1$ and go to S1.

\textbf{Remark.} In the restoration phase, our aim is to decrease the value of $h(x)$. The restoration algorithm is similar to the one given by Long et al. [23].

3 Global Convergence

In this section, we first show that Algorithm A is well-defined, and then show the global convergence. To this end, the following assumptions are necessary.

\textbf{A 1} The iterate $\{x^k\}$ remains in a closed, bounded subset $X$.

\textbf{A 2} The objective function $f(x)$ and the constraint function $G(x)$ are twice continuously differentiable in $\mathbb{R}^n$.

\textbf{A 3} There exist two constants $0 < a \leq b$ such that $a\|d\|^2 \leq d^TH_kd \leq b\|d\|^2$ for any $d \in \mathbb{R}^n$.

In what follows, we analyze the feasibility of Algorithm A. To this end, it is necessary to extend the definition of infeasible stationary point for nonlinear programming [24] to nonlinear semidefinite programming.

\textbf{Definition 3.1} Let $\bar{x} \in \mathbb{R}^n$ be an infeasible point of NLSDP (1.1), if

$$\min_{d \in \mathbb{R}^n} \max_{\lambda_1(G(\bar{x}) + DG(\bar{x})d), 0} = \max_{\lambda_1(G(\bar{x})), 0} = h(\bar{x}),$$

then $\bar{x}$ is called an infeasible stationary of NLSDP (1.1).
Lemma 3.1  Supposed that the assumptions A1-A3 hold, if Algorithm A terminates at \( x^k \), then \( x^k \) is either an infeasible stationary point or a KKT point of \( NLSDP \) (1.1).

Proof. The proof is divided into two cases.

Case A. If Algorithm A terminates in S1, then \( \hat{d}^k = 0 \) and \( z_k \neq 0 \). We know from \( LSDP(x^k) \) (2.6) that \( z_k = h(x^k) \), so \( h(x^k) \neq 0 \), which implies \( x^k \) is an infeasible point of \( NLSDP \) (1.1).

In the following, we prove that \( x^k \) is an infeasible stationary point of \( NLSDP \) (1.1), namely, \( x^k \) satisfies:

\[
\min_{d \in \mathbb{R}^n} \max\{\lambda_1(G(x^k) + DG(x^k)d), 0\} = \max\{\lambda_1(G(x^k)), 0\} = h(x^k).
\]

By contradiction, suppose that the conclusion is not true. So there exists \( d^{k,0} \in \mathbb{R}^n \) such that

\[
\hat{z} := \max\{\lambda_1(G(x^k) + DG(x^k)d^{k,0}), 0\} < h(x^k).
\]

Clearly, \((d^{k,0T}, \hat{z})^T\) is a feasible solution of \( LSDP(x^k) \) (2.6). Note that \( z_k \) is a solution of \( LSDP(x^k) \) (2.6), so we obtain

\[
z_k \leq \hat{z} < h(x^k),
\]

this contradicts \( z_k = h(x^k) \). Therefore, \( x^k \) is an infeasible stationary point of \( NLSDP \) (1.1).

Case B. If Algorithm A terminates in S2, then the solution \( d^k \) of \( QSD(x^k, H_k) \) (2.7) is zero, i.e., \( d^k = 0 \). Further, \( d^k = 0 \) satisfies KKT condition of \( QSDP(x^k, H_k) \) (2.7), that is to say, there exists \( Y_k \in S^m \), such that

\[
\nabla f(x^k) + DG(x^k)Y_k = 0,
\]

\[
G(x^k) \preceq z_k I_m,
\]

\[Y_k \succeq 0, \quad <G(x^k) - z_k I_m, Y_k> = 0.
\]

In what follows, we prove that \( z_k = 0 \). By contradiction, supposed that \( z_k \neq 0 \), obviously, \((0^T, z_k)^T\) is a solution of \( LSDP(x^k) \) (2.6) from (3.5). Therefore, \( x^k \) is an infeasible point of \( NLSDP \) (1.1). Since Algorithm A does not stop in S1, \( z_k < h(x^k) \).

On the other hand, it follows from (3.5) that

\[
\lambda_1(G(x^k)) \leq z_k.
\]

In view of \( z_k > 0 \), we obtain \( h(x^k) = \max\{\lambda_1(G(x^k)), 0\} \leq z_k \). This contradict \( z_k < h(x^k) \). Therefore, \( z_k = 0 \).

Substituting \( z_k = 0 \) into (3.5), and combining with (3.4) and (3.6), we know that \( x^k \) is a KKT point of \( NLSDP \) (1.1). □

Lemma 3.2  If \( d^k \) satisfies the inequality (2.9), then the line search (2.14) is performed.

Proof. It is sufficient to show that there exists \( t \in (0, 1) \) such that (2.14) hold.
In view of \( \nabla f(x^k)^T d^k < -\frac{1}{2}(d^k)^T H_k d^k \), so in combination with the positive definiteness of \( H_k \), we know that there exists \( d^k \neq 0 \) such that \( \nabla f(x^k)^T d^k < 0 \). For convinence, denote
\[
    f(x^{l(k)}) = \max_{0 \leq j \leq m(k)} \{ f(x^{k-j}) \}. \tag{3.7}
\]

By contradiction, if the conclusion is not true, then for all \( t \in (0, 1) \), we have
\[
f(x^k + td^k) - f(x^{l(k)}) > -t\alpha(d^k)^T H_k d^k \geq 2t\alpha \nabla f(x^k)^T d^k. \tag{3.8}
\]

From (3.7), it is obvious that \( f(x^{l(k)}) \geq f(x^k) \), so combining with (3.8), we have
\[
f(x^k + td^k) - f(x^k) \geq f(x^k + td^k) - f(x^{l(k)}) > 2t\alpha \nabla f(x^k)^T d^k, \tag{3.9}
\]
equivalently,
\[
\frac{f(x^k + td^k) - f(x^k)}{t} > 2\alpha \nabla f(x^k)^T d^k. \tag{3.10}
\]

Let \( t \to 0^+ \), taking the limit for the both sides, it follows that
\[
\nabla f(x^k)^T d^k \geq 2\alpha \nabla f(x^k)^T d^k.
\]

This implies \( \alpha \in [\frac{1}{2}, \infty) \) due to \( \nabla f(x^k)^T d^k < 0 \). This contradicts \( \alpha \in (0, \frac{1}{2}) \). Hence, the desired result holds. \( \square \)

**Lemma 3.3** Supposed that the assumptions A1-A3 hold, then there exists \( \bar{t} > 0 \) such that \( t_k \geq \bar{t} \) for \( k \) sufficiently large.,

**Proof.** According to Algorithm A, without loss of generality, suppose that the search direction \( d^k \) satisfies (2.10), that is,
\[
\nabla f(x^k)^T d^k \leq -\frac{1}{2}(d^k)^T H_k d^k.
\]

By Taylor expansion, (3.7) and the assumptions A1-A3, we have
\[
\begin{align*}
    f(x^k + t_k d^k) - f(x^{l(k)}) &+ t_k\alpha(d^k)^T H_k d^k \\
    &= f(x^k) + t_k \nabla f(x^k)^T d^k + \frac{1}{2}t_k^2(d^k)^T \nabla^2 f(y_k)^T d^k - f(x^{l(k)}) + t_k\alpha(d^k)^T H_k d^k \\
    &\leq f(x^k) + t_k \nabla f(x^k)^T d^k + \frac{1}{2}t_k^2(d^k)^T \nabla^2 f(y_k)^T d^k - f(x^k) + t_k\alpha(d^k)^T H_k d^k \\
    &= t_k \nabla f(x^k)^T d^k + \frac{1}{2}t_k^2(d^k)^T \nabla^2 f(y_k)^T d^k + t_k\alpha(d^k)^T H_k d^k \\
    &\leq -\frac{1}{2}t_k(d^k)^T H_k d^k + \frac{1}{2}t_k^2(d^k)^T \nabla^2 f(y_k)^T d^k + t_k\alpha(d^k)^T H_k d^k \\
    &\leq -at_k(\frac{1}{2} - \alpha)\|d^k\|^2 + \frac{1}{2}t_k^2 M\|d^k\|^2,
\end{align*}
\]

where \( y_k \) is between \( x^k \) and \( x^k + t_k d^k \), \( M \) is a positive integer such that \( \|\nabla^2 f(x)\| \leq M \).

Let \( \bar{t} = \frac{a(1-2\alpha)}{M} > 0 \), so (2.10) holds for \( t_k \geq \bar{t} \) and \( \alpha \in (0, \frac{1}{2}) \). \( \square \)

**Lemma 3.4** Supposed that the assumptions A1-A3 hold, \( \{x^k\} \) is an infinite sequence generated by Algorithm A, then \( \lim_{k \to \infty} h(x^k) = 0 \).
Proof. Since \( m(k + 1) \leq m(k) + 1 \), we have

\[
h(x^{(k+1)}) = \max_{0 \leq j \leq m(k+1)} \{ h(x^{(k+1-j)}) \} \leq \max_{0 \leq j \leq m(k)+1} \{ h(x^{k+1-j}) \} = \max \{ h(x^{k+1}), h(x^{(k)}) \} = h(x^{(k)}),
\]

this implies that the sequence \( \{ h(x^{(k)}) \} \) is not increasing for \( k \). Combining with \( h(x^{(k)}) \geq 0 \), we conclude that \( \{ h(x^{(k)}) \} \) is convergent.

By Algorithm A, we have

\[
h(x^{k+1}) \leq \beta \max_{0 \leq j \leq m(k)} \{ h(x^{k-j}) \} = \beta h(x^{(k)}).
\]

Replace \( k \) by \( l(k) - 1 \). we obtain

\[
h(x^{(k)}) \leq \beta h(x^{(l(k)-1)}),
\]

which together with \( \beta \in (\frac{1}{2}, 1) \) gives \( \lim_{k \to \infty} h(x^{(k)}) = 0 \). Further, by (3.12), we can conclude that \( \lim_{k \to \infty} h(x^{k}) = 0 \). \( \square \)

Theorem 3.1 Supposed that the assumptions A1-A3 hold, \( \{ x^k \} \) is an infinite sequence generated by Algorithm A, \( d^k \) is the solution of \( QSDP(x^k, H_k) (2.7) \). If the multiplier corresponding to \( d^k \) is uniform bounded, then there exists \( \bar{K} \subseteq \{ 1, 2, \cdots \} \) such that \( \lim_{k \in \bar{K}} d^k = 0 \).

Proof. By the assumption A1, we know that \( \{ x^k \} \) is bounded, so there exists an infinite index set \( K \subseteq \{ 1, 2, \cdots \} \), such that \( \{ x^k \}_K \) is convergent. Let \( \lim_{k \in K} x^k = x^* \).

We consider the following two cases:

Case 1. The index set \( K_0 = \{ k \in K | \nabla f(x^k)^T d^k \leq -\frac{1}{2}(d^k)^T H_k d^k \} \) is infinite.

By (2.14), we obtain

\[
f(x^{k+1}) = f(x^k + t_k d^k) \leq f(x^{(k)}) - t_k \alpha(d^k)^T H_k d^k \leq f(x^{(l)}) , \forall k \in K_0.
\]

Since \( m(k+1) \leq m(k) + 1 \), we obtain

\[
f(x^{(l+1)}) \leq \max_{0 \leq j \leq m(k)+1} \{ f(x^{(l+1-j)}) \} = \max \{ f(x^{(l+1)}), f(x^{(l)}) \} = f(x^{(l)}).
\]

This implies that the sequence \( \{ f(x^{(k)}) \} \) is not increasing. Combining with the boundedness of \( \{ f(x^{(k)}) \} \), it follows that \( \{ f(x^{(k)}) \}_{K_0} \) is convergent.

For \( \{ l(k) - 1, \ k \in K_0 \} \), we obtain

\[
f(x^{(l(k))}) \leq f(x^{(l(k)-1)}) - t_{l(k)-1} \alpha(d^{(l(k)-1)})^T H_{l(k)-1} d^{(l(k)-1)}.
\]

Since \( \{ f(x^{(l)}) \} \) is convergent, we have

\[
\lim_{K_0} t_{l(k)-1} \alpha(d^{(l(k)-1)})^T H_{l(k)-1} d^{(l(k)-1)} = 0,
\]

By Lemma 3.3, we know that there exists \( \bar{t} > 0 \) such that \( t_{l(k)-1} \geq \bar{t} > 0 \), so by the assumption A3, we obtain

\[
\lim_{K_0} d^{(l(k)-1)} = 0.
\]
The uniform continuity of \( f(x) \) implies that
\[
\lim_{k \to 0} f(x^{(k)} - 1) = \lim_{k \to 0} f(x^{(k)}).
\]  
(3.18)

Let \( \hat{l}(k) = l(k + M + 2) \), it is not difficult to prove by induction that for any given \( j \geq 1 \),
\[
\lim_{k \to 0} ||d(\hat{l}(k) - j)|| = 0,
\]  
(3.19)
\[
\lim_{k \to 0} f(x^{(\hat{l}(k) - j)}) = \lim_{k \to 0} f(x^{(l)(k)}).
\]  
(3.20)

For any \( k \in \mathcal{K}_0 \), we obtain \( x^{k+1} = x^{(\hat{l})(k)} - \sum_{j=1}^{\hat{l}(k) - k - 1} t_{\hat{l}(k) - j} d(\hat{l}(k) - j) \). Note that \( \hat{l}(k) - k - 1 \leq M + 1 \) and (3.19), we get \( \lim_{k \to 0} ||x^{k+1} - x^{(\hat{l})(k)}|| = 0 \). So it follows from the convergence of \( \{ f(x^{(l)(k)}) \} \) and the uniform continuity of \( f(x) \) that
\[
\lim_{k \to 0} f(x^{k+1}) = \lim_{k \to 0} f(x^{(\hat{l})(k)}).
\]

So let \( k (\in \mathcal{K}_0) \to \infty \), taking the limit in (3.14), we have
\[
\lim_{k \to 0} t_k \alpha (d^k)^T H_k d^k = 0.
\]  
(3.21)

Similar to the proof of (3.17), we obtain \( \lim_{k \to 0} d^k = 0 \). Hence, let \( \tilde{\mathcal{K}} = \mathcal{K}_0 \) and the conclusion follows.

**Case 2.** The index set \( \mathcal{K}_0 = \{ k \in \mathcal{K} | \nabla f(x^k)^T d^k \leq -\frac{1}{2} (d^k)^T H_k d^k \} \) is finite, which implies that \( \mathcal{K}_1 = \{ k \in \mathcal{K} | \nabla f(x^k)^T d^k > -\frac{1}{2} (d^k)^T H_k d^k \} \) is infinite.

By contradiction, supposed that the conclusion is not true, then \( \lim_{k \to 1} d^k \neq 0 \). So there exist \( \mathcal{K}_2 \subseteq \mathcal{K}_1 \) and a constant \( \varepsilon > 0 \), such that \( ||d^k|| > \varepsilon \) for \( k \in \mathcal{K}_2 \).

Since \( d^k \) is the solution of \( QSDP(x^k, H_k) \) (2.7), by KKT condition of \( QSD(x^k, H_k) \) (2.7) , it follows that there exists a positive semidefinite matrix \( Y_k \) such that
\[
\nabla f(x^k) + H_k d^k + DG(x^k)^* Y_k = 0,
\]  
(3.22)
\[
\text{Tr}((G(x^k) + DG(x^k)^* d^k - z_k I_m)Y_k) = 0,
\]  
(3.23)

According to the assumption of Theorem 3.1, there exists \( \tilde{M} > 0 \) such that \( \|Y_k\|_F \leq \tilde{M} \).

By Lemma 3.4, we know \( \lim_{k \to \infty} h(x^k) = 0 \), hence there exists \( k_0 \geq 0 \), such that
\[
h(x^k) \leq \frac{1}{2Mm} a \varepsilon^2, \text{ for } k (\in \mathcal{K}_2) > k_0,
\]  
(3.24)
combining with \( ||d^k|| > \varepsilon \) and the assumption A3, we obtain
\[
h(x^k) \leq \frac{1}{2Mm} (d^k)^T H_k d^k.
\]  
(3.25)

It follows from (2.2) that
\[
\text{Tr}(DG(x^k)d^k Y_k) = \text{Tr}(\sum_{i=1}^{n} d_i \partial G(x^k)\partial x_i) Y_k) = \sum_{i=1}^{n} \text{Tr}(\partial G(x^k)\partial x_i) Y_k d_i^k = \sum_{i=1}^{n} < \partial G(x^k)\partial x_i, Y_k > d_i^k.
\]  
(3.26)
It follows from (3.23) that
\[
\text{Tr}((DG(x^k)d^k)Y_k) = \text{Tr}((G(x^k) - z_k I_m)Y_k),
\]
so (3.26) and (3.27) give rise to
\[
\sum_{i=1}^n \frac{\partial G(x^k)}{\partial x_i}, Y_k > d_i^k = \text{Tr}((G(x^k) - z_k I_m)Y_k).
\]
(3.28)
By (3.22) and (3.28), we have
\[
\nabla f(x^k)^T d^k = -(d^k)^T H_k d^k - (DG(x^k)^* Y_k)^T d^k
\]
\[= -(d^k)^T H_k d^k - \sum_{i=1}^n \frac{\partial G(x^k)}{\partial x_i}, Y_k > d_i^k
\]
\[= -(d^k)^T H_k d^k + \text{Tr}((G(x^k) - z_k I_m)Y_k).
\]
(3.29)
By Neumann Inequality, we obtain
\[
\text{Tr}((G(x^k) - z_k I_m)Y_k) \leq \sum_{i=1}^m \lambda_i(G(x^k) - z_k I_m)\lambda_i(Y_k)
\]
\[\leq \sum_{i=1}^m \lambda_i(G(x^k) - z_k I_m)\|Y_k\|_F
\]
\[\leq \sum_{i=1}^m \lambda_i(G(x^k) - z_k I_m)\tilde{M}
\]
\[\leq \sum_{i=1}^m \lambda_i(G(x^k))\tilde{M},
\]
(3.30)
the last inequality above is due to \(z_k \geq 0\). According to the definition (2.8) of \(h(x^k)\) and (3.30), we obtain
\[
\text{Tr}((G(x^k) - z_k I_m)Y_k) \leq \tilde{M}mh(x^k) \leq \frac{1}{2}(d^k)^T H_k d^k.
\]
(3.31)
Substituting (3.31) into (3.29), it follows that
\[
\nabla f(x^k)^T d^k \leq -\frac{1}{2}(d^k)^T H_k d^k,
\]
which contradicts the definition of \(K_1\). Hence, the conclusion is true.

Theorem 3.2 Supposed that \(\{x^k\}\) is an infinite sequence generated by Algorithm A, and the assumptions in Theorem 3.1 hold, then any accumulation point of \(\{x^k\}\) is a KKT point of NLSDP (1.1).

Proof. Supposed that \(x^*\) is an accumulation point of \(\{x^k\}\), then there exists \(K \subseteq \{1, 2, \cdots\}\), such that \(\lim_{k \in K} x^k = x^*\). In view of the assumption A3, without loss of generality, we suppose that \(\lim_{k \in K} H_k = H_s\).
By Lemma 3.4, we have \( \lim_{k \in K} h(x^k) = h(x^*) = 0 \), which means that \( x^* \) is a feasible point of \textit{NLSDP} (1.1).

By Theorem 3.1, there exists \( \tilde{K} \subseteq \{1, 2, \cdots\} \) such that \( \lim_{\tilde{K}} d^k = d^* = 0 \). By the proof of Theorem 3.1, we know that \( \tilde{K} \subseteq K \).

According to KKT conditions of \textit{QSDP} (2.7), we obtain

\[
\nabla f(x^k) + H_k d^k + DG(x^k)^* Y_k = 0,
\]

\[
Y_k \succeq 0, \quad \text{Tr}((G(x^k) + DG(x^k)d^k - z_k I_m)Y_k) = 0.
\]

Let \( k(\in \tilde{K}) \to \infty \), taking the limit, we obtain

\[
\nabla f(x^*) + DG(x^*)^* Y_* = 0,
\]

\[
Y_* \succeq 0, \quad <G(x^*), Y_*> = 0.
\]

This implies that \( x^* \) is a KKT point of \textit{NLSDP} (1.1). \( \square \)

4 Numerical experiments

In this section, preliminary numerical experiments of Algorithm A is implemented. Algorithm A was coded by Matlab (2014a) and run on the computer with Windows 7 (64 bite), Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz, RAM: 4.00GB.

In the numerical experiments, the parameters are chosen as follows: \( \alpha = 0.25, \beta = 0.85, \sigma = 0.5, M = 3 \). And the termination criteria of Algorithm A is: \( \|d^k\| \leq 10^{-4} \).

The test problem is chosen from [11].

Problem 1. Nearest Correlation Matrix (NCM) Problem:

\[
\begin{align*}
\min & \quad f(X) = \frac{1}{2}\|X - C\|_F^2 \\
\text{s.t} & \quad X \preceq \epsilon I, \\
& \quad X_{ii} = 1, i = 1, 2, ..., m,
\end{align*}
\]

(4.1)

where \( C \in S^m \) is a given matrix, \( X \in S^m \), \( \epsilon \) is a scalar.

In the implementation, \( \epsilon = 10^{-3} \), \( C \) is generated randomly, which diagonal elements are 1. We test ten times for every fixed dimensionality.

We compare Algorithm A with the ones in [11] (denoted by Algo. YYH) and [14] (denoted by Algo. YYY).

The numerical results are listed in Table 1. The meaning of the notations in Table 1 are described as follows:

- \( n \) : the dimensionality of independent variable;
- \( m \) : the dimensionality of \( \mathcal{G}(x) \);
- \( A - Iter \) : the average number of evaluation of iterations.

| Table 1. Numerical results of NCM |
5 Concluding remarks

In this paper, we have presented a new SSDP algorithm for nonlinear semidefinite programming. Two subproblems, which are constructed skillfully, are solved to generate the search directions. The nonmonotone line search ensures that the objective function or constraint violation function is sufficiently reduced. The global convergence of the proposed algorithm is shown under some mild conditions. The preliminary numerical results show that the proposed algorithm is effective.

References


Approximation by Sublinear and Max-product Operators using Convexity

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Abstract
Here we consider quantitatively using convexity the approximation of a function by general positive sublinear operators with applications to Max-product operators. These are of Bernstein type, of Favard-Szász-Mirakjan type, of Baskakov type, of Meyer-Köning and Zeller type, of sampling type, of Lagrange interpolation type and of Hermite-Fejér interpolation type. Our results are both: under the presence of smoothness and without any smoothness assumption on the function to be approximated which fulfills a convexity property.

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1 Background

We make

Remark 1 Let \( f \in C ([a,b]) \), \( x_0 \in (a,b) \), \( 0 < h \leq \min (x_0 - a, b - x_0) \), and \( |f(t) - f(x_0)| \) is convex in \( t \in [a,b] \).

By Lemma 8.1.1, p. 243 of [1] we have that

\[
|f(t) - f(x_0)| \leq \frac{\omega_1(f,h)}{h} |t - x_0|, \quad \forall t \in [a,b], \tag{1}
\]

where

\[
\omega_1(f,h) := \sup_{\substack{x,y \in [a,b] \\mid |x-y| \leq h}} |f(x) - f(y)|, \tag{2}
\]

the first modulus of continuity of \( f \).
We also make

**Remark 2** Let \( f \in C^n ([a, b]) \), \( n \in \mathbb{N} \), \( x_0 \in (a, b) \), \( 0 < h \leq \min (x_0 - a, b - x_0) \), and \( |f^{(n)} (t) - f^{(n)} (x_0)| \) is convex in \( t \in [a, b] \). We have that

\[
f(t) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (t - x_0)^k + I_t,
\]

where

\[
I_t = \int_{x_0}^{t} \left( \int_{x_0}^{t_1} \left( \int_{x_0}^{t_{n-1}} \left( f^{(n)}(t_n) - f^{(n)}(x_0) \right) dt_n \right) dt_{n-1} \right) \cdots dt_1.
\]

Assuming \( f^{(k)}(x_0) = 0 \), \( k = 1, \ldots, n \), we get

\[
f(t) - f(x_0) = I_t.
\]

By Lemma 8.1.1, p. 243 of [1] we have

\[
|f^{(n)}(t) - f^{(n)}(x_0)| \leq \frac{\omega_1(f^{(n)}, h)}{h} |t - x_0|, \quad \forall t \in [a, b].
\]

Furthermore it holds

\[
|I_t| \leq \frac{\omega_1(f^{(n)}, h)}{h} |t - x_0|^{n+1} (n+1)!, \quad \forall t \in [a, b].
\]

Hence we derive that

\[
|f(t) - f(x_0)| \leq \frac{\omega_1(f^{(n)}, h)}{h} |t - x_0|^{n+1} (n+1)!, \quad \forall t \in [a, b].
\]

We have proved the following results:

**Theorem 3** Let \( f \in C([a, b]) \), \( x \in (a, b) \), \( 0 < h \leq \min (x - a, b - x) \), and \( |f(\cdot) - f(x)| \) is convex over \([a, b]\). Then

\[
|f(\cdot) - f(x)| \leq \frac{\omega_1(f, h)}{h} |\cdot - x|, \quad \text{over} \; [a, b].
\]

**Theorem 4** Let \( f \in C^n([a, b]) \), \( n \in \mathbb{N} \), \( x \in (a, b) \), \( 0 < h \leq \min (x - a, b - x) \), and \( |f^{(n)}(\cdot) - f^{(n)}(x)| \) is convex over \([a, b]\). Assume that \( f^{(k)}(x) = 0 \), \( k = 1, \ldots, n \). Then

\[
|f(\cdot) - f(x)| \leq \frac{\omega_1(f^{(n)}, h)}{h} |\cdot - x|^{n+1} (n+1)! \, , \quad \text{over} \; [a, b].
\]

We give
Definition 5 Call \( C_+ ([a,b]) := \{ f : [a,b] \to \mathbb{R}_+ \text{ and continuous} \} \). Let \( L_N \) from \( C_+ ([a,b]) \) into \( C_+ ([a,b]) \) be a sequence of operators satisfying the following properties (see also [6], p. 17):

(i) (positive homogeneous)

\[
L_N (\alpha f) = \alpha L_N (f), \quad \forall \alpha \geq 0, \quad \forall f \in C_+ ([a,b]),
\]

(ii) (Monotonicity)

if \( f, g \in C_+ ([a,b]) \) satisfy \( f \leq g \), then

\[
L_N (f) \leq L_N (g), \quad \forall N \in \mathbb{N},
\]

(iii) (Subadditivity)

\[
L_N (f + g) \leq L_N (f) + L_N (g), \quad \forall f, g \in C_+ ([a,b]).
\]

We call \( L_N \) positive sublinear operators.

Remark 6 As in [6], p. 17, we get that for \( f, g \in C_+ ([a,b]) \)

\[
|L_N (f) (x) - L_N (g) (x)| \leq L_N (|f - g|) (x), \quad \forall x \in [a,b].
\]

From now on we assume that \( L_N (1) = 1, \forall N \in \mathbb{N} \). Hence it holds

\[
|L_N (f) (x) - f (x)| \leq L_N (|f (\cdot) - f (x)|) (x), \quad \forall x \in [a,b], \quad \forall N \in \mathbb{N},
\]

see also [6], p. 17.

We obtain the following results:

Theorem 7 Let \( f \in C_+ ([a,b]), x \in (a,b), 0 < h \leq \min (x-a, b-x), \) and \( |f (\cdot) - f (x)| \) is a convex function over \( [a,b] \). Let \( \{ L_N \}_{N \in \mathbb{N}} \) positive sublinear operators from \( C_+ ([a,b]) \) into itself, such that \( L_N (1) = 1, \forall N \in \mathbb{N} \). Then

\[
|L_N (f) (x) - f (x)| \leq \frac{\omega_1 (f,h)}{h} L_N (|\cdot - x|) (x), \quad \forall N \in \mathbb{N}.
\]

Proof. By (9) and (15). \( \blacksquare \)

Theorem 8 Let \( f \in C^n ([a,b], \mathbb{R}_+), n \in \mathbb{N}, x \in (a,b), 0 < h \leq \min (x-a, b-x), \) and \( |f^{(n)} (\cdot) - f^{(n)} (x)| \) is convex over \( [a,b] \). Assume that \( f^{(k)} (x) = 0, k = 1, \ldots, n \). Let \( \{ L_N \}_{N \in \mathbb{N}} \) positive sublinear operators from \( C_+ ([a,b]) \) into itself, such that \( L_N (1) = 1, \forall N \in \mathbb{N} \). Then

\[
|L_N (f) (x) - f (x)| \leq \frac{\omega_1 (f^{(n)}, h)}{h (n+1)!} L_N (|\cdot - x|^{n+1}) (x), \quad \forall N \in \mathbb{N}.
\]
Proof. By (10) and (15). ■

We continue with

**Theorem 9** Let \( f \in C^1([a,b]) \), \( x \in (a,b) \), \( 0 < L_N (|x|) (x) \leq \min(x-a,b-x) \)
\( \forall N \in \mathbb{N} \), and \(|f(\cdot) - f(x)|\) is a convex function over \([a,b] \). Here \( L_N \) are positive sublinear operators from \( C^1([a,b]) \) into itself, such that \( L_N (1) = 1 \), \( \forall N \in \mathbb{N} \). Then
\[
|L_N (f) (x) - f (x)| \leq \omega_1 (f, L_N (|x|) (x)) , \forall N \in \mathbb{N} .
\]
If \( L_N (|x|) (x) \to 0 \), then \( L_N (f) (x) \to f (x) \), as \( N \to +\infty \).

**Proof.** By (16). ■

**Theorem 10** Let \( f \in C^N ([a,b], \mathbb{R}_+) \), \( n \in \mathbb{N} \), \( x \in (a,b) \), \( 0 < L_N (|x|^{n+1}) (x) \leq \min(x-a,b-x) \)
\( \forall N \in \mathbb{N} \), and \(|f^{(n)}(\cdot) - f^{(n)}(x)|\) is convex over \([a,b] \). Assume that \( f^{(k)} (x) = 0 \), \( k = 1, ..., n \). Here \( \{L_N\}_{N \in \mathbb{N}} \) are positive sublinear operators from \( C^N ([a,b]) \) into itself, such that \( L_N (1) = 1 \), \( \forall N \in \mathbb{N} \). Then
\[
|L_N (f) (x) - f (x)| \leq \frac{\omega_1 (f^{(n)}, L_N (|x|^{n+1}) (x))}{(n+1)!} , \forall N \in \mathbb{N} .
\]
If \( L_N (|x|^{n+1}) (x) \to 0 \), then \( L_N (f) (x) \to f (x) \), as \( N \to +\infty \).

**Proof.** By (17). ■

Next we combine both Theorems 7, 8:

**Theorem 11** Let \( f \in C^N ([a,b], \mathbb{R}_+) \), \( n \in \mathbb{Z}_+ \), \( x \in (a,b) \), \( 0 < h \leq \min(x-a,b-x) \)
and \(|f^{(n)}(\cdot) - f^{(n)}(x)|\) is convex over \([a,b] \). Assume that \( f^{(k)} (x) = 0 \), \( k = 1, ..., n \). Let \( \{L_N\}_{N \in \mathbb{N}} \) are positive sublinear operators from \( C^N ([a,b]) \) into itself, such that \( L_N (1) = 1 \), \( \forall N \in \mathbb{N} \). Then
\[
|L_N (f) (x) - f (x)| \leq \frac{\omega_1 (f^{(n)}, h)}{h(n+1)!} L_N (|x|^{n+1}) (x) , \forall N \in \mathbb{N} ; n \in \mathbb{Z}_+ .
\]

The initial conditions \( f^{(k)} (x) = 0 \), \( k = 1, ..., n \), are void when \( n = 0 \).

In this article we study under convexity quantitatively the approximation properties of Max-product operators to the unit. These are special cases of positive sublinear operators. We present also results regarding the convergence to the unit of general positive sublinear operators under convexity. Special emphasis is given to our study about approximation under the presence of smoothness. Our work is inspired from [6].

Under our convexity conditions the derived convergence inequalities are elegant and compact with very small constants.
2 Main Results

Here we apply Theorem 11 to Max-product operators.

We make

Remark 12 We start with the Max-product Bernstein operators ([6], p. 10)

\[ B_N^{(M)}(f)(x) = \frac{\sum_{k=0}^{N} p_{N,k} \frac{x}{N} f \left( \frac{k}{N} \right)}{\sum_{k=0}^{N} p_{N,k}(x)}, \quad \forall \ N \in \mathbb{N}, \]  \hspace{1cm} (21)

\[ p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}, \quad x \in [0,1], \quad \forall \ N \in \mathbb{N}, \]  \hspace{1cm} (22)

Clearly \( B_N^{(M)} \) is a positive sublinear operators from \( C_+([0,1]) \) into itself with \( B_N^{(M)}(1) = 1 \).

By [6], p. 31, we have

\[ B_N^{(M)}(|x|) \leq \frac{6}{\sqrt{N+1}}, \quad \forall \ x \in [0,1], \forall \ N \in \mathbb{N}. \]  \hspace{1cm} (23)

And by [2] we get:

\[ B_N^{(M)}(|x|^m) \leq \frac{6}{\sqrt{N+1}}, \quad \forall \ x \in [0,1], \forall \ N \in \mathbb{N}. \]  \hspace{1cm} (24)

Denote by

\[ C_n^m([0,1]) = \{ f : [0,1] \to \mathbb{R}_+, n \text{-times continuously differentiable} \}, \quad n \in \mathbb{Z}_+. \]

We present

Theorem 13 Let \( f \in C_n^m([0,1]), \forall \ x \in (0,1) \) and \( N^* \in \mathbb{N} : 0 < \frac{1}{\sqrt{N+1}} \leq \min (x,1-x), \) and \( |f^{(n)}(\cdot) - f^{(n)}(x)| \) is convex over \([0,1]\). Assume that \( f^{(k)}(x) = 0, k = 1,\ldots, n \). Then

\[ \left| B_N^{(M)}(f)(x) - f(x) \right| \leq \frac{6 \omega_1 \left( f^{(n)} \left( \frac{x}{\sqrt{N+1}} \right) \frac{1}{(n+1)!} \right)}{\frac{1}{(n+1)!}}, \quad \forall \ N \in \mathbb{N} : N \geq N^*. \]  \hspace{1cm} (25)

It holds \( \lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x) \).

Proof. By (20) we get

\[ \left| B_N^{(M)}(f)(x) - f(x) \right| \leq \omega_1 \left( f^{(n)} \frac{1}{h(n+1)!} \right) B_N^{(M)} \left( |x|^n \right) (x) \leq \frac{6 \omega_1 \left( f^{(n)} \frac{1}{h(n+1)!} \right)}{\sqrt{N+1}} = \]
(setting $h := \frac{1}{\sqrt{N+1}}$)

$$6\omega_1 \left( f^{(n)} \frac{1}{\sqrt{N+1}} \right) \frac{1}{(n+1)!},$$

proving the claim. \qed

We make

Remark 14 Here we focus on the truncated Favard-Szász-Mirakjan operators

$$T^{(M)}_N (f) (x) = \frac{\sum_{k=0}^{N} s_{N,k} (x) f \left( \frac{k}{N} \right)}{\sum_{k=0}^{N} s_{N,k} (x)}, \quad x \in [0,1], \quad N \in \mathbb{N}, \quad f \in C_+ ([0,1]),$$

$$s_{N,k} (x) = \binom{N}{k} x^k (1-x)^{N-k}, \text{ see also [6], p. 11.}$$

By [6], p. 178-179 we have

$$T^{(M)}_N (|\cdot - x|) (x) \leq \frac{3}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ N \in \mathbb{N}. \quad (27)$$

And by [2] we get

$$T^{(M)}_N (|\cdot - x|^m) (x) \leq \frac{3}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ m, N \in \mathbb{N}. \quad (28)$$

The operators $T^{(M)}_N$ are positive sublinear from $C_+ ([0,1])$ into itself with $T^{(M)}_N (1) = 1, \ \forall \ N \in \mathbb{N}$.

We give

Theorem 15 Let $f \in C^n_+ ([0,1]), \ n \in \mathbb{Z}_+, \ x \in (0,1)$ and $N^* \in \mathbb{N} : 0 < \frac{1}{\sqrt{N^*}} \leq \min (x,1-x), \ \text{and} \ |f^{(n)} (\cdot) - f^{(n)} (x)| \ \text{is convex over} \ [0,1]$. Assume that $f^{(k)} (x) = 0, \ \kappa = 1, ..., n$. Then

$$\left| T^{(M)}_N (f) (x) - f (x) \right| \leq \frac{3\omega_1 \left( f^{(n)} \frac{1}{\sqrt{N}} \right) \frac{1}{(n+1)!}}{\sqrt{N}}, \quad \forall \ N \in \mathbb{N} : N \geq N^*. \quad (29)$$

It holds $\lim_{N \to +\infty} T^{(M)}_N (f) (x) = f (x)$.

Proof. By (20) we get

$$\left| T^{(M)}_N (f) (x) - f (x) \right| \leq \omega_1 \left( f^{(n)} \frac{1}{h} \frac{1}{(n+1)!} \right) T^{(M)}_N \left( |\cdot - x|^{n+1} \right) (x) \leq \omega_1 \left( f^{(n)} \frac{1}{h} \frac{1}{(n+1)!} \right) \frac{3}{\sqrt{N}} = \omega_1 \left( f^{(n)} \frac{1}{h} \frac{1}{(n+1)!} \right) \frac{3}{\sqrt{N}}$$

$$= \omega_1 \left( f^{(n)} \frac{1}{h} \frac{1}{(n+1)!} \right) \frac{3}{\sqrt{N}}.$$
(setting \( h := \frac{1}{\sqrt{N}} \))

\[
\frac{3\omega_1 \left( f^{(n)} \cdot \frac{1}{\sqrt{N}} \right)}{(n+1)!}, \tag{30}
\]

proving the claim. 

We make

**Remark 16** Next we study the truncated Max-product Baskakov operators (see [6], p. 11)

\[
U_N^{(M)} (f) (x) = \frac{\sum_{k=0}^{N} b_{N,k} (x) \frac{1}{\sqrt{N}}}{\sum_{k=0}^{N} b_{N,k} (x)}, \quad x \in [0,1], \quad f \in C_+ ([0,1]), \quad N \in \mathbb{N}, \tag{31}
\]

where

\[
b_{N,k} (x) = \binom{N + k - 1}{k} \frac{x^k}{(1 + x)^{N+k}}.
\]

From [6], pp. 217-218, we get \((x \in [0,1])\)

\[
\left( U_N^{(M)} (| \cdot - x^m |) \right) (x) \leq \frac{12}{\sqrt{N+1}}, \quad N \geq 2, \quad N \in \mathbb{N}. \tag{32}
\]

And as in [2], we obtain \((m \in \mathbb{N})\)

\[
\left( U_N^{(M)} (| \cdot - x^m |) \right) (x) \leq \frac{12}{\sqrt{N+1}}, \quad N \geq 2, \quad N \in \mathbb{N}, \quad \forall \ x \in [0,1]. \tag{33}
\]

Also it holds \( U_N^{(M)} (1) (x) = 1, \) and \( U_N^{(M)} \) are positive sublinear operators from \( C_+ ([0,1]) \) into itself, \( \forall \ N \in \mathbb{N}. \)

We give

**Theorem 17** Let \( f \in C_+^n ([0,1]), \ n \in \mathbb{Z}_+, \ x \in (0,1) \) and \( N^* \in \mathbb{N} - \{ 1 \} : 0 < \frac{1}{\sqrt{N^*+1}} \leq \min (x,1-x), \) and \(|f^{(n)} (\cdot) - f^{(n)} (x)|\) is convex over \([0,1]. \) Assume that \( f^{(k)} (x) = 0, \ k = 1, \ldots, n. \) Then

\[
\left| U_N^{(M)} (f) (x) - f (x) \right| \leq \frac{12\omega_1 \left( f^{(n)} \cdot \frac{1}{\sqrt{N+1}} \right)}{(n+1)!}, \quad \forall \ N \in \mathbb{N}: N \geq N^*. \tag{34}
\]

It holds \( \lim_{N \to +\infty} U_N^{(M)} (f) (x) = f (x). \)

**Proof.** By (20) we get

\[
\left| U_N^{(M)} (f) (x) - f (x) \right| \leq \frac{\omega_1 \left( f^{(n)} (x) \cdot \frac{1}{h} \right)}{h (n+1)!} U_N^{(M)} \left( | \cdot - x |^{n+1} \right) (x) \leq \frac{12\omega_1 \left( f^{(n)} \cdot \frac{1}{\sqrt{N+1}} \right)}{(n+1)!}, \quad \forall \ N \in \mathbb{N}: N \geq N^*. \tag{33}
\]


\[
\frac{\omega_1 \left( f^{(n)} \right)}{h(n+1)! \sqrt{N+1}} = \frac{12}{h(n+1)! \sqrt{N+1}}
\]

(setting \( h := \frac{1}{\sqrt{N+1}} \))

\[
\frac{12\omega_1 \left( f^{(n)} \right)}{h(n+1)!} = \left( \frac{1}{\sqrt{N+1}} \right),
\]

proving the claim. 

We make

Remark 18

Here we study Max-product Meyer-Köning and Zeller operators (see [6], p. 11) defined by

\[
Z^M_N (f) (x) = \frac{\sum_{k=0}^{\infty} s_{N,k} (x) f \left( \frac{k}{N+k} \right)}{\sum_{k=0}^{\infty} s_{N,k} (x)}, \quad \forall \ N \in \mathbb{N}, \ f \in C_+ ([0,1]),
\]

\( s_{N,k} (x) = \binom{N+k}{k} x^k, \ x \in [0,1]. \)

By [6], p. 253, we get that

\[
Z^M_N (| - x |) (x) \leq \frac{8 \left( 1 + \sqrt{5} \right)}{3} \frac{\sqrt{x} (1-x)}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ N \geq 4.
\]

And by [2], we derive that

\[
Z^M_N (| - x |^m) (x) \leq \frac{8 \left( 1 + \sqrt{5} \right)}{3} \frac{\sqrt{x} (1-x)}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ N \geq 4, \forall \ m \in \mathbb{N}.
\]

The ceiling \( \left\lceil \frac{8 \left( 1 + \sqrt{5} \right)}{3} \right\rceil = 9 \), and using a basic calculus technique (see [4]) we get that \( g (x) := (1-x) \sqrt{x} \) has an absolute maximum over \( (0,1) : g \left( \frac{1}{3} \right) = \frac{2}{3\sqrt{3}}. \)

That is \( (1-x) \sqrt{x} \leq \frac{2}{3\sqrt{3}}, \forall \ x \in [0,1]. \)

Consequently it holds

\[
Z^M_N (| - x |^m) (x) \leq \frac{6}{\sqrt{3} \sqrt{N}}, \quad \forall \ x \in [0,1], \forall \ N \in \mathbb{N} : N \geq 4, \forall \ m \in \mathbb{N}.
\]

Also it holds \( Z^M_N (1) = 1 \), and \( Z^M_N \) are positive sublinear operators from \( C_+ ([0,1]) \) into itself, \( \forall \ N \in \mathbb{N}. \)

We give
**Theorem 19** Let $f \in C^n_+([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ and $N^* \in \mathbb{N} : N^* \geq 4$ with $0 < \frac{1}{\sqrt{N^*}} \leq \min(x,1-x)$, and $|f^{(n)}(\cdot) - f^{(n)}(x)|$ is convex over $[0,1]$. Assume that $f^{(k)}(x) = 0$, $k = 1, \ldots, n$. Then

$$Z_N^{(M)}(f)(x) - f(x) \leq \left(\frac{6}{\sqrt{3}(n+1)!}\right) \omega_1 \left(f^{(n)}, \frac{1}{\sqrt{N}}\right), \quad \forall N \in \mathbb{N} : N \geq N^*.$$  

(40)

It holds

$$\lim_{N \to +\infty} Z_N^{(M)}(f)(x) = f(x).$$

**Proof.** By (20) we get

$$ \left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1 \left(f^{(n)}, h\right)}{h(n+1)!} Z_N^{(M)} \left(|x|^{n+1}\right)(x) \leq$$

$$= \frac{\omega_1 \left(f^{(n)}, h\right)}{h(n+1)!} \frac{6}{\sqrt{3}\sqrt{N}} =$$

(setting $h := \frac{1}{\sqrt{N}}$)

$$\left(\frac{6}{\sqrt{3}(n+1)!}\right) \omega_1 \left(f^{(n)}, \frac{1}{\sqrt{N}}\right),$$

(41)

proving the claim. $\blacksquare$

**Remark 20** Here we mention the Max-product truncated sampling operators (see [6], p. 13) defined by

$$W_N^{(M)}(f)(x) := \sum_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f \left(\frac{k\pi}{N}\right), \quad x \in [0,\pi],$$

(42)

$f : [0,\pi] \to \mathbb{R}_+$, continuous,

and

$$K_N^{(M)}(f)(x) := \sum_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f \left(\frac{k\pi}{N}\right), \quad x \in [0,\pi],$$

(43)

$f : [0,\pi] \to \mathbb{R}_+$, continuous.

By convention we take $\frac{\sin(0)}{0} = 1$, which implies for every $x = \frac{k\pi}{N}$, $k \in \{0,1,\ldots,N\}$ that we have $\frac{\sin(Nx-k\pi)}{Nx-k\pi} = 1$.

We define the Max-product truncated combined sampling operators (see also [5])

$$M_N^{(M)}(f)(x) := \sum_{k=0}^{N} \rho_{N,k}(x) f \left(\frac{k\pi}{N}\right), \quad x \in [0,\pi],$$

(44)
Let \( f \in C_+ ([0, \pi]) \), where

\[
M_N^{(M)} (f) (x) := \begin{cases} 
W_N^{(M)} (f) (x), & \text{if } \rho_{N,k} (x) := \frac{\sin(Nx-k\pi)}{Nx-k\pi} \\
K_N^{(M)} (f) (x), & \text{if } \rho_{N,k} (x) := \left( \frac{\sin(Nx-k\pi)}{Nx-k\pi} \right)^2 .
\end{cases} 
\tag{45}
\]

By [6], p. 346 and p. 352 we get

\[
\left( M_N^{(M)} (|\cdot-x|) \right) (x) \leq \frac{\pi}{2N} ,
\tag{46}
\]

and by [3] \( (m \in \mathbb{N}) \) we have

\[
\left( M_N^{(M)} (|\cdot-x|^m) \right) (x) \leq \frac{\pi^m}{2N}, \quad \forall \, x \in [0, \pi], \; \forall \, N \in \mathbb{N}.
\tag{47}
\]

Also it holds \( M_N^{(M)} (1) = 1 \), and \( M_N^{(M)} \) are positive sublinear operators from \( C_+ ([0, \pi]) \) into itself, \( \forall \, N \in \mathbb{N} \).

We give

**Theorem 21** Let \( f \in C^n ([0, \pi], \mathbb{R}_+) \), \( n \in \mathbb{Z}_+ \), \( x \in (0, \pi) \) and \( N^* \in \mathbb{N} : 0 < \frac{1}{N^*} \leq \min (x, \pi - x), \) and \( |f^{(n)} (\cdot) - f^{(n)} (x)| \) is convex over \([0, \pi]\). Assume that \( f^{(k)} (x) = 0 \), \( k = 1, \ldots, n \). Then

\[
\forall \, N \in \mathbb{N} : N \geq N^*; n \in \mathbb{Z}_+, \\
\text{It holds } \lim_{N \to +\infty} M_N^{(M)} (f) (x) = f (x).
\]

**Proof.** By (20) we have:

\[
\left| M_N^{(M)} (f) (x) - f (x) \right| \leq \omega_1 \left( f^{(n)}, \frac{1}{N} \right),
\tag{48}
\]

\[
\left( \omega_1 \left( f^{(n)}, \frac{h}{n+1} \right) \right) M_N^{(M)} (|\cdot-x|^{n+1}) (x) \leq \omega_1 \left( f^{(n)}, \frac{h}{(n+1)!} \right) \frac{\pi^{n+1}}{h (n+1)!} \frac{2N}{\pi} = \left( \frac{\pi^{n+1}}{2 (n+1)!} \right) \omega_1 \left( f^{(n)}, \frac{1}{N} \right),
\tag{49}
\]

proving the claim. \( \blacksquare \)

We make
Remark 22 The Chebyshev knots of first kind \( x_{N,k} := \cos \left( \frac{2(N-k+1)}{2N+1} \pi \right) \in (-1,1) \), \( k \in \{0,1,\ldots,N\} \), \(-1 < x_{N,0} < x_{N,1} < \ldots < x_{N,N} < 1\), are the roots of the first kind Chebyshev polynomial \( T_{N+1}(x) := \cos ((N+1) \arccos x) \), \( x \in [-1,1] \).

Define \( (x \in [-1,1]) \)

\[
h_{N,k}(x) := (1 - x \cdot x_{N,k}) \left( \frac{T_{N+1}(x)}{(N+1)(x-x_{N,k})} \right)^2, \tag{50}
\]
the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see p. 12 of [6]) are defined by

\[
H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall N \in \mathbb{N}, \tag{51}
\]
for \( f \in C_{+}([-1,1]), \forall x \in [-1,1] \).

By [6], p. 287, we have

\[
H_{2N+1}^{(M)}(|\cdot - x|)(x) \leq \frac{2\pi}{N+1}, \quad \forall x \in [-1,1], \forall N \in \mathbb{N}. \tag{52}
\]

And by [3], we get that

\[
H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^m \pi}{N+1}, \quad \forall x \in [-1,1], \forall m, N \in \mathbb{N}. \tag{53}
\]

Notice \( H_{2N+1}^{(M)}(1) = 1 \), and \( H_{2N+1}^{(M)} \) maps \( C_{+}([-1,1]) \) into itself, and it is a positive sublinear operator. Furthermore it holds \( \bigvee_{k=0}^{N} h_{N,k}(x) > 0 \), \( \forall x \in [-1,1] \). We also have \( h_{N,k}(x_{N,k}) = 1 \), and \( h_{N,k}(x_{N,j}) = 0 \), if \( k \neq j \), and \( H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j}) \), for all \( j \in \{0,1,\ldots,N\} \), see [6], p. 282.

We give

Theorem 23 Let \( f \in C^{n}([-1,1],\mathbb{R}_{+}) \), \( n \in \mathbb{Z}_{+}, x \in (-1,1) \) and let \( N^{*} \in \mathbb{N} : 0 < \frac{1}{N^{*}+1} \leq \min(x+1,1-x) \), and \( |f^{(n)}(\cdot) - f^{(n)}(x)| \) is convex over \([-1,1]\). Assume that \( f^{(k)}(x) = 0 \), \( k = 1,\ldots,n \). Then

\[
\forall N \geq N^{*}, N \in \mathbb{N}; n \in \mathbb{Z}_{+}.
\]

It holds \( \lim_{N \rightarrow +\infty} H_{2N+1}^{(M)}(f)(x) = f(x) \).

Proof. By (20) we get

\[
\left| H_{2N+1}^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_{1}(f^{(n)},h)}{h(n+1)!} H_{2N+1}^{(M)}(|\cdot - x|^{n+1})(x), \tag{53}
\]

\begin{align*}
\omega_1 \left( f^{(n)}(h) \right) \frac{2^{n+1}\pi}{h(n+1)!} \frac{2^{n+1}\pi}{N+1} = \\
\quad \text{(setting } h := \frac{1}{N+1})
\end{align*}

proving the claim. \hfill \blacksquare

**Remark 24** Let \( f \in C_+([-1,1]) \). Let the Chebyshev knots of second kind \( x_{N,k} = \cos \left( \left( \frac{N-k}{N-1} \right) \pi \right) \in [-1,1], k = 1, \ldots, N, N \in \mathbb{N} - \{1\} \), which are the roots of \( \omega_N(x) = \sin((N-1)t\sin x), x = \cos t \in [-1,1] \). Notice that \( x_{N,1} = -1 \) and \( x_{N,N} = 1 \).

Define

\begin{equation}
I_{N,k}(x) := \frac{(-1)^{k-1}\omega_N(x)}{(1 + \delta_{k,1} + \delta_{k,N})(N-1)(x-x_{N,k})},
\end{equation}

\( N \geq 2, k = 1, \ldots, N, \) and \( \omega_N(x) = \prod_{k=1}^{N} (x - x_{N,k}) \) and \( \delta_{i,j} \) denotes the Kronecher's symbol, that is \( \delta_{i,j} = 1, \) if \( i = j, \) and \( \delta_{i,j} = 0, \) if \( i \neq j. \)

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints \( \pm 1, \) are defined by ([6], p. 12)

\begin{equation}
L_{N}^{(M)}(f)(x) = \sqrt{\frac{N}{k=1} l_{N,k}(x) f(x_{N,k})}, \quad x \in [-1,1].
\end{equation}

By [6], pp. 297-298 and [3], we get that

\begin{equation}
L_{N}^{(M)}(|-x|^m)(x) \leq \frac{2^{m+1}\pi^2}{3(N-1)},
\end{equation}

\( \forall x \in (-1,1) \) and \( \forall m \in \mathbb{N}; \forall N \in \mathbb{N}, N \geq 4. \)

We see that \( L_{N}^{(M)}(f)(x) \geq 0 \) is well defined and continuous for any \( x \in [-1,1]. \) Following [6], p. 289, because \( \sum_{k=1}^{N} l_{N,k}(x) = 1, \forall x \in [-1,1], \) for any \( x \) there exists \( k \in \{1, \ldots, N\} : l_{N,k}(x) > 0, \) hence \( \sqrt{\sum_{k=1}^{N} l_{N,k}(x)} > 0. \) We have that \( l_{N,k}(x_{N,k}) = 1, \) and \( l_{N,k}(x_{N,j}) = 0, \) if \( k \neq j. \) Furthermore it holds \( L_{N}^{(M)}(f)(x_{N,j}) = f(x_{N,j}), \) all \( j \in \{1, \ldots, N\}, \) and \( L_{N}^{(M)}(1) = 1. \)

By [6], pp. 289-290, \( L_{N}^{(M)} \) are positive sublinear operators.

Finally we present

**Theorem 25** Let \( f \in C^n([-1,1], \mathbb{R}_+), n \in \mathbb{Z}_+, x \in (-1,1) \) and let \( N^* \in \mathbb{N} : N^* \geq 4, \) with \( 0 < \frac{N^*}{N^* - 1} \leq \min(x + 1,1 - x), \) and \( |f^{(n)}(\cdot) - f^{(n)}(x)| \) is convex over \([-1,1]. \) Assume that \( f^{(k)}(x) = 0, k = 1, \ldots, n. \) Then

\begin{equation}
\left| L_{N}^{(M)}(f)(x) - f(x) \right| \leq \omega_1 \left( f^{(n)} \left( \frac{1}{N+1} \right), \frac{2^{n+1}\pi}{3(n+1)!} \right),
\end{equation}

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∀ N ∈ ℕ : N ≥ N^* ≥ 4; n ∈ ℤ_+.
It holds \( \lim_{N \to +\infty} L_N^{(M)}(f)(x) = f(x) \).

Proof. Using (20) we get:

\[
\left| L_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1(f^{(n)}, h)}{h(n+1)!} L_N^{(M)}(|x|^{-n+1})(x) \leq \omega_1 \left( f^{(n)}, h \right) h \left( \frac{2^{n+2} \pi^2}{3(N-1)} \right).
\]

(setting \( h := \frac{1}{\sqrt{n-1}} \))

\[
\left( \frac{2^{n+2} \pi^2}{3(n+1)!} \right) \omega_1 \left( f^{(n)}, \frac{1}{N-1} \right) ,
\]
proving the claim. ■

References


Symmetric identities for Carlitz’s generalized twisted $q$-Bernoulli numbers and polynomials associated with $p$-adic invariant integral on $\mathbb{Z}_p$

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Abstract: In this paper, we study the symmetry for the Carlitz’s generalized twisted $q$-Bernoulli polynomials $\beta_{n,x,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the Carlitz’s generalized twisted $q$-Bernoulli polynomials $\beta_{n,x,q,\zeta}(x)$ using the symmetric properties for the $p$-adic invariant integral on $\mathbb{Z}_p$.

Key words: Symmetric properties, power sums, Bernoulli numbers and polynomials, Carlitz’s generalized twisted $q$-Bernoulli numbers and polynomials, $p$-adic invariant integral on $\mathbb{Z}_p$.

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1. Introduction

Bernoulli polynomials, $q$-Bernoulli polynomials, the second kind Bernoulli polynomials, Euler polynomials, tangent polynomials, and Bell polynomials were studied by many authors (see [1, 3, 4, 5, 6, 7, 8, 9, 10]). Recently, Y. He obtained several identities of symmetry for Carlitz’s $q$-Bernoulli numbers and polynomials in complex field (see [1]). D. Kim et al. [3] studied some identities of symmetry for generalized Carlitz’s $q$-Bernoulli numbers and polynomials by using the $p$-adic integrals on $\mathbb{Z}_p$ in $p$-adic field. The purpose of this paper is to obtain some interesting identities of the power sums and Carlitz’s generalized twisted $q$-Bernoulli polynomials $\beta_{n,x,q,\zeta}(x)$ using the symmetric properties for the $p$-adic invariant integral on $\mathbb{Z}_p$.

Let $\mathbf{p}$ be a fixed prime number. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad (\text{cf. [1-4]}).$$

Hence, $\lim_{q \to 1}[x] = x$ for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$ the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x), \quad (\text{cf. [2, 3, 4]}). \quad (1.1)$$

Let a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{N} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p, \quad a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N}\},$$
where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < dp^N \). It is easy to see that
\[
\int_X g(x) d\mu_q(x) = \int_{Z_p} g(x) d\mu_q(x).
\]
(1.2)

Let \( T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N} \), where \( C_{p^N} = \{ \zeta \mid \zeta^{p^N} = 1 \} \) is the cyclic group of order \( p^N \). For \( \zeta \in T_p \), we denote by \( \phi_\zeta : Z_p \to C_p \) the locally constant function \( x \mapsto \zeta^x \) (cf. [6, 10]).

2. Symmetric identities for Carlitz’s generalized twisted \( q \)-Bernoulli numbers and polynomials

Mathematicians investigated interesting properties of symmetry for special polynomials using \( p \)-adic invariant integral on \( Z_p \) (see [1, 3, 4, 5]). If we take \( \chi^0 = 1 \), then [5] is the special case of this paper. Let \( \chi \) be Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( (d, p) = 1 \). For \( q \in C_p \) with \( |q - 1|_p < p^{-\frac{N}{2}} \), the twisted \( q \)-Bernoulli polynomials \( \beta_{n,q,\zeta}(x) \) are defined by
\[
\beta_{n,q,\zeta}(x) = \int_{Z_p} \phi_\zeta(y) q^n [x + y]_q^N d\mu_1(y).
\]

Let \( \beta_{n,q,\zeta}(x) \) be natural numbers. Then, by (1.1) and (1.2), we obtain
\[
\begin{align*}
\frac{1}{w_1} \int_X \chi(y) \zeta^{w_1} q^{w_1} e^{[w_1 w_2 x + w_2 j + w_1 y]_t} d\mu_1(y) \\
&= \lim_{N \to \infty} \frac{1}{w_1} \frac{1}{dw_2 p^N} \sum_{y=0}^{dw_2 - 1} \chi(y) \zeta^{w_1} q^{w_1} e^{[w_1 w_2 x + w_2 j + w_1 y]_t} q^{w_1 y} \\
&= \lim_{N \to \infty} \frac{1}{w_1} \frac{1}{dw_2 p^N} \sum_{y=0}^{dw_2 - 1} \chi(y) \zeta^{w_1} q^{w_1} e^{[w_1 w_2 x + w_2 j + w_1 y]_t} q^{w_1 y}.
\end{align*}
\]

From (2.1), we can derive the following equation (2.2):
\[
\begin{align*}
\frac{1}{w_1} \sum_{j=0}^{dw_1 - 1} \chi(j) \zeta^{w_1} q^{w_1 j} \int_X \chi(y) \zeta^{w_1} q^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_t} d\mu_1(y) \\
&= \lim_{N \to \infty} \frac{1}{dw_1 w_2 p^N} \sum_{j=0}^{dw_1 - 1} \sum_{i=0}^{dw_2 - 1} \sum_{y=0}^{p^N - 1} \chi(i) \chi(j) \zeta^{w_2 j} \zeta^{w_1 i} q^{w_2 j} q^{w_1 i} \\
&\quad \times e^{[w_1 w_2 x + w_2 j + w_1 y + dw_1 w_2 y]_t},
\end{align*}
\]

By the same method as (2.2), we obtain
\[
\begin{align*}
\frac{1}{w_2} \sum_{j=0}^{dw_2 - 1} \chi(j) \zeta^{w_2} q^{w_2 j} \int_X \chi(y) \zeta^{w_2} q^{w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 y]_t} d\mu_1(y) \\
&= \lim_{N \to \infty} \frac{1}{dw_1 w_2 p^N} \sum_{j=0}^{dw_2 - 1} \sum_{i=0}^{dw_1 - 1} \sum_{y=0}^{p^N - 1} \chi(i) \chi(j) \zeta^{w_1} \zeta^{w_2} q^{w_1 i} q^{w_2 j} \\
&\quad \times e^{[w_1 w_2 x + w_1 j + w_2 y + dw_1 w_2 y]_t}.
\end{align*}
\]
Therefore, by (2.2) and (2.3), we have the following theorem.

**Theorem 1.** For \( w_1, w_2 \in \mathbb{N} \), we have

\[
\frac{1}{w_1} \sum_{j=0}^{d_{w_1}-1} \chi(j)\zeta^{w_2j}q^{w_2j} \int_X \chi(y)\zeta^{w_1y}q^{w_1y} e^{\left[w_1w_2x+w_2j+w_1y\right]t} \, d\mu_1(y)
= \frac{1}{w_2} \sum_{j=0}^{d_{w_2}-1} \chi(j)\zeta^{w_1j}q^{w_1j} \int_X \chi(y)\zeta^{w_2y}q^{w_2y} e^{\left[w_1w_2x+w_1j+w_2y\right]t} \, d\mu_1(y).
\]

By substituting Taylor series of \( e^{xt} \) into (2.4) and after calculations, we obtain the following corollary.

**Corollary 2.** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
\frac{[w_1]_q}{w_1} \sum_{j=0}^{d_{w_1}-1} \chi(j)\zeta^{w_2j}q^{w_2j} \int_X \chi(y)\zeta^{w_1y}q^{w_1y} \left[w_2x + \frac{w_2}{w_1} j + y\right]^n \, d\mu_1(y)
= \frac{[w_2]_q}{w_2} \sum_{j=0}^{d_{w_2}-1} \chi(j)\zeta^{w_1j}q^{w_1j} \int_X \chi(y)\zeta^{w_2y}q^{w_2y} \left[w_1x + \frac{w_1}{w_2} j + y\right]^n \, d\mu_1(y).
\]

By Corollary 2, we have the following theorem.

**Theorem 3.** For \( w_1, w_2 \in \mathbb{N}, n \geq 0 \), we have

\[
\frac{[w_1]_q}{w_1} \sum_{j=0}^{d_{w_1}-1} \chi(j)\zeta^{w_2j}q^{w_2j} \beta_{n,1},q^{w_1},q^{w_2} \left(w_2x + \frac{w_2}{w_1} j\right)
= \frac{[w_2]_q}{w_2} \sum_{j=0}^{d_{w_2}-1} \chi(j)\zeta^{w_1j}q^{w_1j} \beta_{n,1},q^{w_2},q^{w_2} \left(w_1x + \frac{w_1}{w_2} j\right).
\]

By (2.5), we can derive the following equation:

\[
\int_X \chi(y)\zeta^{w_1y}q^{w_1y} \left[w_2x + \frac{w_2}{w_1} j + y\right]^n \, d\mu_1(y)
= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q}\right)^i \left[j\right]^{n-i} q^{w_2(n-i)j} \int_X \chi(y)\zeta^{w_1y}q^{w_1y} \left[w_2x + y\right]^{n-i} \, d\mu_1(y) \quad (2.5)
\]

Again, by (2.5), and Theorem 3, we have

\[
\frac{[w_1]_q}{w_1} \sum_{j=0}^{d_{w_1}-1} \chi(j)\zeta^{w_2j}q^{w_2j} \int_X \chi(y)\zeta^{w_1y}q^{w_1y} \left[w_2x + \frac{w_2}{w_1} j + y\right]^n \, d\mu_1(y)
= \sum_{j=0}^{d_{w_1}-1} \chi(j)\zeta^{w_2j}q^{w_2j} \sum_{i=0}^{n} \binom{n}{i} \left[\frac{[w_2]_q}{[w_1]_q}\right]^{n-i} \left[j\right]^{n-i} q^{w_2(n-i)j} \beta_{n-i,1},q^{w_1},q^{w_2} (w_2x) \quad (2.6)
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left[\frac{[w_2]_q}{[w_1]_q}\right]^{n-i} \beta_{n-i,1, q^{w_1}, q^{w_2}} (w_2x) \sum_{j=0}^{d_{w_1}-1} \zeta^{w_2j}q^{w_2(n-i)j} \left[j\right]^{n-i}.
\]

Again, by (2.5), and Theorem 3, we have

\[
S_{n,j}(w_1, \zeta, q\chi) = \sum_{j=0}^{w_1-1} \chi(j)\zeta^j q^{(n-i+h)j} [j]_q^i.
\]

where

\[
S_{n,i}(w_1, \zeta, q\chi) = \sum_{j=0}^{w_1-1} \chi(j)\zeta^j q^{(n-i+h)j} [j]_q^i.
\]
Therefore, by (2.6) and (2.7), we have the following theorem.

By the same method as (2.6), we get

$$\frac{[w_2^n]}{w_2} \sum_{j=0}^{n-1} \chi(j)\zeta^{w_1j}q^{w_1j} \int_X \chi(y)\zeta^{w_2y}q^{w_2y} \left[ w_1x + \frac{w_1}{w_2} j + y \right]^n dq_1(y)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{[w_2^n][w_2^n]}{w_2} \beta_{n-i,\chi,q^{w_2}x^{w_2}} (w_1x) S_n,i(dw_2,\zeta^{w_1},q^{w_1}|x).$$

(2.7)

Therefore, by (2.6) and (2.7), we have the following theorem.

**Theorem 4.** For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\sum_{i=0}^{n} \binom{n}{i} \frac{[w_2^n][w_1^n]}{w_1^n} \beta_{n-i,\chi,q^{w_1}x^{w_1}} (w_2x) S_n,i(dw_1,\zeta^{w_2},q^{w_2}|x)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{[w_1^n][w_2^n]}{w_2^n} \beta_{n-i,\chi,q^{w_2}x^{w_2}} (w_1x) S_n,i(dw_2,\zeta^{w_1},q^{w_1}|x).$$

**Remark 5.** Let $w_1, w_2 \in \mathbb{N}, n \geq 0$, and $\chi$ be the trivial character. Then we have

$$\sum_{i=0}^{n} \binom{n}{i} \frac{[w_2^n][w_1^n]}{w_1^n} \beta_{n-i,\chi,q^{w_1}x^{w_1}} (w_2x) S_n,i(w_1|\zeta^{w_2},q^{w_2})$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{[w_1^n][w_2^n]}{w_2^n} \beta_{n-i,\chi,q^{w_2}x^{w_2}} (w_1x) S_n,i(w_2|\zeta^{w_1}q^{w_1}).$$

**REFERENCES**


An efficient optimal algorithm for high frequency in wavelet based image reconstruction

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Abstract

Wavelet algorithms for high-resolution image reconstruction has been shown effectively, it relies on the decomposition of low/high frequency, and hard/soft thresholding arguments are often used to denoise for high frequency. In this paper, instead of using this kind of thresholding arguments, we apply the gradient based shrinkage thresholding optimization for high-frequency, in this way, we can keep the useful information in the original signal as much as possible, coupling the shrinkage thresholding optimization with the wavelet algorithm, we get an efficient reconstruction algorithm. Numerical results show we obtain a higher resolution, better peak signal-to-noise ratios and lower relative errors.

Key words: Wavelet; high-resolution; image reconstruction; shrinkage thresholding; high frequency.

1 Introduction

Increasing the resolution is important and necessary for many applications, lots of studies have been done on the high-resolution image reconstruction [13, 14, 18, 20, 21, 22, 23, 24, 27].

Among the methods in image processing, wavelet method is a well developed technology [6, 9, 10, 12, 26]. In this method, global patterns are represented by densely distributed coefficients obtained from low-pass filtering, while local features are represented by coefficients obtained from high-pass filtering. This makes it easy for us to distinct between smooth and sharp image components. In this way wavelet frames can effectively separate smooth image components and non smooth ones, and the wavelet-based procedure is essentially to approximate iteratively the densely distributed coefficients folded by the given low-pass filter. To overcome the incompatibility of symmetry and exact reconstruction, bi-orthogonal wavelet system is thus proposed in image processing, see [1, 8, 25].

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The relatively complex hard/soft thresholding methods [11] are often used to denoise for high frequency information, but some useful information will lose in the processing because of its cut off action. Preserving useful high frequency part while removing noise is the main goal in image denoising, some techniques developed in the past years has shown their advantage than the hard- and soft-thresholding in the wavelet field, for example, the wavelet packet method, it is based on the further decomposition of wavelet coefficients by packets, and this leads to an essentially translation invariant wavelet packet system.

To get an efficient algorithm while keep useful information in high frequency as much as possible, we consider the optimization strategy instead of hard/soft thresholding method for the high frequency components, this strategy is based on the classic variation technology, and has been previously used in image reconstruction, because of its computational complex, a fast iterative shrinkage-thresholding algorithms are proposed in [2, 4], this kind of method, which can be viewed as an extension of the classical gradient algorithm, is attractive due to its simplicity, it is adequate for solving large-scale problems even with dense matrix data in image reconstruction, to improve the convergence rate, a more fast iterative shrinkage-thresholding algorithm with a significantly better global convergence rate is introduced in [3, 28], this algorithm improves the convergence rate from $O(1/k)$ to $O(1/k^2)$, it relies on computing the next iteration based on the values not only in the previous one, but also in two previously computed steps.

In this paper, we are intent to improve the wavelet algorithm in image reconstruction. We begin with the bi-orthogonal wavelet systems, and obtain the decomposition formula, which represent a perfect reconstruction equation for the symbols of the low-pass and the high-pass filters, theoretical analysis shows that the noise is contains in high-frequency part, and the hard/soft thresholding argument will inevitably delete some useful information, instead of using this kind of thresholding argument for high-frequency components of the original image, we take advantage of shrinkage thresholding algorithm for the optimization of high-frequency, it has been proved that it has notable effect in image denoising, to get the algorithm more efficient, we apply some accelerating iteration argument in shrinkage thresholding algorithm.

The outline of the paper is as follows. the algorithms are derived in section 2. Numerical examples are given in section 3 to illustrate the effectiveness of the algorithms. Some concluding remarks are given in section 4.

2 Reconstruction algorithm

In this section, we construct a shrinkage thresholding optimization coupling with the wavelet based algorithm for high resolution image reconstruction.
2.1 Iterative scheme

Refer to [5], we obtain that using the periodic boundary condition and ordering the discretized values of $f$ and $g$ in a row-by-row fashion, we obtain $M1M2 \times M1M2$ linear system of the form:

$$Lf = g$$  \hspace{1cm} (1)

where $f$ is original image, $g$ low-resolution image, $L = Lx \otimes Ly$ is the blurring matrix which is made up from each sensor, and $Lx, Ly$ have circulant structure as follow:

$$Lx = \frac{1}{L} \cdot \text{circulant}(a),$$

where

$$a = [1, \cdots, 1, \frac{1}{2}, 0, \cdots, 0, \frac{1}{2}, 1, \cdots, 1]^t,$$

where $\text{circulant}(a)$ represents circulant matrix, and the first $L/2$ entries in $a$ are equal to 1, the last $L/2 − 1$ entries are equal to 1. The matrix $Ly$ can be define similarly, these matrix are circulant matrices, then we get that the matrix $L$ is a block-circulant-circulant-block (BCCB)matrix [17].

By the biorthogonal wavelet theory [7, 19], the symbols of the refinement masks and wavelet masks satisfy the following equation

$$\widehat{a}^d \hat{a} + \sum_{\nu \in Z^2_k \setminus \{(0,0)\}} \widehat{b}_\nu^d \hat{b}_\nu^d = 1$$ \hspace{1cm} (2)

where $K$ is sensor size.

The equation (2) is not only for the reconstruction of function but also for image reconstruction, the matrix representation of the perfect reconstruction from biorthogonal system can be written as

$$L^dL + \sum_{\nu \in Z^2_k \setminus \{(0,0)\}} M^d_\nu M_\nu = I,$$ \hspace{1cm} (3)

here denote by $L, L^d, M^d_\nu, M_\nu$ the matrices generated by the symbols of the refinement and wavelet masks $\hat{a}, \widehat{a}^d, \hat{b}_\nu, \hat{b}_\nu^d$, respectively.

Since $g = Lf$ is just the observed high-resolution image, and the other $M_\nu f, \nu \neq (0,0)$, represent the high-frequency components of $f$, from equation (3) we obtain an iterative algorithm

$$f_{n+1} = L^d g + \left( \sum_{\nu \in Z^2_k \setminus \{(0,0)\}} M^d_\nu M_\nu \right) f_n$$ \hspace{1cm} (4)

In the usual denoising procedure, the high frequency components are often penalized by a factor, this smoothes the original signals, so a nonlinear denoising scheme can be built into equation (4), and thus obtain an iterative algorithm

$$f_{n+1} = L^d g + \sum_{\nu \in Z^2_k \setminus \{(0,0)\}} M^d_\nu T(M_\nu f_n).$$ \hspace{1cm} (5)
where $T$ is a denoising operator, a hard/soft thresholding wavelet denoising algorithm is presented in [7], in which a further decomposition by the translation invariant wavelet packets is used, this can remedy the smoothing effect on the original signals in some sense, some useful information in high frequency is still lost, this motivates us to consider an efficient optimal method to keep information as much as possible.

2.2 Shrinkage thresholding optimization for high frequency

Let $b = M_n f_n$, we consider the following formulation:

$$ x^* = \min_x F(x), \quad F(x) = ||Ax - b||^2, \quad (6) $$

where $A = L^d L$, and $L$ is the blurring matrix in the last section, the norm $|| \cdot ||$ is the inner product, and $x$ is the vector we are looking for, this is the classical least square problem. The optimization problem (6) can be cast as a second order cone programming problem and thus could be solved via interior point methods.

Usually this problem is not only in large scale but also involves dense matrix data, which often precludes the use and potential advantage of sophisticated interior point methods. This motivated a simpler gradient-based algorithms for solving it, the gradient algorithm generates a sequence $\{x_k\}$ via

$$ x_k = x_{k-1} - t_k \nabla F(x_{k-1}), $$

where $t_k > 0$ is a suitable stepsize. It can be viewed as an approximal regularization of the linearized function $F$ at $x_{k-1}$, and can be written equivalently as

$$ x_k = \min_x \{ F(x_{k-1}) + (x - x_{k-1}, \nabla F(x_{k-1})) + \frac{1}{2t_k} ||x - x_{k-1}||^2 \}. $$

After ignoring constant terms, this can be rewritten as

$$ x_k = \min_x \{ \frac{1}{2t_k} ||x - (x_{k-1} - t_k \nabla F(x_{k-1}))||^2 \}, $$

the computation of $x_k$ reduces to solving a one-dimensional minimization problem for each of its components, which produces

$$ x_k = T_{M_k} (x_{k-1} - t_k \nabla F(x_{k-1})), $$

considering the expression of $F(x)$ in (6), we get:

$$ x_k = T_{M_k} (x_{k-1} - 2t_k A^T (Ax_{k-1} - b)), \quad (7) $$

where $T_\alpha : R^2 \to R^2$ is the thresholding operator defined by

$$ T_\alpha (x_i) = (|x_i| - \alpha)_+ sgn(x_i). \quad (8) $$
This algorithm (7) is a kind of iterative shrinkage thresholding algorithm similar as that in [15].
It has been proved that for large-scale problems this first order methods are only practical option, and the sequence \( x_k \) converges quite slowly to its solution, that is
\[
F(x_k) - F(x^*) = O(1/k),
\]
namely, it shares a sublinear global rate of convergence.

To improve the efficiency of the iterative shrinkage thresholding algorithm (ISTA) with better global rate of convergence, Beck etc.[3] consider an improved fast iteration, that is the \( x_k \) in (7) is not dependent on the previous point \( x_{k-1} \), but rather on the point \( y_k \) which is a linear combination of the previous two point \( \{x_{k-1}, x_{k-2}\} \), with this modification, they get a fast ISTA, and the convergence rate is
\[
F(x_k) - F(x^*) = O(1/k^2).
\]
In this way, we get \( x^* \) from \( b \) according to the above iterative shrinkage thresholding algorithm.

### 2.3 Summary of Algorithms

For convenience, we call our shrinkage thresholding algorithm in wavelet based reconstruction algorithm as STWL, to compare with the hard/soft thresholding wavelet reconstructed algorithm (abbr. TWL) in [7]. Now we embed the Shrinkage thresholding optimization algorithm into the iteration scheme (5), denote the previous two iterations as \( \{f_{n-1}, f_{n-2}\} \), then our algorithm for the model equation (5) can be summarized as follows:

1. Choose an initial approximation \( f_0 \) (e.g., \( f_0 = L^d g \));
2. Iterate until convergence:
   Outer circulation:
   \[
   f_{n+1} = L^d g + \sum_{\nu \in Z_2^d \setminus \{(0,0)\}} M_{\nu}^T \hat{T}(M_{\nu} f_n).
   \]

   Begin inner loop:
   To get optimal high frequency part \( \hat{T}(M_{\nu} f_n) \). Let \( b_{\nu} = M_{\nu} f_n, y_{1,\nu} = M_{\nu} f_n, h_{0,\nu} = 1, t_1 = 1 \), then a fast iteration for (7) is
   \[
   h_{k,\nu} = T_{M_k}(y_{k,\nu} - 2t_k A^T(y_{k,\nu} - b_{\nu})),
   \]
   \[
   t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},
   \]
   \[
   y_{k+1,\nu} = h_{k,\nu} + \left( \frac{t_k - 1}{t_{k+1}} \right)(h_{k,\nu} - h_{k-1,\nu}),
   \]
where $T_{M_k}$ is defined as in (8), $\lambda$ is estimated by the method given in [11] which uses the median of the absolute value of the entries in the vector $M_{\nu} f_n$.

End inner loop.

Return the optimized results $y_{n,\nu}^*$ for high frequency $\tilde{T}(M_{\nu} f_n)$, that is

$$f_{n+1} = L^d g + \sum_{\nu \in \mathbb{Z}_2^d \setminus \{(0,0)\}} M_{\nu} y_{n,\nu}^*.$$ 

End outer circulation.

Remark: When the operator $\tilde{T}$ in (9) is realized by soft/hard thresholding operator $T$, then this reduces to the soft/hard thresholding wavelet reconstruction algorithm as that in [7].

3 Experimental results

In this part, we present the efficiency and accuracy of our shrinkage thresholding wavelet based reconstruction algorithm (abbr. STWL), and compare with the hard/soft thresholding wavelet reconstructed method (abbr. TWL). As usual, we evaluate the methods using the peak signal-to-noise ratio (PSNR), relative error (RE) and cpu time cost they are defined by

$$RE = \frac{\| f - f_c \|_2}{\| f \|_2},$$

and

$$PSNR = 10 \log_{10} \frac{\| f \|_2^2}{\| f - f_c \|_2^2}$$

for 1D signals, while as

$$PSNR = 10 \log_{10} \frac{255^2 N M}{\| f - f_c \|_2^2}$$

for 2D images, respectively, with the size of the signals (images) is $N \times M$. Where $f$ is original image, and $f_c$ is restored image.

In our tests, $N = 1$ for 1D signals while $N = M$ for 2D images. Here we take $2 \times 2$ and $4 \times 4$ sensor arrays in 2D.

3.1 1D denoisy signal recovery

We take the original signal data from the WaveLab toolbox at http://statweb.stanford.edu/wavelab/ developed by Donoho’s research group. Fig. 1(a) shows the original signal $f$. Fig. 1(b) depicts the contaminated signal with white noise at signal-to-noise ratio ($SNR = 25$), here we use matlab function awgn to add noise to the original signal. The results of denoising by the above two algorithms with periodic conditions are shown in Fig. 1(c) and (d), respectively. From the data results of experiments in table 1, it shows that our algorithm STWL has a better performance, and has a better time efficiency than TWL.
Fig. 1: (a) Original signal; (b) Contaminated by white noise at SNR = 25; (c) Reconstructed signal from the algorithm TWL; (d) Reconstructed signal from our algorithm STWL.

Table 1: Corresponding PSNR, RE and timecost values using algorithm TWL in [7] and our algorithm STWL.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>( SNR = 25 )</th>
<th>( SNR = 30 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PSNR</td>
<td>RE</td>
</tr>
<tr>
<td>TWL</td>
<td>47.1560</td>
<td>0.0946</td>
</tr>
<tr>
<td>STWL</td>
<td>53.3382</td>
<td>0.0695</td>
</tr>
</tbody>
</table>

3.2 High-resolution image reconstruction

In this section, we use the classical "boat", the "Lena" and the "cameraman" images with size of 256 \( \times \) 256 as the original images for our tests, and consider 2 \( \times \) 2 sensor arrays and 4 \( \times \) 4 sensor arrays respectively, and the Gaussian white noise is added to these original images.
The dual pair and \( m \) see [12] for details. are \( a \) and \( b \) the eigenvalues of the matrix shrinkage thresholding method, the Lipschitz constant is computable in the examples since the eigenvalues of the matrix \( A^T A \) can be easily calculated using the two-dimensional cosine transform [17]. For simplicity, we will only consider the matrices for the periodic case.

### 3.2.1 2 \times 2 sensor array

For \( 2 \times 2 \) sensor arrays, the corresponding refinement mask \( m \) is the piecewise linear spine,

\[
m(-1) = \frac{1}{4}, m(0) = \frac{1}{2}, m(1) = \frac{1}{4},
\]

and \( m(\alpha) = 0 \) for all other \( \alpha \). The nonzero terms of the dual mask of \( m \) used in this paper are

\[
m^d(-2) = -\frac{1}{8}, m^d(-1) = \frac{1}{4}, m^d(0) = \frac{3}{4}, m^d(1) = \frac{1}{4}, m^d(2) = -\frac{1}{8}.
\]

The dual pair of the wavelet masks are \( r_\alpha := (-1)^\alpha m^d(1 - \alpha) \) and \( r^d(\alpha) := (-1)^\alpha m(1 - \alpha) \), see [12] for details.

The tensor product dual pair of the refinement symbols are given by \( \hat{a}(\omega) = \hat{m}(\omega_1)\hat{m}(\omega_2) \), \( \hat{a}^d(\omega) = \hat{m}^d(\omega_1)\hat{m}^d(\omega_2) \), and the corresponding wavelet symbols are \( \hat{b}_{(0,1)}(\omega) = \hat{m}(\omega_1)\hat{r}(\omega_2) \), \( \hat{b}^d_{(0,1)}(\omega) = \hat{m}^d(\omega_1)\hat{r}^d(\omega_2) \), \( \hat{b}_{(1,0)}(\omega) = \hat{r}(\omega_1)\hat{m}(\omega_2) \), \( \hat{b}^d_{(1,0)}(\omega) = \hat{r}^d(\omega_1)\hat{m}^d(\omega_2) \), \( \hat{b}_{(1,1)}(\omega) = \hat{r}(\omega_1)\hat{r}(\omega_2) \), \( \hat{b}^d_{(1,1)}(\omega) = \hat{r}^d(\omega_1)\hat{r}^d(\omega_2) \), where \( \omega = (\omega_1, \omega_2) \).

Although we give here only the details of the refinable functions and their corresponding wavelets with dilation \( 2I \), the whole theory can be carried over to the general isotropic integer dilation matrices.

The wavelet matrices are formed by the tensor product, and we consider

\[
Z_2^2 = \{(0,0), (0,1), (1,0), (1,1)\}.
\]

In particular, we have

\[
L = L_2 \otimes L_2, \quad M_{(0,1)} = L_2 \otimes M_2, \quad M_{(1,0)} = M_2 \otimes L_2, \quad M_{(1,1)} = M_2 \otimes M_2,
\]

\[
L^d = L_2^d \otimes L_2^d, \quad M_{(0,1)}^d = L_2^d \otimes M_2^d, \quad M_{(1,0)}^d = M_2^d \otimes L_2^d, \quad M_{(1,1)}^d = M_2^d \otimes M_2^d.
\]

where

\[
L_2 = \text{circulant}(\frac{1}{2}, \frac{1}{4}, 0, \cdots, 0, \frac{1}{4}), \quad L_2^d = \text{circulant}(\frac{3}{4}, \frac{1}{4}, -\frac{1}{8}, 0, \cdots, 0, -\frac{1}{8}, \frac{1}{4})
\]

\[
M_2 = \text{circulant}(\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{8}, 0, \cdots, 0, \frac{1}{8}), \quad M_2^d = \text{circulant}(\frac{1}{4}, 0, \cdots, 0, \frac{1}{4}, -\frac{1}{2}).
\]
Fig. 2 demonstrate the reconstructed high-resolution image for the "boat", the "Lena" and the "cameraman" images respectively, in these figures, (a1)-(c1) are the original images, (a2)-(c2) are with noise $PSNR = 40dB$, (a3)-(c3) are the denoisy images with the algorithm TWL, and (a4)-(c4) are obtained by our algorithm STWL. Table 2 gives the PSNR, RE, and the runtime of the reconstructed images for different levels of Gaussian noise, our algorithm shows less RE, less runtime and better PSNR, we can conclude that our algorithm STWL is better than the original algorithm TWL in [7].

Fig. 2: (a1)-(c1) the original images; (a2)-(c2) the noisy images; (a3)-(c3) the reconstructed image by algorithm TWL; (a4)-(c4) the reconstructed images by our algorithm STWL.
Table 2: Comparison of PSNR, RE and cputime values using algorithm TWL and STWL for $2 \times 2$ sensor arrays with different kinds noise level.

<table>
<thead>
<tr>
<th>Image</th>
<th>Evaluation</th>
<th>TWL $SNR = 35$</th>
<th>TWL $SNR = 40$</th>
<th>STWL $SNR = 35$</th>
<th>STWL $SNR = 40$</th>
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<td></td>
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<td>RE</td>
<td>PSNR</td>
<td>RE</td>
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<tr>
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<td>85.0012</td>
</tr>
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<td>5.9124</td>
<td>5.6472</td>
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3.2.2 $4 \times 4$ sensor array

In this case, we give the refinable and wavelet masks with dilation 4$I$ that used to generate the matrices for $4 \times 4$ sensor arrays.

For $4 \times 4$ sensor arrays, the corresponding mask is

\[
m(\alpha) = \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \alpha = -2, \cdots, 2,
\]

with $m(\alpha) = 0$ for all other $\alpha$. The nonzero terms of a dual refinement mask of $m$ is

\[
m^d(\alpha) = \frac{1}{16}, \frac{5}{16}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, -\frac{1}{4}, \alpha = -3, \cdots, 3.
\]

The nonzero terms of the corresponding wavelet masks are

\[
r_1(\alpha) = \frac{1}{8}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{8}, \alpha = -2, \cdots, 2,
\]

\[
r_2(\alpha) = \frac{1}{16}, -\frac{1}{8}, -\frac{1}{16}, -\frac{5}{16}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \alpha = -2, \cdots, 4,
\]

\[
r_3(\alpha) = \frac{1}{16}, \frac{1}{8}, -\frac{1}{16}, 0, \frac{7}{16}, -\frac{1}{8}, -\frac{1}{16}, \alpha = -2, \cdots, 4.
\]

The dual wavelet masks are

\[
r_1^d(\alpha) = (-1)^{1-\alpha}r_3(1-\alpha), r_2^d(\alpha) = (-1)^{1-\alpha}m(1-\alpha), r_3^d(\alpha) = (-1)^{1-\alpha}r_1(1-\alpha)
\]
The observed high-resolution image \( g \) is generated by applying the bivariate lowpass filter on the true image \( f \), again, we consider periodic boundary condition. The matrices

\[
L, L^d, M, M^d, \nu \in \mathbb{Z}_4^2 \setminus \{(0, 0)\}
\]

can be generated by the corresponding filters.

Fig. 3 shows the reconstructed high-resolution image for the "boat", the "Lena" and the "cameraman" images, (a1)-(c1) are blurred with noise \( PSNR = 40dB \), (a2)-(c2) are obtained from the algorithm TWL, and (a1)-(c1) are obtained from our algorithm STWL. From Table 3, we can also find that our algorithm shows less RE, less cputime and better PSNR, since the problem is more difficult than the \( 2 \times 2 \) sensor case, we need more cputime consuming, we can see that the performance of our algorithm STWL is much better than the original algorithm TWL.

Table 3: Comparison of PSNR, RE and cputime values using algorithm TWL and STWL for \( 4 \times 4 \) sensor arrays with different kinds noise level.

<table>
<thead>
<tr>
<th>Image</th>
<th>Evaluation</th>
<th>TWL ( SNR = 30 )</th>
<th>TWL ( SNR = 40 )</th>
<th>STWL ( SNR = 30 )</th>
<th>STWL ( SNR = 40 )</th>
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<td>69.9279</td>
<td>69.9769</td>
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<td>0.0281</td>
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<tr>
<td></td>
<td>cputime</td>
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<td>20.0773</td>
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</table>

4 Conclusions

In this paper, we constructed a shrinkage thresholding algorithm in wavelet based image reconstruction, instead of using the hard/soft thresholding algorithm we apply the iterative shrinkage thresholding algorithm for the optimization for high frequency. Our new algorithm works effectively both in one-dimensional and two-dimensional situations, numerical tests show that this algorithm gives higher resolution, larger signal-to noise ratios, lower relative errors and less cputime.
Fig. 3: (a1)-(c1) is the noisy images, (a2)-(c2) is reconstructed from the algorithm TWL; (a3)-(c3) is reconstructed from our algorithm STWL.

Acknowledgments

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References


Abstract

This research is about inequalities in a local fractional environment over a negative domain. The author presents the following types of analytic local fractional inequalities: Opial, Hilbert-Pachpatte, comparison of means, Poincare and Sobolev. The results are with respect to uniform and $L_p$ norms, involving left and right Riemann-Liouville fractional derivatives.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.

Keywords and phrases: Local fractional derivative, Riemann-Liouville fractional derivative, Opial inequality, Hilbert Pachpatte, Poincare inequality, fractional inequalities.

1 Introduction

Many sources motivate us to write this work. The first one comes next. It is the famous Opial inequality ([13]):

$$\int_0^a |y'(x)y(x)| \, dx \leq \frac{a}{2} \int_0^a |y'(x)|^2 \, dx,$$

(1)

where $y(x)$ is absolutely continuous function and $y(0) = 0$. The above inequality is proved sharp.

The well known Ostrowski ([14]) inequality also motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y) \, dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right) (b-a) \| f' \|_\infty,$$

(2)

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Next $D^\rho_a f$ indicates the left Caputo fractional derivative of order $\rho > 0$, anchored at $a \in \mathbb{R}$, see [10], p. 50.
The author in [7], pp. 82-83, proved the following left Caputo fractional Landau inequality: Let $0 < \nu \leq 1$, $f \in AC^{2}([0,b])$ (i.e. $f' \in AC([0,b])$, absolutely continuous functions), $\forall \ b > 0$. Suppose $\|f\|_{\infty,R_{+}} < +\infty$, $D_{x_{0}}^{\nu+1} f \in L_{\infty} (R_{+})$, and

$$\|D_{x_{0}}^{\nu+1} f\|_{\infty,[a,+\infty)} \leq \|D_{x_{0}}^{\nu+1} f\|_{\infty,R_{+}}, \ \forall \ a \geq 0. \hspace{1cm} (3)$$

Then

$$\|f'\|_{\infty,R_{+}} \leq (\nu + 1) \left( \frac{2}{\nu} \right)^{\frac{1}{1-\nu}} (\Gamma (\nu + 2))^{-\frac{1}{1-\nu}} \left( \|f\|_{\infty,R_{+}} \right)^{\frac{1}{1-\nu}} \left( \|D_{x_{0}}^{\nu+1} f\|_{\infty,R_{+}} \right)^{\frac{1}{1-\nu}} , \hspace{1cm} (4)$$

that is $\|f'\|_{\infty,R_{+}}$ is finite.

The last inequality is another inspiration.

The author’s monographs [2], [3], [4], [5], [6], [8], motivate and support largely this work too. See also [1].

Under the point of view of local fractional differentiation the author examines the broad area of analytic inequalities and produces a variety of well-known inequalities in a local fractional setting over a negative domain to all possible directions.

2 Background

We mention

**Definition 1 ([11])** Let $x, x' \in [a,b]$, $f \in C([a,b])$. The Riemann-Liouville (R-L) fractional derivative of a function $f$ of order $q$ ($0 < q < 1$) is defined as

$$D_{x}^{q} f (x') = \begin{cases} D_{x}^{q} f (x') , & x' > x, \\ D_{x}^{q} f (x') , & x' < x \end{cases}$$

$$= \frac{1}{\Gamma (1-q)} \left\{ \begin{array}{ll} \frac{d}{dt} \int_{x_{0}}^{x'} (x' - t)^{-q} f (t) dt , & x' > x, \\ -\frac{d}{dt} \int_{x_{0}}^{x} (t - x')^{-q} f (t) dt , & x' < x, \end{array} \right\} , \hspace{1cm} (5)$$

the left and right R-L fractional derivatives, respectively.

We need

**Definition 2 ([11], [12])** The local fractional derivative of order $q$ ($0 < q < 1$) of a function $f \in C([a,b])$ is defined as

$$D_{x}^{q} f (x) = \lim_{x' \to x} D_{x}^{q} (f (x') - f (x)). \hspace{1cm} (6)$$

More generally we define
Remark 5 Here \( x', x \in [a, b] \), and \( a \leq x + t \leq b \), equivalently \( a - x \leq t \leq b - x \). From \( a \leq x \leq b \), we get \( a - x \leq 0 \leq b - x \). We assume here that \( F(x, \cdot ; q, N) \in C^1([a - x, b - x]) \). Clearly, then it holds
\[
D^{N+q} f(x) = F(x, 0; q, N),
\]
and \( D^{N+q} f(x) \) exists in \( \mathbb{R} \).

We would need:

Theorem 6 ([9]) Let \( f \in C^N([a, b]), N \in \mathbb{Z}^+ \). Here \( x, x' \in [a, b] \), and \( D(x, \cdot ; q, N) \in C^1([a - x, b - x]) \). Then
\[
f(x') = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x' - x)^n + \frac{D^{N+q} f(x)}{\Gamma(q + 1)} |x' - x|^q + \frac{1}{\Gamma(q + 1)} \int_0^{x' - x} \frac{dF(x, t; q, N)}{dt} |(x' - x) - t|^q dt.
\]

Corollary 7 (to Theorem 6, \( N = 0 \)) Let \( f \in C([a, b]), x, x' \in [a, b] \), and \( F(x, \cdot ; q, 0) \in C^1([a - x, b - x]) \). Then
\[
f(x') = f(x) + \frac{D^{q} f(x)}{\Gamma(q + 1)} |x' - x|^q + \frac{1}{\Gamma(q + 1)} \int_0^{x' - x} \frac{dF(x, t; q, 0)}{dt} |(x' - x) - t|^q dt.
\]
We make

**Remark 8** Let \( f \in C^N ([a,b]) \), \( N \in \mathbb{Z}_+ \). Here \( x, x' \in [a,b] : x' < x \), and \( F(x, \cdot ; q, N) \in C^1 ([a-x, b-x]) \), \( 0 < q < 1 \). By Theorem 6 we get

\[
\begin{align*}
    f (x') &= \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x' - x)^n + \frac{D^{N+q}f(x)}{\Gamma(q+1)} (x-x')^q - \\
    &\quad \frac{1}{\Gamma(q+1)} \int_{a-x}^{x'} \frac{dF(x, t; q, N)}{dt} (t-x')^q dt.
\end{align*}
\]

Clearly then we get:

**Remark 9** Let \( f \in C^N ([a,0]) \), \( a < 0 \), \( N \in \mathbb{Z}_+ \), \( F(0, \cdot ; q, N) \in C^1 ([a,0]) \), \( 0 < q < 1 \). Then, for any \( x \in [a,0] \), we derive

\[
\begin{align*}
    f (x) &= \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)} (-x)^q - \\
    &\quad \frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0, t; q, N)}{dt} (t-x)^q dt.
\end{align*}
\]

In this article we will use a lot (14).

Assume that \( f^{(n)}(0) = 0 \), \( n = 0, 1, \ldots, N \), and \( D^{N+q}f(0) = 0 \) (= \( F(0, 0; q, N) = D^q_0 f(0) \)).

Then

\[
    -f (x) = \frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0, t; q, N)}{dt} (t-x)^q dt,
\]

\( \forall x \in [a,0] \).

Here it is

\[
    F(0, t; q, N) = D^q_0 (f(t)) \in C^1 ([a,0]),
\]

where \( D^q_0 \) is the right Riemann-Liouville fractional derivative.

Let \( a \leq x \leq w \leq 0 \), then

\[
    -f (w) = \frac{1}{\Gamma(q+1)} \int_{w}^{0} \frac{dF(0, t; q, N)}{dt} (t-w)^q dt.
\]
Consider $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. Then

$$|f(w)| = \frac{1}{\Gamma(q+1)} \int_0^1 \left| \frac{dF(0, t; q, N)}{dt} \right| (t - w)^{q_1} dt \leq$$

$$\frac{1}{\Gamma(q+1)} \left( \int_0^1 \left| \frac{dF(0, t; q, N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} \left( \int_0^1 (t - w)^{q p_1} dt \right)^{\frac{1}{p_1}} \leq$$

$$\frac{1}{\Gamma(q+1)} \frac{(-w)^{q p_1 + 1}}{(q p_1 + 1)^{\frac{1}{p_1}}} \left( \int_0^1 \left| \frac{dF(0, t; q, N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} \leq$$

$$\frac{1}{\Gamma(q+1)} \frac{(-w)^{q p_1 + 1}}{(q p_1 + 1)^{\frac{1}{p_1}}} \left( z(w) \right)^{\frac{1}{p_1}}, \quad (18)$$

where

$$z(w) := \int_0^w \left| \frac{dF(0, t; q, N)}{dt} \right|^{q_1} dt,$$  \hspace{1cm} (19)

all $a \leq x \leq w \leq 0$, and $z(0) = 0$.

From $-z(w) = \int_{0}^{w} \left| \frac{dF(0, t; q, N)}{dt} \right|^{q_1} dt$, we get

$$-z'(w) = (-z(w))' = \left| \frac{dF(0, w; q, N)}{dw} \right|^{q_1}, \quad (20)$$

and

$$\left| \frac{dF(0, w; q, N)}{dw} \right| = (-z'(w))^{\frac{1}{p_1}}. \quad (21)$$

Therefore we obtain

$$\int_{0}^{w} \left| f(w) \right| \left| \frac{dF(0, w; q, N)}{dw} \right| \leq$$

$$\frac{1}{\Gamma(q+1)(q p_1 + 1)^{\frac{1}{p_1}}} \left( -w \right)^{q p_1 + 1} \left( z(w) \right)^{\frac{1}{p_1}} (-z'(w))^{\frac{1}{p_1}}. \quad (22)$$

Hence it holds

$$\int_{0}^{w} \left| f(w) \right| \left| \frac{dF(0, w; q, N)}{dw} \right| dw \leq \quad (23)$$

$$\frac{1}{\Gamma(q+1)(q p_1 + 1)^{\frac{1}{p_1}}} \int_{0}^{w} \left( -w \right)^{q p_1 + 1} \left( z(w) (-z'(w)) \right)^{\frac{1}{p_1}} dw \leq$$

$$\frac{1}{\Gamma(q+1)(q p_1 + 1)^{\frac{1}{p_1}}} \left( \int_{0}^{w} \left( -w \right)^{q p_1 + 1} dw \right)^{\frac{1}{p_1}} \left( \int_{0}^{w} z(w) (-z'(w)) dw \right)^{\frac{1}{p_1}} \leq$$

$$\frac{1}{\Gamma(q+1)(q p_1 + 1)^{\frac{1}{p_1}}} \left( \frac{-w^{q p_1 + 2}}{q p_1 + 2} \right)^{\frac{1}{p_1}} \left( \frac{-z^2(w)}{2} \right)^{\frac{1}{p_1}} \leq \quad (24)$$

Hence, it holds.
We have proved that
\[
\int_0^x |f(w)| \left| \frac{dF(0,w;q,N)}{dw} \right| dw \leq \frac{(-x)^{q_1 + \frac{p_1}{2}}}{2^{\frac{1}{2q_1}}} \left( \int_0^x \left| \frac{dF(0,w;q,N)}{dw} \right|^{q_1} dw \right)^{\frac{1}{q_1}}.
\]
(25)

We have established the following negative domain $L^p$-Opial type local right fractional inequality:

**Theorem 10** Let $p_1, q_1 > 1$ : $\frac{1}{p_1} + \frac{1}{q_1} = 1$; $f \in C^N([a,0])$, $N \in \mathbb{Z}_+$, $a < 0$, $x \in [a,0]$; $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. Assume that $f^{(n)}(0) = 0$, $n = 0, 1, \ldots, N$, and $D^{N+q}f(0) = 0$ \((= F(0,0;q,N) = D^q_0 f(0))\). [Here it is $F(0,t;q,N) = D^q_0 (f(t)) \in C^1([a,0])$, where $D^q_0$ is the right Riemann-Liouville fractional derivative]. Then

\[
\int_0^x |f(t)| \left| \frac{dF(0,t;q,N)}{dt} \right| dt \leq \frac{(-x)^{q + \frac{p_1}{2}}}{2^{\frac{1}{2q_1}}} \left( \int_0^x \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}},
\]
(26)

\(\Leftarrow\) it holds

\[
\int_0^x |f(t)| \left| \frac{dD^q_0 (f(t))}{dt} \right| dt \leq \frac{(-x)^{q + \frac{p_1}{2}}}{2^{\frac{1}{2q_1}}} \left( \int_0^x \left| \frac{dD^q_0 (f(t))}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}},
\]
(27)

\(\forall x \in [a,0].\)

The case $p_1 = q_1 = 2$ follows:

**Corollary 11** All as in Theorem 10, with $p_1 = q_1 = 2$. Then

\[
\int_0^x |f(t)| \left| \frac{dF(0,t;q,N)}{dt} \right| dt \leq \frac{(-x)^{q + 1}}{2^{\frac{1}{q}}} \left( \int_0^x \left| \frac{dF(0,t;q,N)}{dt} \right|^2 dt \right)^{\frac{1}{2}},
\]
(28)

\(\Leftarrow\)
it holds
\[
\int_0^x \left| f (t) \right| \left| D^q_0 (f (t)) \right| dt \leq \frac{(-x)^{q+1}}{2 \Gamma (q + 1) \sqrt{(q + 1)(2q + 1)}} \left( \int_0^x \left( \frac{d D^q_0 (f (t))}{dt} \right)^2 dt \right),
\]
\forall x \in [a, 0].

We make

Remark 12 Let \( f_1, f_2 \) according to the assumptions of Theorem 10. Then
\[
-f_1 (x_1) = \frac{1}{\Gamma (q + 1)} \int_{x_1}^0 \frac{d F_1 (0, t_1; q, N)}{dt_1} (t_1 - x_1)^q dt_1,
\]
\forall x_1 \in [a_1, 0], a_1 < 0;
\[
-f_2 (x_2) = \frac{1}{\Gamma (q + 1)} \int_{x_2}^0 \frac{d F_2 (0, t_2; q, N)}{dt_2} (t_2 - x_2)^q dt_2,
\]
\forall x_2 \in [a_2, 0], a_2 < 0.

Here it is
\[
F_i (0, t_i; q, N) = D^q_0 (f_i (t_i)) \in C^1 ([a_i, 0]), \quad i = 1, 2;
\]
where \( D^q_0 \) is the right Riemann-Liouville fractional derivative.

Consider \( p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1. \)

Hence
\[
\left| f_i (x_i) \right| \leq \frac{1}{\Gamma (q + 1)} \int_{x_i}^0 \left| \frac{d F_1 (0, t_i; q, N)}{dt_1} \right| (t_i - x_i)^q dt_1,
\]
\forall i = 1, 2; \forall x_i \in [a_i, 0].

We get by Hölder’s inequality:
\[
\left| f_1 (x_1) \right| \leq \frac{1}{\Gamma (q + 1)} \left( \int_{x_1}^0 (t_1 - x_1)^{qp_1} dt_1 \right)^{\frac{1}{p_1}} \left( \int_{x_1}^0 \left| \frac{d F_1 (0, t_1; q, N)}{dt_1} \right|^{q_1} dt_1 \right)^{\frac{1}{q_1}} \leq \frac{(-x_1)^{qp_1 + 1}}{\Gamma (q + 1) (qp_1 + 1)^{p_1}} \left\| \frac{d F_1 (0, t_1; q, N)}{dt_1} \right\|_{q_1, [a_1, 0]},
\]
\forall x_1 \in [a_1, 0].

Similarly, we obtain
\[
\left| f_2 (x_2) \right| \leq \frac{1}{\Gamma (q + 1)} \left( \int_{x_2}^0 (t_2 - x_2)^{qp_2} dt_2 \right)^{\frac{1}{p_2}} \left\| \frac{d F_2 (0, t_2; q, N)}{dt_2} \right\|_{p_1, [a_2, 0]},
\]
\forall x_2 \in [a_2, 0].
∀ \( x_2 \in [a_2, 0] \).

Therefore we have

\[
|f_1(x_1)| \leq \frac{1}{(\Gamma(q + 1))^\frac{1}{r}(q p_1 + 1)\frac{1}{p_1}}\left(\int_{a_1}^{0} \frac{|f_2(x_2)|}{|x_2|^{\frac{1}{q_1} - 1}} \, dx_2\right) \leq \frac{1}{(\Gamma(q + 1))^\frac{1}{r}(q p_1 + 1)\frac{1}{p_1}}\left(\int_{a_1}^{0} \frac{|f_2(x_2)|}{|x_2|^{\frac{1}{q_1} - 1}} \, dx_2\right)
\]

(35)

用地的不等式对 \( a, b \geq 0 \), \( a^{\frac{1}{r}} b^{\frac{1}{s}} \leq \frac{a}{p_1} + \frac{b}{q_1} \)

\[
\frac{1}{(\Gamma(q + 1))^\frac{1}{r}(q p_1 + 1)\frac{1}{p_1}}\left(\int_{a_1}^{0} \frac{|f_2(x_2)|}{|x_2|^{\frac{1}{q_1} - 1}} \, dx_2\right)
\]

∀ \( x_1 \in [a_1, 0] \), \( i = 1, 2 \).

So far we have established

\[
|f_1(x_1)| \leq \frac{1}{(\Gamma(q + 1))^\frac{1}{r}(q p_1 + 1)\frac{1}{p_1}}\left(\int_{a_1}^{0} \frac{|f_2(x_2)|}{|x_2|^{\frac{1}{q_1} - 1}} \, dx_2\right)
\]

(37)

∀ \( x_1 \in [a_1, 0] \), \( i = 1, 2 \).

The denominator of left hand side of (37) can be zero only when \( x_1 = 0 \) and \( x_2 = 0 \). By integrating (37) over \([a_1, 0] \times [a_2, 0]\) we get

\[
\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)|}{|x_2|^{\frac{1}{q_1} - 1}} \, dx_1 \, dx_2 \leq \frac{\alpha_{1} \alpha_{2}}{(\Gamma(q + 1))^\frac{1}{r}(q p_1 + 1)\frac{1}{p_1}(q q_1 + 1)\frac{1}{q_1}}
\]

(38)

We have proved the following negative domains local right fractional Hilbert-Pachpatte inequality:

**Theorem 13** Let \( p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1; i = 1, 2 \) for \( f_i \in C^n([a_1, 0]), N \in \mathbb{Z}_+ \), \( a_i < 0; F_i(0, q, N) \in C^1([a_1, 0]), 0 < q < 1 \). Assume that \( f_{i}^{(n)}(0) = 0, n = 0, 1, \ldots, N, N + f_{i}(0) = 0, i = 1, 2 \) (i.e. \( F_i(0, q, N) = D^N D^N f_i(0) = 0 \)). Then it is \( F_i(0, t_1, q, N) = D^N D^N f_i(t_1) \in C^1([a_1, 0]), \) where \( D^N_D \) is the right Riemann-Liouville fractional derivative. Then

\[
\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)|}{|x_2|^{\frac{1}{q_1} - 1}} \, dx_1 \, dx_2 \leq \frac{\alpha_{1} \alpha_{2}}{(\Gamma(q + 1))^\frac{1}{r}(q p_1 + 1)\frac{1}{p_1}(q q_1 + 1)\frac{1}{q_1}}
\]

(39)
\[
\frac{dF_1 (0, t_1; q, N)}{dt_1} \|_{q_1, [a_1, 0]} \leq \frac{dF_2 (0, t_2; q, N)}{dt_2} \|_{p_1, [a_2, 0]},
\]

it holds

\[
\int_{a_1}^{0} \int_{a_2}^{0} \left[ \left( -x \right)^{\alpha_1 + \beta} + \left( -x \right)^{\alpha_2 + \beta} \right] dx_1 dx_2 \leq \frac{\alpha_1 \alpha_2}{\Gamma (q + 1)^2 (qp_1 + 1)^{\frac{1}{q}} (qq_1 + 1)^{\frac{1}{q}}}.
\]

We make

**Remark 14** Let \( f \in C^N ([a, 0]), a < 0, N \in \mathbb{Z}_+, F (0, \cdot; q, N) \in C^1 ([a, 0]), 0 < q < 1. \) Then for any \( x \in [a, 0] \), we have

\[
f (x) = \sum_{n=0}^{N} \frac{f^{(n)} (0)}{n!} x^n + \frac{D^{N+q} f (0)}{\Gamma (q + 1)} (-x)^q \quad (41)
\]

Assume that \( f^{(n)} (0) = 0, n = 0, 1, \ldots, N \). Here \( D^{N+q} f (0) = F (0, 0; q, N) = D_q^0 f (0) \), where \( D_q^0 \) is the right Riemann-Liouville fractional derivative.

So far we have

\[
f (x) = \frac{D^{N+q} f (0)}{\Gamma (q + 1)} (-x)^q + R (x),
\]

where

\[
R (x) := - \frac{1}{\Gamma (q + 1)} \int_{x}^{0} \frac{dF (0, t; q, N)}{dt} (t - x)^q dt.
\]

We also assume that \( D_q^0 f \in C^1 ([a, 0]) \).

We can rewrite

\[
R (x) = - \frac{1}{\Gamma (q + 1)} \int_{x}^{0} \left( \frac{d}{dt} D_q^0 f (t) \right) (t - x)^q dt.
\]

We notice that

\[
|R (x)| \leq \frac{1}{\Gamma (q + 1)} \int_{x}^{0} \left| \frac{d}{dt} D_q^0 f (t) \right| (t - x)^q dt \leq \frac{1}{\Gamma (q + 1)} \left[ \frac{d}{dt} D_q^0 f (t) \right]_{\infty, [a, 0]} \frac{(-x)^{q+1}}{q+1}.
\]
That is
\[ |R(x)| \leq \frac{(-x)^{q+1}}{\Gamma(q+2)} \left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty, [a,0]}, \tag{45} \]

\( \forall x \in [a,0] \).

Hence, it holds
\[
\int_a^0 f(x) \, dx = \frac{D^{N+q} f(0)}{\Gamma(q+1)} \int_a^0 (-x)^q \, dx + \int_a^0 R(x) \, dx = \tag{46}
\]

\[
\frac{D^{N+q} f(0)}{\Gamma(q+1)} \frac{(-a)^{q+1}}{q+1} + \int_a^0 R(x) \, dx = \frac{D^{N+q} f(0)}{\Gamma(q+2)} (-a)^{q+1} + \int_a^0 R(x) \, dx.
\]

Therefore, we get
\[
\int_a^0 f(x) \, dx = \frac{D^{N+q} f(0)}{\Gamma(q+2)} (-a)^{q+1} = \int_a^0 R(x) \, dx. \tag{47}
\]

Consequently, we derive
\[
\left| \int_a^0 f(x) \, dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^{q+1} \right| \leq \int_a^0 |R(x)| \, dx \leq \tag{48}
\]

\[
\frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty, [a,0]}}{\Gamma(q+2)} \int_a^0 (-x)^{q+1} \, dx = \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty, [a,0]}}{\Gamma(q+2)} (-a)^{q+2} + \frac{1}{q+2} \int_a^0 \left( \frac{d}{dt} D_0^q f(t) \right)_{\infty, [a,0]} (-a)^{q+2} \cdot \tag{49}
\]

\[
\Gamma(q+3)
\]

We have proved the following negative domain local right fractional comparison of means results:

**Theorem 15** Let \( f \in C^N([a,0]), \ a < 0, \ N \in \mathbb{Z}_+, \ D_0^q f \in C^1([a,0]), \ 0 < q < 1. \)

Assume \( f^{(n)}(0) = 0, \ n = 0, 1, \ldots, N. \) Then
\[
\left| \int_a^0 f(x) \, dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^{q+1} \right| \leq \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty, [a,0]}}{\Gamma(q+3)} (-a)^{q+2}, \tag{50}
\]

\[ \Leftrightarrow \]
\[
\left| \frac{1}{(-a)} \int_a^0 f(x) \, dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^q \right| \leq \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty, [a,0]}}{\Gamma(q+3)} (-a)^{q+1}. \tag{51}
\]

We make

**Remark 16** All as in Theorem 10. Then
\[
-f(x) = \frac{1}{\Gamma(q+1)} \int_x^0 dF(0,t;q,N) \frac{dt}{(t-x)^q} dt, \tag{52}
\]
∀ \, x \in [a, 0].

Let \( p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1 \). Thus

\[
|f(x)| \leq \frac{1}{\Gamma(q + 1)} \left( \int_x^0 \left| \frac{dF(0, t; q, N)}{dt} \right|^q dt \right)^{\frac{1}{q}} \left( \int_x^0 (t - x)^{q_{p_1}} dt \right)^{\frac{1}{p_1}} \leq \\
\frac{1}{\Gamma(q + 1)} \left( \int_x^0 \left| \frac{dF(0, t; q, N)}{dt} \right|^q dt \right)^{\frac{1}{q}} \left( \frac{(-x)^{q_{p_1} + 1}}{(pq_{p_1} + 1)^{\frac{1}{p_1}}} \right)^{\frac{1}{p_1}}.
\]

That is

\[
|f(x)| \leq \frac{(-x)^{q_{p_1} + 1}}{(pq_{p_1} + 1)^{\frac{1}{p_1}}} \left( \int_x^0 \left| \frac{dF(0, t; q, N)}{dt} \right|^q dt \right)^{\frac{1}{q}} \bigg|_{q_1, [a, 0]}. \tag{53}
\]

∀ \, x \in [a, 0].

Therefore

\[
|f(x)|^{q_1} \leq \frac{(-x)^{q_1(q + 1) - 1}}{(pq_{p_1} + 1)^{\frac{1}{p_1}}} \left( \int_x^0 \left| \frac{dF(0, t; q, N)}{dt} \right|^q dt \right)^{\frac{1}{q}} \bigg|_{q_1, [a, 0]}. \tag{54}
\]

Consequently, it holds

\[
\int_a^0 |f(x)|^{q_1} \, dx \leq \\
\frac{(-a)^{q_1(q + 1)}}{\Gamma(q + 1)^{q_1(q + 1)^{\frac{1}{p_1}}} (pq_{p_1} + 1)^{\frac{1}{p_1}}} \left( \int_x^0 \left| \frac{dF(0, t; q, N)}{dt} \right|^q dt \right)^{\frac{1}{q}} \bigg|_{q_1, [a, 0]}. \tag{55}
\]

That is

\[
\|f\|_{q_1, [a, 0]} \leq \frac{(-a)^{(q + 1)}}{\Gamma(q + 1)(pq_{p_1} + 1)^{\frac{1}{p_1}} (q_1(q + 1))^{\frac{1}{p_1}}} \left( \int_x^0 \left| \frac{dF(0, t; q, N)}{dt} \right|^q dt \right)^{\frac{1}{q}} \bigg|_{q_1, [a, 0]} . \tag{56}
\]

We have proved the following negative domain local right fractional Poincare inequality:

**Theorem 17** All as in Theorem 10. Then

\[
\|f\|_{q_1, [a, 0]} \leq \frac{(-a)^{(q + 1)}}{\Gamma(q + 1)(pq_{p_1} + 1)^{\frac{1}{p_1}} (q_1(q + 1))^{\frac{1}{p_1}}} \left( \|D_{\alpha, N} f\|^{q_1}_{q_1, [a, 0]} \right) . \tag{57}
\]

We make
Remark 18 All as in Theorem 10, plus $r > 0$. By (54) we have

$$|f(x)| \leq \frac{(-x)^{\frac{q+1}{p_1} + \frac{1}{p_1}}}{\Gamma (q + 1) (qp_1 + 1)^{p_1}} \left\| \frac{dF(0,t; q, N)}{dt} \right\|_{q_1,[a,0]}, \quad (59)$$

$$\forall x \in [a,0].$$

Hence it holds

$$|f(x)|^r \leq \frac{(-x)^{r(q + \frac{1}{p_1})}}{\Gamma (q + 1) (qp_1 + 1)^{p_1}} \left\| (D^q_0(f))^r \right\|_{q_1,[a,0]}, \quad (60)$$

Consequently, we get

$$\int_a^0 |f(x)|^r \, dx \leq \frac{(-a)^{r(q + \frac{1}{p_1}) + 1}}{\Gamma (q + 1) (qp_1 + 1)^{p_1}} \left\| (D^q_0(f))^r \right\|_{q_1,[a,0]} \quad (61)$$

We have proved the following negative domain local right fractional Sobolev type inequality:

**Theorem 19** All as in Theorem 10, plus $r > 0$. Then

$$\|f\|_{r,[a,0]} \leq \frac{(-a)^{q + \frac{1}{p_1} + \frac{1}{p_1}}}{\Gamma (q + 1) (qp_1 + 1)^{p_1}} \left\| (D^q_0(f))^r \right\|_{q_1,[a,0]} \quad (62)$$

**References**


Approximate controllability for semilinear integro-differential control equations in Hilbert spaces

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Abstract
This paper deals with the approximate controllability for a class of semilinear integro-differential functional control equations, which is provided under general sufficient conditions on the system operator, controller and nonlinear terms. Our used tool is applying results similar to Fredholm alternative for nonlinear operators under restrictive assumptions. Finally, a simple example to which our main result can be applied is given.

Keywords: approximate controllability, semilinear control equations, integro-differential control equations, controller, Fredholm alternative.

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1 Introduction
In this paper, we deal with the approximate controllability for semilinear integro-differential functional control equations in the form
\[
\begin{cases}
\frac{dx(t)}{dt} = Ax(t) + \int_0^t k(t-s)g(s, x(s), u(s))ds + Bu(t), & 0 < t \leq T, \\
x(0) = x_0
\end{cases}
\]
in a Hilbert space $H$, where $k$ belongs to $L^2(0,T)(T > 0)$ and $g$ is a nonlinear mapping as detailed in Section 2. The principal operator $A$ generates an analytic semigroup $(S(t))_{t \geq 0}$ and $B$ is a bounded linear operator from another Hilbert space $U$ to $H$.

The controllability problem is a question of whether is possible to steer a dynamic system from an initial state to an arbitrary final state using the set of admissible controls. Naito [13] was the first to deal with the range condition argument of controller in order to obtain the approximate controllability of a semilinear control system. In [3, 9, 17, 18], they have studied continuously about controllability of semilinear systems dominated by linear parts (in case $g \equiv 0$) by assuming that $S(t)$ is compact operator for each $t > 0$ as matters connected with [13]. Another approach used to obtain sufficient conditions for approximate solvability of nonlinear equations is a fixed point theorem combined with technique of operator transformations by configuring the resolvent as seen in [2].

The controllability for various nonlinear equations has been studied by many authors, for example, see [5, 6, 12] for local controllability of neutral functional differential systems with unbounded delay, [10, 14] for neutral evolution integrodifferential systems with state dependent delay.

Sukavanam and Tomar [15] studied the approximate controllability for the general retarded initial value problem by assuming that the Lipschitz constant of the nonlinear term is less than 1, and Wang [17] for general retarded semilinear equations assuming the growth condition of the nonlinear term and the compactness of the semigroup.

In this paper, authors want to use a different method than the previous one. Our used tool is the theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda T(x) - F(x) = y$ in dependence on the real number $\lambda$, where $T$ and $F$ are nonlinear operators defined a Banach space $X$ with values in a Banach space $Y$. In order to obtain the approximate controllability for a class of semilinear integro-differential functional control equations, it is necessary to suppose that $T$ acts as the identity operator while $F$ related to the nonlinear term of (1.1) is completely continuous.

In Section 2, we introduce regularity properties for (1.1). Since we apply the Fredholm theory in the proof of the main theorem, we assume some compactness of the embedding between intermediate spaces. Then by virtue of Aubin [1], we can show that the solution mapping of a control space to the terminal state space is completely continuous. Based on Section 2, it is shown the sufficient conditions on the controller and nonlinear terms for approximate controllability for (1.1) by using the Fredholm theory. Finally, a simple example to which our main result can be applied is given.

## 2 Semilinear functional equations

Let $V$ and $H$ be complex Hilbert spaces forming a Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$
by identifying the antidual of $H$ with $H$. Therefore, for the brevity, we may regard that $\|u\|_* \leq |u| \leq \|u\|$ for all $u \in V$, where the notations $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$ denote the norms of $H$, $V$ and $V^*$, respectively as usual. Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding’s inequality

$$\Re a(u, u) \geq c_0 \|u\|_*^2 - c_1 |u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$  

Let $A$ be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u, \ v \in V.$$  

Then $A$ is a bounded linear operator from $V$ to $V^*$. The realization of $A$ in $H$ which is the restriction of $A$ to

$$D(A) = \{u \in V : Au \in H\}$$  

is also denoted by $A$. For the sake of simplicity we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \Re \lambda \geq 0\}$ is contained in the resolvent set of $A$. It is known that $A$ generates an analytic semigroup $S(t)$ in both $H$ and $V^*$. As seen in Lemma 3.6.2 of [16], there exists a constant $M > 0$ such that

$$|S(t)x| \leq M|x| \text{ and } \|S(t)x\|_* \leq M\|x\|_*,$$  

(2.1)

The following initial value problem for the abstract linear parabolic equation

$$\begin{cases}
\frac{dx(t)}{dt} = Ax(t) + k(t), & 0 < t \leq T, \\
x(0) = x_0.
\end{cases}$$  

(2.2)

By virtue of Theorem 3.3 of [4](or Theorem 3.1 of [9]), we have the following result on the corresponding linear equation (2.2).

**Proposition 2.1.** Suppose that the assumptions for the principal operator $A$ stated above are satisfied. Then the following properties hold:

1) For $x_0 \in V$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution $x$ of (2.2) belonging to

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$  

and satisfying

$$\|x\|_{L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)} \leq C_1(\|x_0\| + \|k\|_{L^2(0, T; H)}),$$  

(2.3)

where $C_1$ is a constant depending on $T$.

2) Let $x_0 \in H$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then, there exists a unique solution $x$ of (2.2) belonging to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$  

and satisfying

$$\|x\|_{L^2(0, T; V) \cap W^{1,2}(0, T; V^*)} \leq C_1(\|x_0\| + \|k\|_{L^2(0, T; V^*)}),$$  

(2.4)

where $C_1$ is a constant depending on $T$. 

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By virtue of Proposition 2.1, we have the following lemma.

**Lemma 2.1.** Suppose that $k \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \leq t \leq T$. Then there exists a constant $C_2$ such that

$$
||x||_{L^2(0,T;H)} \leq C_2 T ||k||_{L^2(0,T;H)},
$$

(2.5)

and

$$
||x||_{L^2(0,T;V)} \leq C_2 \sqrt{T} ||k||_{L^2(0,T;H)}.
$$

(2.6)

Consider the following initial value problem for the abstract semilinear parabolic equation

$$
\begin{cases}
\frac{d}{dt} x(t) = Ax(t) + \int_0^t k(t-s)g(s, x(s), u(s))ds + Bu(t), \\
x(0) = x_0.
\end{cases}
$$

(2.7)

Let $U$ be a Hilbert space and the controller operator $B$ be a bounded linear operator from $U$ to $H$.

Let $g : \mathbb{R}^+ \times V \times U \to H$ be a nonlinear mapping satisfying the following:

**Assumption (F).**

(i) For any $x \in V$, $u \in U$ the mapping $g(\cdot, x, u)$ is strongly measurable;

(ii) There exist positive constants $L_0, L_1, L_2$ such that

(a) $u \mapsto g(t, x, u)$ is an odd mapping $(g(\cdot, x, -u) = -g(\cdot, x, u))$;

(b) for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $u, \hat{u} \in U$,

$$
|g(t, x, u) - g(t, \hat{x}, \hat{u})| \leq L_1||x - \hat{x}|| + L_2||u - \hat{u}||,
$$

$$
|g(t, 0, 0)| \leq L_0.
$$

For $x \in L^2(0, T; V)$, we set

$$
f(t, x, u) = \int_0^t k(t-s)g(s, x(s), u(s))ds
$$

where $k$ belongs to $L^2(0, T)$.

**Lemma 2.2.** Let Assumption (F) be satisfied. Assume that $x \in L^2(0, T; V)$ for any $T > 0$. Then $f(\cdot, x, u) \in L^2(0, T; H)$ and

$$
||f(\cdot, x, u)||_{L^2(0,T;H)} \leq L_0 ||k||_{L^2(0,T)} T / \sqrt{2} + ||k||_{L^2(0,T)} \sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}).
$$

(2.8)

Moreover if $x$, $\hat{x} \in L^2(0, T; V)$, then

$$
||f(\cdot, x, u) - f(\cdot, \hat{x}, \hat{u})||_{L^2(0,T;H)} \leq ||k||_{L^2(0,T)} \sqrt{T}(L_1||x - \hat{x}||_{L^2(0,T;V)} + L_2||u - \hat{u}||_{L^2(0,T;U)}).
$$

(2.9)
The proof is easily from Assumption (F), and using the Hölder inequality.

By virtue of Theorem 2.1 of [8], we have the following result on (2.7).

**Proposition 2.2.** Let Assumption (F) be satisfied. Then there exists a unique solution \(x\) of (2.7) such that
\[
x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)
\]
for any \(x_0 \in H\). Moreover, there exists a constant \(C_3\) such that
\[
||x||_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} \leq C_3(||x_0|| + ||u||_{L^2(0,T;U)}).
\]  
(2.10)

**Corollary 2.1.** Assume that the embedding \(D(A) \subset V\) is completely continuous. Let Assumption (F) be satisfied, and \(x_u\) be the solution of equation (2.7) associated with \(u \in L^2(0,T;U)\). Then the mapping \(u \mapsto x_u\) is completely continuous from \(L^2(0,T;U)\) to \(L^2(0,T;V)\).

**Proof.** If \(u\) is bounded in \(L^2(0,T;U)\), then so is \(x_u\) in \(L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)\) by (2.8). Since \(D(A)\) is compactly embedded in \(V\) by assumption, the embedding
\[
L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)
\]
is completely continuous in view of Theorem 2 of [1], the mapping \(u \mapsto x_u\) is completely continuous from \(L^2(0,T;U)\) to \(L^2(0,T;V)\). \(\square\)

3 Approximate controllability

Throughout this section, we assume that \(D(A)\) is compactly embedded in \(V\). Let \(x(T;f,u)\) be a state value of the system (2.7) at time \(T\) corresponding to the nonlinear term \(f\) and the control \(u\). We define the reachable sets for the system (2.7) as follows:
\[
\mathcal{R}_T(f) = \{ x(T;f,u) : u \in L^2(0,T;U) \},
\]
\[
\mathcal{R}_T(0) = \{ x(T;0,u) : u \in L^2(0,T;U) \}.
\]

**Definition 3.1.** The system (2.7) is said to be approximately controllable in the time interval \([0,T]\) if for every desired final state \(x_1 \in H\) and \(\epsilon > 0\) there exists a control function \(u \in L^2(0,T;U)\) such that the solution \(x(T;f,u)\) of (2.7) satisfies
\[
|x(T;f,u) - x_1| < \epsilon,
\]
that is, if \(\mathcal{R}_T(f) = H\) where \(\mathcal{R}_T(f)\) is the closure of \(\mathcal{R}_T(f)\) in \(H\), then the system (2.9) is called approximately controllable at time \(T\).

Let us introduce the theory of the degree for completely continuous perturbations of the identity operator, which is the infinite dimensional version of Borsuk’s theorem. Let \(0 \in D\) be a bounded open set in a Banach space \(X\), \(\overline{D}\) its closure and \(\partial D\) its boundary. The number \(d(I - T;D,0)\) is the degree of the mapping \(I - T\) with respect to the set \(D\) and the point \(0\) (see Fučik et al. [7] or Lloid [11]).
Theorem 3.1. (Borsuk’s theorem) Let \( D \) be a bounded open symmetric set in a Banach space \( X \), \( 0 \in D \). Suppose that \( T : \overline{D} \to X \) be odd completely continuous operator satisfying \( T(x) \neq x \) for \( x \in \partial D \). Then \( d(I - T; D, 0) \) is odd integer. That is, there exists at least one point \( x_0 \in D \) such that \((I - T)(x_0) = 0\).

Definition 3.2. Let \( T \) be a mapping defined by on a Banach space \( X \) with value in a real Banach space \( Y \). The mapping \( T \) is said to be a \((K, L, \alpha)\)-homeomorphism of \( X \) onto \( Y \) if

(i) \( T \) is a homeomorphism of \( X \) onto \( Y \);
(ii) there exist real numbers \( K > 0 \), \( L > 0 \), and \( \alpha > 0 \) such that

\[ L ||x||_X^\alpha \leq ||T(x)||_Y \leq K ||x||_X^\alpha, \quad \forall x \in X. \]

Lemma 3.1. Let \( T \) be an odd \((K, L, \alpha)\)-homeomorphism of \( X \) onto \( Y \) and \( F : X \to Y \) a continuous operator satisfying

\[ \limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+. \]

Then if \( |\lambda| \notin \left[ \frac{N}{K}, \frac{N}{L} \right] \cup \{0\} \) then

\[ \lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \]

Proof. Suppose that there exist a constant \( M > 0 \) and a sequence \( \{x_n\} \subset X \) such that

\[ ||\lambda T(x_n) - F(x_n)||_Y \leq M \] (3.1)
as \( x_n \to \infty \). From (3.1) it follows that

\[ \frac{\lambda T(x_n) - F(x_n)}{||x_n||_X^\alpha} \to 0. \]

Hence, we have

\[ \limsup_{n \to \infty} \frac{||\lambda||_Y ||T(x_n)||_Y}{||x_n||_X^\alpha} = N, \]

and so, \( |\lambda|K \geq N \geq |\lambda|L \). It is a contradiction with \( |\lambda| \notin \left[ \frac{N}{K}, \frac{N}{L} \right] \).

Proposition 3.1. Let \( T \) be an odd \((K, L, \alpha)\)-homeomorphism of \( X \) onto \( Y \) and \( F : X \to Y \) an odd completely continuous operator. Suppose that for \( \lambda \neq 0 \),

\[ \lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \] (3.2)

Then \( \lambda T - F \) maps \( X \) onto \( Y \).
Proof. We follow the proof Theorem 1.1 in Chapter II of Fučík et al. [7]. Suppose that there exists \( y \in Y \) such that \( \lambda T(x) = y \). Then from (3.2) it follows that \( FT^{-1} : Y \rightarrow Y \) is an odd completely continuous operator and

\[
\lim_{\|y\|_Y \to \infty} \|y - FT^{-1}\left(\frac{y}{\lambda}\right)\|_Y = \infty.
\]

Let \( y_0 \in Y \). There exists \( r > 0 \) such that

\[
\|y - FT^{-1}\left(\frac{y}{\lambda}\right)\|_Y > \|y_0\|_Y \geq 0
\]

for each \( y \in Y \) satisfying \( \|y\|_Y = r \). Let \( Y_r = \{y \in Y : \|y\|_Y < r\} \) be an open ball. Then by view of Theorem 3.1, we have \( d[y - FT^{-1}\left(\frac{y}{\lambda}\right); Y_r, 0] \) is an odd number. For each \( y \in Y \) satisfying \( \|y\|_Y = r \) and \( t \in [0, 1] \), there is

\[
\|y - FT^{-1}\left(\frac{y}{\lambda}\right) - ty_0\|_Y \geq \|y - FT^{-1}\left(\frac{y}{\lambda}\right)\|_Y - \|y_0\|_Y > 0
\]

and hence, by the homotopic property of degree, we have

\[
d[y - FT^{-1}\left(\frac{y}{\lambda}\right); Y_r, y_0] = d[y - FT^{-1}\left(\frac{y}{\lambda}\right); Y_r, 0] \neq 0.
\]

Hence, by the existence theory of the Leray-Schauder degree, there exists a \( y_1 \in Y_r \) such that

\[
y_1 - FT^{-1}\left(\frac{y_1}{\lambda}\right) = y_0.
\]

We can choose \( x_0 \in X \) satisfying \( \lambda T(x_0) = y_1 \), and so, \( \lambda T(x_0) - F(x_0) = y_0 \). Thus, it implies that \( \lambda T - F \) is a mapping of \( X \) onto \( Y \). \( \square \)

Corollary 3.1. Let \( T \) be an odd \((K, L, \alpha)\)-homeomorphism of \( X \) onto \( Y \) and \( F : X \rightarrow Y \) an odd completely continuous operator satisfying

\[
\lim_{\|x\|_X \to \infty} \sup \frac{\|F(x)\|_Y}{\|x\|_X^\alpha} = N \in \mathbb{R}^+.
\]

Then if \( |\lambda| \notin \left[\frac{N}{K}, \frac{N}{L}\right] \cup \{0\} \) then \( \lambda T - F \) maps \( X \) onto \( Y \). Therefore, if \( N = 0 \), then for all \( \lambda \neq 0 \) the operator \( \lambda T - F \) maps \( X \) onto \( Y \).

First we consider the approximate controllability of the system (2.7) in case where the controller \( B \) is the identity operator on \( H \) under Assumption (F) on the nonlinear operator \( f \) in Section 2. Hence, noting that \( H = U \), we consider the linear system given by

\[
\begin{cases}
\frac{d}{dt} y(t) = Ay(t) + u(t), \\
y(0) = x_0,
\end{cases}
\]

(3.3)
and the following semilinear control system
\[
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + f(t, x(t), v(t)) + v(t), \\
x(0) &= x_0.
\end{aligned}
\tag{3.4}
\]

**Theorem 3.2.** Assume that
\[
\lim \sup_{||u|| \to \infty} \frac{||f(\cdot, x_u, u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} < 1.
\tag{3.5}
\]

Under the Assumption (F) we have \(R_T(0) \subset R_T(f)\).

Therefore, if the linear system (3.3) with \(f = 0\) is approximately controllable, then so is the semilinear system (3.4).

**Proof.** Let \(x(t)\) be solution of (3.4) corresponding to a control \(u\). First, we show that there exist a \(v \in L^2(0,T;H)\) such that
\[
\begin{aligned}
v(t) &= u(t) - f(t, x(t), v(t)), \quad 0 < t \leq T, \\
v(0) &= u(0).
\end{aligned}
\]

Let us define an operator \(F : L^2(0,T;H) \to L^2(0,T;H)\) as
\[
Fv = -f(\cdot, x_v, v).
\]

Then by Corollary 2.1, \(F\) is a compact mapping from \(L^2(0,T;H)\) to itself, and we have
\[
\lim_{||v|| \to \infty} ||\lambda I(v) - F(v)||_{L^2(0,T;H)} = \infty,
\]
where the identity operator \(I\) on \(L^2(0,T;H)\) is an odd \((1,1,1)\)-homeomorphism. Thus, from (3.5) and Corollary 3.1, if \(\lambda \geq 1\) then \(\lambda I - F\) maps \(L^2(0,T;H)\) onto itself. Hence, we have showed that there exists a \(v \in L^2(0,T;H)\) such that \(v(t) = u(t) - f(t, y(t), v(t))\).

Let \(y\) and \(x\) be solutions of (3.3) and (3.4) corresponding to controls \(u\) and \(v\), respectively. Then, equation (3.4) is rewritten as
\[
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + f(t, x(t), v(t)) + v(t), \quad 0 < t \leq T \\
&= Ax(t) + f(t, x(t), v(t)) + u(t) - f(t, y(t), v(t)) \\
&= Ax(t) + u(t)
\end{aligned}
\]

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with \(x(0) = x_0\), which means

\[
x(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s,x(s),v(s)) + v(s)\}ds
\]

\[
= S(t)x_0 + \int_0^t S(t-s)u(s)ds = y(t),
\]

where \(y\) be solution of (3.3) corresponding to a control \(u\). Therefore, we have proved that \(R_T(0) \subset R_T(f)\). \(\square\)

**Corollary 3.2.** Let us assume that

\[
||k||_{L^2(0,T)}\sqrt{T}(L_1C_3 + L_2) < 1,
\]

where \(C_3\) is the constant in Proposition 2.2. Under the Assumption (F), we have

\[R_T(0) \subset R_T(f)\]

in case where \(B \equiv I\).

**Proof.** By Lemma 2.2 and Proposition 2.2, we have

\[
||F(u)||_{L^2(0,T;H)} = ||f(\cdot,x,u)||_{L^2(0,T;H)}
\]

\[
\leq L_0||k||_{L^2(0,T)}\sqrt{T}/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)})
\]

\[
\leq L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}\{L_1C_3(||x_0|| + ||u||_{L^2(0,T;U)}) + L_2||u||_{L^2(0,T;U)}\}
\]

Hence, we have

\[
\limsup_{||u|| \to \infty} \frac{||F(u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;U)}} \leq ||k||_{L^2(0,T)}\sqrt{T}(L_1C_3 + L_2).
\]

Thus, from Theorem 3.2, it follows that if \(\lambda \geq 1\) then \(\lambda I - F\) maps \(L^2(0,T;H)\) onto itself, and so, by the same argument as in the proof of theorem it holds that \(R_T(0) \subset R_T(f)\). \(\square\)

From now on, we consider the initial value problem for the semilinear parabolic equation (2.7). Let \(U\) be some Hilbert space and the controller operator \(B\) be a bounded linear operator from \(U\) to \(H\).

**Assumption (B)** There exists a constant \(\beta > 0\) such that \(R(f) \subset R(B)\) and

\[
||Bu|| \geq \beta||u||, \quad \forall u \in L^2(0,T;U).
\]

Consider the linear system given by

\[
\begin{cases}
\frac{d}{dt} y(t) = Ay(t) + Bu(t), \\
y(0) = x_0.
\end{cases}
\]  

(3.6)
Theorem 3.3. Under the Assumptions (3.5), (B) and (F), we have

\[ R_T(0) \subset R_T(f). \]

Therefore, if the linear system (3.6) with \( f = 0 \) is approximately controllable, then so is the semilinear system (2.7).

Proof. Let \( y \) be a solution of the linear system (3.6) with \( f = 0 \) corresponding to a control \( u \), and let \( x \) be a solution of the semilinear system (3.4) corresponding to a control \( v \). Set \( v(t) = u(t) - B^{-1}f(t, x(t), v(t)) \). Then, system (2.9) is rewritten as

\[
\frac{d}{dt} x(t) = Ax(t) + f(t, x(t), v(t)) + Bv(t), \quad 0 < t \leq T
\]

with \( x(0) = x_0 \). Hence, we have

\[
x(t) = S(t)x_0 + \int_0^t S(t-s) \{ f(s, x(s), v(s)) + v(s) \} ds
\]

\[
= S(t)x_0 + \int_0^t S(t-s) u(s) ds = y(t).
\]

Thus, we obtain that \( R_T(0) \subset R_T(f) \).

References


Convergence theorems and approximating endpoints for multivalued Suzuki mappings in hyperbolic spaces

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Abstract

The objective of this paper is to determine a modified SP-iteration process for multi-valued mappings and to establish the convergence theorems for sequences generated by modified SP-iteration processes involving multi-valued Suzuki mappings converging to endpoints in uniformly convex hyperbolic spaces. The numerical example for supporting our main result is also presented.

Keywords: modified SP-iteration; \(\Delta\)-convergence theorem; strong convergence theorem; endpoint; hyperbolic space.

MSC: Primary 47H10; Secondary 54H25.

1 Introduction

The distance from \(u\) in a metric space \((X, d)\) to a nonempty subset \(E\) of \(X\) is defined by

\[
\text{dist}(u, E) := \inf \{d(u, v) : v \in E\}.
\]

It is denoted by \(K(E)\) the family of nonempty compact subsets of \(E\). The Hausdorff distance on \(K(E)\) is defined by

\[
H(U, V) := \max \{\sup_{u \in U} \text{dist}(u, V), \sup_{v \in V} \text{dist}(v, U)\} \text{ for all } U, V \in K(E).
\]

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For an element $x$ in $E$, if $x \in T(x)$, then $x$ is said to be a fixed point of $T$. Moreover, if $\{x\} = T(x)$, then $x$ is said to be an endpoint of $T$. It denote by $\text{Fix}(T)$ the set of all fixed points of $T$ and by $\text{End}(T)$ the set of all endpoints of $T$. We can see that for every a multi-valued mapping $T$, $\text{End}(T) \subseteq \text{Fix}(T)$ and whenever $t$ is a single-valued mapping, $\text{End}(T) = \text{Fix}(T)$.

The notion of endpoints for multi-valued mappings is significant notion which put between the notion of fixed points for single-valued mappings and the notion of fixed points for multi-valued mappings.

Aubin and Siegel [3] were first studied the existence of endpoints for special kind of contractive mappings on complete metric spaces. The endpoint results for several types of contractive mappings have been quickly developed and many of papers have showed (see, e.g.,[9],[18],[20],[21]).


Recently, Panyanak [16] established the convergence theorems to an endpoint for modified Ishikawa iteration of multi-valued nonexpansive mappings in uniformly convex Banach spaces.

Motivated and inspired by above mention, we prove the convergence results to an endpoint for modified SP-iteration of multi-valued Suzuki mappings in uniformly convex hyperbolic spaces. The numerical example for supporting our main result is also presented.

2 Preliminaries

For this paper, we work in the setting of a hyperbolic space which is defined by Kohlenbach [12].

Definition 2.1 A hyperbolic space [12] is a metric space $(X, d)$ together with a mapping $W : X^2 \times [0, 1] \to X$ satisfying the following statements:

(W1) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y);$  
(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y);$  
(W3) $W(x, y, \alpha) = W(y, x, (1 - \alpha));$  
(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w),$  
for all $x, y, u, z, w \in X$ and $\alpha, \beta \in [0, 1].$

If $x, y \in X$ and $\alpha \in [0, 1]$, then we use the notion $(1 - \alpha)x \oplus \alpha y$ for $W(x, y, \alpha).$

A hyperbolic space $(X, d, W)$ is said to be uniformly convex [14] if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is well known as a modulus of uniformly convexity of
The asymptotic center of a bounded sequence $f$ is unique asymptotic center of $K$ of $r$ for every subsequence $f$. The asymptotic radius of a bounded sequence $\Delta$ in a Banach space has a regular subsequence (see [8]). The proof is case, we write $\Delta$-convergence.

Definition 2.2 [7] A multi-valued mapping $T : E \to CB(E)$ is called to be a Suzuki mapping if

$$\frac{1}{2} \text{dist}(x, T(x)) \leq d(x, y) \implies H(T(x), T(y)) \leq d(x, y)$$

for all $x, y \in X$.

Definition 2.3 [1] A multi-valued mapping $T : E \to CB(E)$ is said to satisfy condition $(E_\mu)$ provided that

$$\text{dist}(x, T(y)) \leq \mu\text{dist}(x, T(x)) + d(x, y), \text{ for all } x, y \in E.$$ We say that $T$ satisfies condition $(E)$ whenever $T$ satisfies condition $(E_\mu)$ for some $\mu \geq 1$.

Lemma 2.4 [6] If $E$ is a nonempty closed convex subset of $X$ and $T : E \to CB(E)$ is a multi-valued Suzuki mapping, then $T$ satisfies the condition $(E_3)$.

We need the following definition of convergence in hyperbolic spaces [5] which is called $\Delta$-convergence.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $X$. Define a function $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n), \text{ for all } x \in X.$$ The asymptotic radius of a bounded sequence $\{x_n\}$ with respect to a nonempty subset $K$ of $X$ is defined and denoted by

$$r_K(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$ The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a nonempty subset $K$ of $X$ is defined and denoted by

$$AC_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}), \text{ for all } y \in K\}.$$ Recall that a sequence $\{x_n\}$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta$-lim$_{n \to \infty} x_n = x$ and call $x$ the $\Delta$-lim of $\{x_n\}$.

The sequence $\{x_n\}$ is called to be regular relative to $E$ if $r(E, \{x_n\}) = r(E, \{x_n\})$ for every subsequence $\{x_n\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has a regular subsequence (see [8]). The proof is metric in nature and carries over to the present setting without change.
Lemma 2.5 [4] Let \((X, d, W)\) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity \(\eta\) and \(E\) is a closed convex subset of \(X\) if \(\{x_n\}\) is a bounded sequence in \(E\), then the asymptotic center of \(\{x_n\}\) is in \(E\).

Lemma 2.6 [10] Let \((X, d, W)\) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity \(\eta\). Let \(x \in X\) and \(\{\alpha_n\}\) be a sequence in \([a, b]\) for some \(a, b \in (0, 1)\). If \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that \(\limsup_{n \to \infty} d(x_n, x) \leq c\), \(\limsup_{n \to \infty} d(y_n, x) \leq c\), \(\lim_{n \to \infty} d(W(x_n, y_n, \alpha_n), x) = c\) for some \(c \geq 0\), then
\[
\lim_{n \to \infty} d(x_n, y_n) = 0.
\]

Lemma 2.7 [11] Every bounded sequence in a complete CAT(0)(and hence hyperbolic) space has a \(\Delta\)-convergent subsequence.

Lemma 2.8 [7] If \(\{x_n\}\) is a bounded sequence in complete uniformly convex hyperbolic space \((X, d, W)\) with \(A(\{x_n\}) = \{p\}\), \(\{u_n\}\) is a subsequence of \(\{x_n\}\) with \(A(\{u_n\}) = \{u\}\) and the sequence \(\{d(x_n, u)\}\) converges, then \(p = u\).

Definition 2.9 [8] Let \(E\) be a nonempty subset of a metric space \((X, d)\) and \(x \in X\). The radius of \(E\) relative to \(x\) is defined by
\[
r_x(E) := \sup\{d(x, y) : y \in E\}.
\]

The diameter of \(E\) is defined by
\[
diam(E) := \sup\{d(x, y) : x, y \in E\}.
\]

Definition 2.10 [2] Let \(T : E \to CB(E)\) be a multi-valued mapping. A sequence \(\{x_n\}\) in \(E\) is called an approximate fixed point sequence (resp. an approximate endpoint sequence) for \(T\) if \(\lim_{n \to \infty} dist(x_n, T(x_n)) = 0\) (resp. \(\lim_{n \to \infty} r_{x_n}(T(x_n)) = 0\)). A mapping \(T\) is said to have the approximate fixed point property (resp. the approximate endpoint property) if it has an approximate fixed point sequence (resp. an approximate endpoint sequence) in \(E\).

Lemma 2.11 [15] Let \(E\) be a nonempty subset of \(X\), \(\{x_n\}\) be a sequence in \(E\) and \(T : E \to K(E)\) be a multi-valued mapping. Then \(r_{x_n}(T(x_n)) \to 0\) if and only if \(\text{dist}(x_n, T(x_n)) \to 0\) and \(\text{diam}(T(x_n)) \to 0\).

Lemma 2.12 [13] Let \(E\) be a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space \(X\) with monotone modulus of uniform convexity and \(T : E \to K(E)\) be a multi-valued Suzuki mapping. Then \(T\) has an endpoint if and only if \(T\) has the approximate endpoint property.

Next, we also need the following definitions that will be used in the next section.

A sequence \(\{x_n\}\) in \(E\) is said to be Fejér monotone with respect to \(E\) if
\[
d(x_{n+1}, q) \leq d(x_n, q) \quad \text{for all} \quad q \in E \quad \text{and} \quad n \in \mathbb{N}.
\]
Definition 2.13 [13] Let $E$ be a nonempty subset of a hyperbolic space $X$. A mapping $T : E \to K(E)$ is said to satisfy condition (J) if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$, $h(r) > 0$ for $r \in (0, \infty)$ such that

$$r_x(T(x)) \geq h(\text{dist}(x, \text{End}(T)))$$

for all $x \in E$.

The mapping $T$ is called \textit{semicompact} if for any sequence $\{x_n\}$ in $E$ such that

$$\lim_{n \to \infty} r_{x_n}(T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{n \to \infty} x_{n_j} = q$.

3 Main results

For this part, we start by introducing the notion of the modified SP-iteration process for multi-valued mappings. Notice that it is an improvement of the one so called the SP-iteration process given in Phuengrattana and Suantai [17]. They [17] also showed that SP-iteration process is a generalized version and the sequence generated by the SP-iteration process converges faster than Ishikawa for the class of nondecreasing and continuous functions.

Let $X$ be a hyperbolic space and $E$ be a nonempty convex subset of $X$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$ and $T : E \to K(E)$ be a multi-valued mapping. The sequence generated by the modified SP-iteration is defined by $z_1 \in E$,

$$\begin{align*}
y_n &= W(u_n, z_n, \gamma_n) \\
w_n &= W(v_n, y_n, \beta_n) \\
z_{n+1} &= W(x_n, w_n, \alpha_n),
\end{align*}$$

(2)

where $u_n \in T(z_n)$ such that $d(z_n, u_n) = r_{z_n}(T(z_n))$, $v_n \in T(y_n)$ such that $d(v_n, y_n) = r_{y_n}(T(y_n))$ and $x_n \in T(w_n)$ such that $d(x_n, w_n) = r_{w_n}(T(w_n))$.

We need the following important Lemmas that will be used in the sequel.

Lemma 3.1 Let $E$ be a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping. If $\{z_n\}$ is a sequence in $E$, then the following holds:

$$z_n \xrightarrow{\Delta} z, \text{dist}(z_n, T(z_n)) \to 0 \text{ and } \text{diam}(T(z_n)) \to 0 \text{ imply } z \in \text{End}(T).$$

Proof. From Lemma 2.5, we obtain that $z \in E$. For each $n \in \mathbb{N}$, we can choose $w_n \in T(z_n)$ such that $d(z_n, w_n) = \text{dist}(z_n, T(z_n))$. By passing through a subsequence, we may assume that $\{z_n\}$ is regular relative to $E$. Let $A(E, \{z_n\}) = \{z\}$ and $r = r(E, \{z_n\})$. By similar way in the proof of Lemma 2.12, we obtain that $z \in \text{End}(T)$.

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Lemma 3.2 Let $E$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $\text{End}(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). Then $\lim_{n \to \infty} d(z_n, q)$ exists for each $q \in \text{End}(T)$.

Proof. Let $T$ be a multi-valued Suzuki mapping and $q \in \text{End}(T)$. Therefore,

\[ \frac{1}{2} \dist(q, T(q)) = 0 \leq d(q, y_n), \]

\[ \frac{1}{2} \dist(q, T(q)) = 0 \leq d(q, w_n), \]

and

\[ \frac{1}{2} \dist(q, T(q)) = 0 \leq d(q, z_n), \]

for all $n \in \mathbb{N}$. This implies that

\[ H(T(q), T(y_n)) \leq d(q, y_n), \]

\[ H(T(q), T(w_n)) \leq d(q, w_n), \]

and

\[ H(T(q), T(z_n)) \leq d(q, z_n). \]

Using (2) and (8), we obtain that

\[ d(y_n, q) = d(W(u_n, z_n, \gamma_n), q) \]

\[ \leq (1 - \gamma_n)d(u_n, q) + \gamma_n d(z_n, q) \]

\[ = (1 - \gamma_n)\dist(u_n, T(q)) + \gamma_n d(z_n, q) \]

\[ \leq (1 - \gamma_n)H(T(z_n), T(q)) + \gamma_n d(z_n, q) \]

\[ \leq (1 - \gamma_n)d(z_n, q) + \gamma_n d(z_n, q) \]

\[ \leq d(z_n, q). \]

Next, using (2), (6) and (9)

\[ d(w_n, q) = d(W(v_n, y_n, \beta_n), q) \]

\[ \leq (1 - \beta_n)d(v_n, q) + \beta_n d(y_n, q) \]

\[ = (1 - \beta_n)\dist(v_n, T(q)) + \beta_n d(y_n, q) \]

\[ \leq (1 - \beta_n)H(T(y_n), T(q)) + \beta_n d(y_n, q) \]

\[ \leq (1 - \beta_n)d(y_n, q) + \beta_n d(y_n, q) \]

\[ \leq d(y_n, q) \leq d(z_n, q). \]
Again, using (2), (7) and (10)

\[ d(z_{n+1}, q) = d(W(x_n, w_n, \alpha_n), q) \]
\[ \leq (1 - \alpha_n)d(x_n, q) + \alpha_n d(w_n, q) \]
\[ = (1 - \alpha_n)\text{dist}(x_n, T(q)) + \alpha_n d(w_n, q) \]
\[ \leq (1 - \alpha_n)H(T(w_n), T(q)) + \alpha_n d(w_n, q) \]
\[ \leq (1 - \alpha_n)d(w_n, q) + \alpha_n d(w_n, q) \]
\[ \leq d(w_n, q) \]
\[ \leq d(z_n, q). \quad (11) \]

This shows that sequence \( \{d(z_n, q)\} \) is decreasing and bounded below. Thus \( \lim_{n \to \infty} d(z_n, q) \) exists for each \( q \in \text{End}(T) \).

Next, we prove \( \Delta \)-convergence theorem for a multi-valued mapping in hyperbolic spaces.

**Theorem 3.3** Let \( E \) be a nonempty closed convex subset of a complete uniformly convex hyperbolic space \( X \) with monotone modulus of uniform convexity and \( T : E \to K(E) \) be a multi-valued Suzuki mapping with \( \text{End}(T) \neq \emptyset \). Let \( \{z_n\} \) be a sequence generated by the modified SP-iteration process (2). Then \( \{z_n\} \) \( \Delta \)-converges to an endpoint of \( T \).

**Proof.** First we will prove that \( r_{z_n}(T(z_n)) \to 0 \). Let \( q \in \text{End}(T) \). Since \( T \) is a multi-valued Suzuki mapping and

\[ \frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, z_n) \]

for all \( n \in \mathbb{N} \), then

\[ H(T(q), T(z_n)) \leq d(q, z_n). \]

From Lemma 3.2, we know that for each \( q \in \text{End}(T) \), \( \lim_{n \to \infty} d(z_n, q) \) exists. Let \( \lim_{n \to \infty} d(z_n, q) = t \geq 0 \). If \( t = 0 \), then

\[ d(z_n, u_n) \leq d(z_n, q) + d(q, u_n) \]
\[ = d(z_n, q) + \text{dist}(T(q), u_n) \]
\[ \leq d(z_n, q) + H(T(q), T(z_n)) \]
\[ \leq d(z_n, q) + d(z_n, q). \]

Taking \( n \to \infty \) on above inequality, we have

\[ \lim_{n \to \infty} r_{z_n}(T(z_n)) = \lim_{n \to \infty} d(z_n, u_n) = 0. \]
If $t > 0$, then
\[ d(y_n, q) = d(W(u_n, z_n, \gamma_n), q) \]
\[ \leq (1 - \gamma_n)d(u_n, q) + \gamma_n d(z_n, q) \]
\[ = (1 - \gamma_n)\text{dist}(u_n, T(q)) + \gamma_n d(z_n, q) \]
\[ \leq (1 - \gamma_n)H(T(z_n), T(q)) + \gamma_n d(z_n, q) \]
\[ \leq (1 - \gamma_n)d(z_n, q) + \gamma_n d(z_n, q) \]
\[ \leq d(z_n, q). \]

Letting limsup as $n \to \infty$ on the both sides of above inequality, we have
\[ \limsup_{n \to \infty} d(y_n, q) \leq \limsup_{n \to \infty} d(z_n, q) \leq t. \] \hfill (12)

From (11), we have $d(z_{n+1}, q) \leq d(w_n, q)$. Then we obtain that
\[ t \leq \liminf_{n \to \infty} d(z_{n+1}, q) \leq \liminf_{n \to \infty} d(w_n, q). \] \hfill (13)

From the proof in (10), we have $d(w_n, q) \leq d(y_n, q)$. Taking liminf as $n \to \infty$ on above inequality and using (13),
\[ t \leq \liminf_{n \to \infty} d(y_n, q). \] \hfill (14)

Combine (12) and (14), we obtain that
\[ \lim_{n \to \infty} d(W(u_n, z_n, \gamma_n), q) = \lim_{n \to \infty} d(y_n, q) = t. \] \hfill (15)

Since
\[ d(u_n, q) = \text{dist}(u_n, T(q)) \]
\[ \leq H(T(z_n), T(q)) \leq d(z_n, q), \]

this implies that
\[ \limsup_{n \to \infty} d(u_n, q) \leq t. \] \hfill (16)

By (15), (16), $\lim_{n \to \infty} d(z_n, q) = t$ together with Lemma 2.6, we have
\[ \lim_{n \to \infty} d(u_n, z_n) = 0. \] \hfill (17)

From the condition of the modified SP-iteration, so
\[ \lim_{n \to \infty} r_{z_n}(T(z_n)) = \lim_{n \to \infty} d(u_n, z_n) = 0. \] \hfill (18)

Hence by the both cases we can conclude that $r_{z_n}(T(z_n)) \to 0$. It follows from Lemma 2.11, we have dist$(z_n, T(z_n)) \to 0$ and diam$(T(z_n)) \to 0$. 

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To show that \( \{z_n\} \) \( \Delta \)-converges to an endpoint of \( T \). Now we prove that
\[
W_\omega(z_n) := \cup_{s_n \in [z_n]} AC(E; \{s_n\}) \subseteq \text{End}(T) \quad \text{and} \quad W_\omega(z_n) \text{ consists of exactly one point. Let } s \in W_\omega(z_n). \text{ Therefore there exists a subsequence } \{s_{n_k}\} \text{ of } \{z_n\} \text{ such that } AC(E; \{s_{n_k}\}) = \{s\}. \text{ From Lemma 2.5 and Lemma 2.7, there exists a subsequence } \{t_{n_k}\} \text{ of } \{s_{n_k}\} \text{ such that } \Delta \text{-lim}_{n \to \infty} t_{n_k} = t \in E. \text{ Since } \text{dist}(t_{n_k}, T(t_{n_k})) \to 0 \text{ and } \text{diam}(T(t_{n_k})) \to 0 \text{ and it follows from Lemma 3.1, we have } t \in \text{End}(T) \text{ and } \lim_{n \to \infty} d(z_n, t) \text{ exists by Lemma 3.2. Thus by Lemma 2.8 we have } s = t \in \text{End}(T). \text{ This shows that } W_\omega(z_n) \subseteq \text{End}(T). \text{ Next, we prove that } W_\omega(z_n) \text{ consists of exactly one point. Let } \{s_n\} \text{ be a subsequence of } \{z_n\} \text{ such that } AC(E; \{s_n\}) = \{s\} \text{ and } AC(E; \{z_n\}) = \{z\}. \text{ Since } s \in W_\omega(z_n) \subseteq \text{End}(T) \text{ and from Lemma 3.2, we know that } \{d(z_n, s)\} \text{ exists. By Lemma 2.8, } z = s. \text{ Therefore the proof is completed.}

Next, we present the following key lemma for proving the strong convergence theorem.

**Lemma 3.4** Let \( E \) be a nonempty closed subset of a complete hyperbolic space \( X \) and \( \{w_n\} \) be a Fejér monotone sequence with respect to \( E \). Then \( \{w_n\} \) converges strongly to an element of \( E \) if and only if \( \lim_{n \to \infty} \text{dist}(w_n, E) = 0 \).

**Proof.** Assume that \( \{w_n\} \) converges strongly to \( q \in E \). Thus \( \lim_{n \to \infty} d(w_n, q) = 0 \). Because \( 0 \leq \text{dist}(w_n, E) \leq d(w_n, q) \), therefore \( \lim_{n \to \infty} \text{dist}(w_n, E) = 0 \).

Conversely, suppose that \( \lim_{n \to \infty} \text{dist}(w_n, E) = 0 \). Since \( \{w_n\} \) is a Fejér monotone sequence with respect to \( E \), we have
\[
d(w_{n+1}, q) \leq d(w_n, q) \quad \text{for all } q \in E.
\]

Thus \( \inf_{q \in E} d(w_{n+1}, q) \leq \inf_{q \in E} d(w_n, q) \), which means that \( \text{dist}(w_{n+1}, E) \leq \text{dist}(w_n, E) \). Therefore \( \lim_{n \to \infty} \text{dist}(w_n, E) \) exists. By hypothesis, we obtain that \( \lim_{n \to \infty} \text{dist}(w_n, E) = 0 \). Next, we show that \( \{w_n\} \) is a Cauchy sequence in \( E \). Let \( r > 0 \). Since \( \lim_{n \to \infty} \text{dist}(w_n, E) = 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[
\text{dist}(w_n, E) < \frac{r}{2} \quad \text{for all } n \geq n_0.
\]

In particular, \( \inf\{d(w_{n_0}, q) : q \in E\} < \frac{r}{2} \). Therefore there exists \( q_0 \in E \) such that \( d(w_{n_0}, q_0) < \frac{r}{2} \). For any \( n, m \geq n_0 \), we have
\[
d(w_{n+m}, w_n) & \leq d(w_{n+m}, q_0) + d(q_0, w_n) \\
& \leq d(w_{n_0}, q_0) + d(q_0, w_{n_0}) \\
& \leq \frac{r}{2} + \frac{r}{2} = r.
\]

This means that a sequence \( \{w_n\} \) is a Cauchy sequence in \( E \). Since \( E \) is a closed subset of a complete hyperbolic space \( X \), we have \( E \) is also complete. Then \( \{w_n\} \) must be convergent to a point in \( E \). ■
Theorem 3.5 Let $E$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $\text{End}(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). If $T$ satisfies condition (J), then $\{z_n\}$ converges strongly to an endpoint of $T$.

Proof. First, we will show that $\text{End}(T)$ is closed. Let $\{z_n\} \subseteq \text{End}(T)$ such that $z_n \to z \in E$. We will prove that $z \in \text{End}(T)$. Since $T$ is a multi-valued Suzuki mapping, therefore $T$ satisfies condition $(E_3)$. Then

$$\text{dist}(z_n, Tz) \leq 3\text{dist}(z_n, T(z_n)) + d(z_n, z) \to 0 \text{ as } n \to \infty.$$  

This implies that $z \in T(z)$. Next, we show that $\{z\} = T(z)$. Take any point $w \in T(z)$. Since $T$ is a multi-valued Suzuki mapping,

$$\frac{1}{2}\text{dist}(z_n, T(z_n)) = 0 \leq d(z_n, z) \text{ implies that } H(T(z_n), T(z)) \leq d(z_n, z).$$

Since $z_n \in \text{End}(T)$, we have

$$d(w, z) \leq d(w, z_n) + d(z_n, z) = \text{dist}(w, T(z_n)) + d(z_n, z) \leq H(T(z), T(z_n)) + d(z_n, z) \leq d(z_n, z) + d(z_n, z) \to 0 \text{ as } n \to \infty.$$  

Hence $w = z$. Because $w \in T(z)$ is arbitrary, then $T(z) = \{z\}$, so $z \in \text{End}(T)$. Thus $\text{End}(T)$ is closed. Next, as in the proof of Theorem 3.3, we have $r_{z_n}(T(z_n)) \to 0$ and it follows from $T$ satisfies condition (J),

$$h(\text{dist}(z_n, \text{End}(T))) \leq r_{z_n}(T(z_n)) \to 0.$$  

This implies that $\lim_{n \to \infty} h(\text{dist}(z_n, \text{End}(T))) = 0$. Since $h : [0, \infty) \to [0, \infty)$ is nondecreasing with $h(0) = 0$, $h(r) > 0$ for $r \in (0, \infty)$, we obtain that $\lim_{n \to \infty} \text{dist}(z_n, \text{End}(T)) = 0$. As in the proof of Lemma 3.2 implies that $\{z_n\}$ is Fejér monotone with respect to $\text{End}(T)$. By applying Lemma 3.4, we obtain the desired result. \hfill $\blacksquare$

Theorem 3.6 Let $E$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $\text{End}(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). If $T$ is semicompact, then $\{z_n\}$ converges strongly to an endpoint of $T$.

Proof. As in the proof of Theorem 3.3, $r_{z_n}(T(z_n)) \to 0$ and $T$ is semicompact, we may assume a subsequence $z_{n_k} \to z$ for some $z \in E$. Again, as in the proof of Theorem 3.3, we obtain that $r_{z_{n_k}}(T(z_{n_k})) \to 0$. By Lemma 2.11, we also get

$$\text{dist}(z_{n_k}, T(z_{n_k})) \leq 3\text{dist}(z_{n_k}, T(z_{n_k})) + d(z_{n_k}, z) \to 0 \text{ as } k \to \infty.$$  

This implies that $z \in T(z_{n_k})$. Since $T(z_{n_k}) = \{z\}$, we obtain that $z \in \text{End}(T)$.
The modified SP-iteration process (2) converging to $T$ as $k \to \infty$. Since $T$ is a multi-valued Suzuki mapping, therefore $T$ satisfies condition (E3). Because of

$$\text{dist}(z, T(z)) \leq d(z, T(z))$$

we obtain that $z \in T(z)$. Next, we show that $\{z\} = T(z)$.

Notice that $\frac{1}{2} \text{dist}(z, T(z)) = 0 \leq d(z, k)$ for all $k \in \mathbb{N}$. Since $T$ is a multi-valued Suzuki mapping, we have

$$H(T(z), T(z)) \leq d(z, z).$$

We now let $u \in T(z)$ and choose $w_{nk} \in T(z_{nk})$ so that $d(u, w_{nk}) = \text{dist}(u, T(z_{nk}))$. For all $k \in \mathbb{N}$, we obtain that

$$d(z, u) \leq d(z, w_{nk}) + d(w_{nk}, u)$$

Taking limit as $k \to \infty$, we get that $z = u$ for all $u \in T(z)$ and so $\{z\} = T(z)$. Hence $z \in \text{End}(T)$. By Lemma 3.2, $\lim_{k \to \infty} d(z, q)$ exists for each $q \in \text{End}(T)$, it follows that $z_n \to z$ as $n \to \infty$. This completes the proof. □

4 Numerical example

In this section, we give an example shows that there exists a mapping which is a multi-valued Suzuki mapping but is not a nonexpansive mapping. Furthermore, we illustrate that a sequence generated by the modified SP-iteration process (2) converges to an endpoint of the multi-valued Suzuki mapping.

Example 4.1 Let $X = \mathbb{R}$ with metric defined by $d(x, y) = |x - y|$ and $E = [0, 3]$. Define $W : X^2 \times [0, 1] \to X$ by $W(x, y, \alpha) := \alpha x + (1 - \alpha)y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. Then $(X, d, W)$ is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $E$ is a nonempty compact convex subset of $X$. Let $T : E \to K(E)$ defined by

$$Tz = \begin{cases} 
\{0\}, & z \neq 3; \\
\{1\}, & z = 3.
\end{cases}$$

By [19] showed that the mapping $T$ is a Suzuki mapping. But $T$ is not a nonexpansive mapping if we take $x = 2.9$ and $y = 3$. Moreover, $\text{End}(T) = \{0\}$. For initial point $z_0 = 0.1$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{\sqrt{3n + 1}}$. Therefore $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$. Set stop parameter to $|z_n - 0| \leq 10^{-12}$, where 0 is an endpoint of $T$. By using MATLAB, we compute the sequence generated by the modified SP-iteration process (2) converging to 0 as in Table 1 and Figure 1.
The modified SP-iteration process

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<th>SP-iteration process</th>
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<tr>
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<tr>
<td>$z_2$</td>
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<tr>
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<tr>
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<td>:</td>
</tr>
<tr>
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</tr>
<tr>
<td>$z_{21}$</td>
<td>0.0000000000003</td>
</tr>
<tr>
<td>$z_{22}$</td>
<td>0.0000000000000</td>
</tr>
</tbody>
</table>

**Table 1:** Sequences generated by SP-iteration process

![Convergence of iterative sequences generated by SP-iteration process](image)

**Figure 1** Convergence of iterative sequences generated by SP-iteration process

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References


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