Foundation of Stochastic Fractional Calculus with Fractional Approximation of Stochastic processes

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Foundation of Stochastic Fractional Calculus

Let $t \in [a,b] \subset \mathbb{R}$, $\omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a probability space. Here $X(t,\omega)$ stands for a stochastic process. Case of $X(\cdot,\omega)$ being continuous on $[a,b]$, $\forall$ $\omega \in \Omega$. Then by Caratheodory Theorem 20.15, p.156, [1], we get that $X(t,\omega)$ is jointly measurable.

Next we define the left and right respectively, Riemann-Liouville stochastic fractional integrals, where $\alpha > 0$ is not an integer:

$$I_{a+}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} X(t,\omega)\, dt, \tag{1}$$

and

$$I_{b-}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} X(t,\omega)\, dt, \tag{2}$$

$\forall$ $x \in [a,b]$, $\forall$ $\omega \in \Omega$, where Γ is the gamma function.

In the following important cases we prove that $I_{a+}^{\alpha}X$, $I_{b-}^{\alpha}X$ are stochastic processes:

i) Assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space, and that $(x-t)^{\alpha-1} X(t,\omega)$ is an integrable function on $[a,x] \times \Omega$, $\forall$ $x \in [a,b]$, then by Fubini's theorem, [12], p. 269, $I_{a+}^{\alpha} X(x,\cdot)$ is an integrable function on Ω, $\forall$ $x \in [a,b]$. Similarly, if $(t-x)^{\alpha-1} X(t,\omega)$ is an integrable function on $[x,b] \times \Omega$, $\forall$ $x \in [a,b]$, then again by Fubini's theorem $I_{b-}^{\alpha} X(x,\cdot)$ is an integrable function on Ω, $\forall$ $x \in [a,b]$. That is $I_{a+}^{\alpha} X(x,\omega)$ and $I_{b-}^{\alpha} X(x,\omega)$ are stochastic processes.

ii) Assume a general probability space $(\Omega, \mathcal{F}, P)$ and the Lebesgue measure spaces on $[a, x]$, $[x, b]$, $\forall\, x \in [a, b]$. These are clearly σ-finite measure spaces. We assume that the jointly measurable stochastic process $X(t, \omega) \geq 0$ on $[a, b] \times \Omega$, hence $(x - t)^{\alpha-1} X(t, \omega) \geq 0$ on $[a, x] \times \Omega$, and $(t - x)^{\alpha-1} X(t, \omega) \geq 0$ on $[x, b] \times \Omega$, $\forall\, x \in [a, b]$, and both are jointly measurable. Then by Tonelli's theorem, [12], p. 270, we get that $I_{a+}^{\alpha} X(x, \cdot)$, $I_{b-}^{\alpha} X(x, \cdot)$ are measurable functions on Ω, $\forall\, x \in [a, b]$. That is $I_{a+}^{\alpha} X$, $I_{b-}^{\alpha} X$ are stochastic processes. The above facts provide the foundation of stochastic fractional calculus in the direct analytical sense. So it is not unusual to consider that $I_{a+}^{\alpha} X$, $I_{b-}^{\alpha} X$ are stochastic processes.

iii) Given that $X(\cdot, \omega)$ is in $L_1([a, b])$ then $I_{a+}^{\alpha} X(\cdot, \omega) \in L_1([a, b])$, $\forall\, \omega \in \Omega$, see [10], p. 13, and $I_{b-}^{\alpha} X(\cdot, \omega) \in L_1([a, b])$, $\forall\, \omega \in \Omega$, see [8], p. 334.

And given that $X(\cdot, \omega) \in L_\infty([a, b])$, then $I_{a+}^{\alpha} X(\cdot, \omega) \in C([a, b])$, when $0 < \alpha < 1$, and $I_{a+}^{\alpha} X(\cdot, \omega) \in AC([a, b])$ (absolutely continuous functions), when $\alpha \geq 1$, $\forall\, \omega \in \Omega$, see [6], p. 388. Similarly, if $X(\cdot, \omega) \in L_\infty([a, b])$, then $I_{b-}^{\alpha} X(\cdot, \omega) \in C([a, b])$, when $0 < \alpha < 1$, and $I_{b-}^{\alpha} X(\cdot, \omega) \in AC([a, b])$, when $\alpha \geq 1$, $\forall\, \omega \in \Omega$, see [9].

Definition 1 *Let non-integer* $\alpha > 0$, $n = \lceil \alpha \rceil$ *(*$\lceil \cdot \rceil$ *is the ceiling of the number),* $t \in [a,b] \subset \mathbb{R}$, $\omega \in \Omega$, *where* $(\Omega, \mathcal{F}, P)$ *is a general probability space. Here* $X(t,\omega)$ *stands for a stochastic process. Assume that* $X(\cdot,\omega) \in AC^n([a,b])$ *(spaces of functions* $X(\cdot,\omega)$ *with* $X^{(n-1)}(\cdot,\omega) \in AC([a,b])$*),* $\forall\, \omega \in \Omega$.

We call stochastic left Caputo fractional derivative

$$D_{*a}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} X^{(n)}(t,\omega)\, dt, \tag{3}$$

$\forall\, x \in [a,b]$, $\forall\, \omega \in \Omega$.

And, we call stochastic right Caputo fractional derivative

$$D_{b-}^{\alpha} X(x,\omega) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} X^{(n)}(z,\omega)\, dz, \tag{4}$$

$\forall\, x \in [a,b]$, $\forall\, \omega \in \Omega$.

Commutative Caputo fractional Korovkin inequalities for Stochastic processes

1 Introduction

Our work is motivated by the following:

Korovkin's Theorem ([12], 1960) Let $(T_j)_{j \in \mathbb{N}}$ be a sequence of positive linear operators from $C([a,b])$ into itself, $[a,b] \subset \mathbb{R}$. In order to have $\lim_{j \to \infty} (T_j f)(t) = f(t)$ (in the sup-norm) for all $f \in C([a,b])$, it is enough to prove it for $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$. The rate of the above convergence for abitrary $f \in C([a,b])$ can be determined exactly from the rates of convergence for f_0, f_1, f_2.

Shisha-Mond inequality ([14]) We have

$$\|T_j(f) - f\| \le \|f\| \cdot \|T_j(1) - 1\| + \omega_1(f, \rho_j) \cdot (1 + \|T_j(1)\|),$$

where

$$\rho_j = \left(\left\|T_j\left((x-y)^2\right)(y)\right\|\right)^{\frac{1}{2}}.$$

In the last inequality $\|\cdot\|$ stands for the supremum norm and ω_1 for the first modulus of continuity. This inequality gives the rate of convergence of T_j to the unit operator I.

Annastassiou in [2]-[4] established a series of sharp inequalities for various cases of the parameters of the problem. However, Weba in [15]-[18] was the first, among many workers in quantitative results of Shisha-Mond type, to produce inequalities for stochastic processes. He assumed that T_j are E-commutative (E means expectation) and stochastically simple. According to his work, if a stochastic process $X(t,\omega)$, $t \in Q$ - a compact convex subset of a real normed vector space, $\omega \in Q$ - probability space, is to be approximated by positive linear operators T_j, then the maximal error in the qth mean is $(q \geq 1)$

$$\|T_j X - X\| = \sup_{t \in Q} \left(E \left| (T_j X)(t,\omega) - X(t,\omega) \right|^q \right)^{\frac{1}{q}}.$$

So, Weba established upper bounds for $\|T_j X - X\|$ involving his own natural general first modulus of continuity of X with several interesting applications.

Anastassiou met ([5]) the pointwise case of $q = 1$. Without stochastic simplicity of T_j he found nearly best and best upper bounds for $|E(T_j X)(x_0) - (EX)(x_0)|$, $x_0 \in Q$.

The author here continues his above work on the approximation of stochastic processes, now at the Caputo stochastic fractional level. He derives pointwise and uniform Caputo fractional stochastic Shisha-Mond type inequalities, see tha main Theorems 4, 7 and the several related corollaries. He gives an extensive application to stochastic Bernstein operators. He finishes with a pointwise and a uniform fractional stochastic Korovkin theorem, derived by Theorems 4, 7. The stochastic convergences, about stochastic processes, of our fractional Korovkin Theorems 15, 16 are enforced only by the convergences of real basic non-stochastic functions.

2 Background

Definition 1 *([10]) Let non-integer* $\alpha > 0$, $n = \lceil \alpha \rceil$ *(*$\lceil \cdot \rceil$ *is the ceiling of the number),* $t \in [a, b] \subset \mathbb{R}$, $\omega \in \Omega$, *where* $(\Omega, \mathcal{F}, P)$ *is a general probability space. Here* $X(t, \omega)$ *stands for a stochastic process. Assume that* $X(\cdot, \omega) \in AC^n([a, b])$ *(spaces of functions* $X(\cdot, \omega)$ *with* $X^{(n-1)}(\cdot, \omega) \in AC([a, b])$ *absolutely continuous functions),* $\forall\ \omega \in \Omega$.

We call stochastic left Caputo fractional derivative

$$D_{*a}^{\alpha} X(x, \omega) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} X^{(n)}(t, \omega)\, dt, \tag{1}$$

$\forall\ x \in [a, b]$, $\forall\ \omega \in \Omega$.

And, we call stochastic right Caputo fractional derivative

$$D_{b-}^{\alpha} X(x, \omega) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} X^{(n)}(z, \omega)\, dz, \tag{2}$$

$\forall\ x \in [a, b]$, $\forall\ \omega \in \Omega$. *Above* Γ *stands for the gamma function.*

$$\left|X^{(n)}(t,\omega)\right| \leq M, \ \forall \ (t,\omega) \in [a,b] \times \Omega,$$

where $M > 0$.

It is not strange to assume that $D^{\alpha}_{*a}X$, $D^{\alpha}_{b-}X$ *are stochastic processes.*

$$\left|D^{\alpha}_{*t}X(x,\omega)\right| \leq \frac{M(x-t)^{n-\alpha}}{\Gamma(n-\alpha+1)} \leq \frac{M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{5}$$

$\forall \ x \in [t,b]$, any $t \in [a,b]$, $\forall \ \omega \in \Omega$,
and

$$\left|D^{\alpha}_{t-}X(x,\omega)\right| \leq \frac{M(t-x)^{n-\alpha}}{\Gamma(n-\alpha+1)} \leq \frac{M(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{6}$$

$\forall \ x \in [a,t]$, any $t \in [a,b]$, $\forall \ \omega \in \Omega$.

Hence, it holds $(\delta > 0)$

$$\sup_{t\in[a,b]} \omega_1\left(E\left(D^{\alpha}_{*t}X\right),\delta\right)_{[t,b]} \leq \frac{2M(b-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}. \tag{10}$$

Similarly, it holds $(\delta > 0)$

$$\sup_{t\in[a,b]} \omega_1\left(E\left(D^{\alpha}_{t-}X\right),\delta\right)_{[a,t]} \leq \frac{2M(b-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}. \tag{11}$$

$$\left|E\left(D_{*t}^{\alpha}X\right)(z)-E\left(D_{*t}^{\alpha}X\right)(t)\right|\leq\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}\left\lceil\frac{|z-t|}{\delta_1}\right\rceil\leq$$

$$\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}\left(1+\frac{(z-t)}{\delta_1}\right), \quad (12)$$

$\forall\ z\in[t,b]$,

and similarly $(\delta_2>0)$,

$$\left|E\left(D_{t-}^{\alpha}X\right)(z)-E\left(D_{t-}^{\alpha}X\right)(t)\right|\leq\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta_2\right)_{[a,t]}\left(1+\frac{t-z}{\delta_2}\right), \quad (13)$$

$\forall\ z\in[a,t]$.

We also set

$$\omega_1\left(E\left(D_t^{\alpha}X\right),\delta\right):=\max\left\{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta\right)_{[t,b]},\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta\right)_{[a,t]}\right\}, \quad (14)$$

where $\delta>0$.

Remark 3 *Let the positive linear operator L mapping $C([a,b])$ into $B([a,b])$ (the bounded functions). By the Riesz representation theorem ([13]) we have that there exists μ_t unique, completed Borel measure on $[a,b]$ with*

$$\mu_t([a,b]) = L(1)(t) > 0, \tag{15}$$

such that

$$L(f)(t) = \int_{[a,b]} f(s)\, d\mu_t(s), \quad \forall\, t \in [a,b], \forall\, f \in C([a,b]). \tag{16}$$

The last means

$$\left| L\left((s-t)^k\right)(t) \right| \le L\left(|s-t|^k\right)(t) \le \left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\left(\frac{k}{\alpha+1}\right)} (L(1)(t))^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \tag{18}$$

all $k = 1, ..., n-1$. It is clear that

$$\left\| L\left(|s-t|^{\alpha+1}\right)(t) \right\|_{\infty,[a,b]} < \infty.$$

Furthermore we derive

$$\left\| L\left((s-t)^k\right)(t) \right\|_{\infty,[a,b]} \le \left\| L\left(|s-t|^k\right)(t) \right\|_{\infty,[a,b]} \le$$

$$\left\| L\left(|s-t|^{\alpha+1}\right)(t) \right\|_{\infty,[a,b]}^{\left(\frac{k}{\alpha+1}\right)} \|L(1)\|_{\infty,[a,b]}^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \tag{19}$$

all $k = 1, ..., n-1$.

From now on we will denote $\|\cdot\|_{\infty,[a,b]} = \|\cdot\|_{\infty}$ the supremum norm.

3 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probabilistic space and $L^1(\Omega, \mathcal{F}, P)$ be the space of all real-valued random variables $Y = Y(\omega)$ with

$$\int_\Omega |Y(\omega)| P(d\omega) < \infty.$$

Let $X = X(t, \omega)$ denote a stochastic process with index set $[a, b] \subset \mathbb{R}$ and real state space $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel subsets of $\mathbb{R}$. Here $C([a, b])$ is the space of continuous real-valued functions on $[a, b]$ and $\mathcal{B}([a, b])$ is the space of bounded real-valued functions on $[a, b]$. Also $C_\Omega([a, b]) = C([a, b], L^1(\Omega, \mathcal{F}, P))$ is the space of L^1-continuous stochastic processes in t and $B_\Omega([a, b]) = \left\{ X : \sup_{t \in [a,b]} \int_\Omega |X(t, \omega)| P(d\omega) < \infty \right\}$, obviously $C_\Omega([a, b]) \subset B_\Omega([a, b])$.

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$, and consider the subspace of stochastic processes $C_\Omega^{\alpha, n}([a, b]) := \{X : X(\cdot, \omega) \in AC^n([a, b]), \forall\ \omega \in \Omega$ and $\left| X^{(n)}(t, \omega) \right| \le M$, $\forall$ $(t, \omega) \in [a, b] \times \Omega$, where $M > 0$; $X^{(k)}(t, \omega) \in C_\Omega([a, b])$, $k = 0, 1, ..., n - 1$; also $D_{*t}^\alpha X$, $D_{t-}^\alpha X$ are stochastic processes for any $t \in [a, b]\}$. That is, for every $\omega \in \Omega$ we have $X(t, \omega) \in C^{n-1}([a, b])$.

Consider the linear operator

$$L : C_\Omega([a, b]) \hookrightarrow B_\Omega([a, b]).$$

If $X \in C_\Omega([a, b])$ is nonnegative and LX, too, then L is called positive. If $EL = LE$, then L is called E-commutative.

4 Main Results

Theorem 4 *Consider the positive E-commutative linear operator $L : C_\Omega([a,b]) \hookrightarrow B_\Omega([a,b])$, and $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$, and let $X \in C_\Omega^{\alpha,n}([a,b])$, with $\delta > 0$.*

Then

$$|(E(LX))(t) - (EX)(t)| \leq |(EX)(t)| \, |(L(1))(t) - 1| + \tag{20}$$

$$\sum_{k=1}^{n-1} \frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!} \left| L\left((s-t)^k\right)(t)\right| + \frac{\omega_1\left(E\left(D_t^\alpha X\right), \delta\right)}{\Gamma(\alpha+1)}$$

$$\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}} \left[(L(1)(t))^{\frac{1}{\alpha+1}} + \frac{\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)} \right],$$

$\forall\, t \in [a,b]$.

Above $\omega_1\left(E\left(D_t^\alpha X\right), \delta\right)$ is as in (14).

From [17, pp. 3-5] we have the following results

(i) $C([a,b]) \subset C_\Omega([a,b])$,

(ii) if $X \in C_\Omega([a,b])$, then $EX \in C([a,b])$,

and

(iii) if L is E-commutative, then L maps the subspace $C([a,b])$ into $B([a,b])$.

Definition 5 *If $0 < \alpha < 1$, then $n = 1$, and $C_\Omega^{\alpha,1}([a,b]) := \{X : X(\cdot,\omega) \in AC([a,b]), \forall\, \omega \in \Omega$ and $\left|X^{(1)}(t,\omega)\right| \le M, \forall\,(t,\omega) \in [a,b] \times \Omega$, where $M > 0$; $X(t,\omega) \in C_\Omega([a,b])$; also $D_{*t}^\alpha X$, $D_{t-}^\alpha X$ are stochastic processes for any $t \in [a,b]\}$.*

Corollary 6 *Consider the positive E-commutative linear operator $L : C_\Omega([a,b]) \hookrightarrow B_\Omega([a,b])$, and $0 < \alpha < 1$ and let $X \in C_\Omega^{\alpha,1}([a,b])$, with $\delta > 0$.*

Then

$$|(E(LX))(t) - (EX)(t)| \le |(EX)(t)|\,|(L(1))(t) - 1| + \frac{\omega_1\left(E\left(D_t^\alpha X\right),\delta\right)}{\Gamma(\alpha+1)}\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}} \left[(L(1)(t))^{\frac{1}{\alpha+1}} + \frac{\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right], \tag{42}$$

$\forall\, t \in [a,b]$.

Theorem 7 *All as in Theorem 4. Then*

$$\|E(LX) - EX\|_\infty \leq \|EX\|_\infty \|L(1) - 1\|_\infty +$$

$$\sum_{k=1}^{n-1} \frac{\left\|EX^{(k)}\right\|_\infty}{k!} \left\|L\left((s-t)^k\right)(t)\right\|_\infty + \sup_{t\in[a,b]} \frac{\omega_1\left(E\left(D_t^\alpha X\right), \delta\right)}{\Gamma(\alpha+1)}$$

$$\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}} \left[\|L(1)\|_\infty^{\frac{1}{\alpha+1}} + \frac{\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right] < \infty. \quad (43)$$

Corollary 8 *All as in Corollary 6. Then*

$$\|E(LX) - EX\|_\infty \leq \|EX\|_\infty \|L(1) - 1\|_\infty +$$

$$\sup_{t\in[a,b]} \frac{\omega_1\left(E\left(D_t^\alpha X\right), \delta\right)}{\Gamma(\alpha+1)} \left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}}$$

$$\left[\|L(1)\|_\infty^{\frac{1}{\alpha+1}} + \frac{\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right]. \quad (44)$$

Corollary 9 *All as in Corollary 6, and* $L(1)=1$. *Then*

$$\|E(LX)-EX\|_\infty \leq \sup_{t\in[a,b]} \frac{\omega_1\left(E\left(D_t^\alpha X\right),\delta\right)}{\Gamma(\alpha+1)}$$

$$\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}}\left[1+\frac{\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right]. \qquad (45)$$

Corollary 10 *All as in Corollary 6, and* $L(1)=1$. *Then*

$$\|E(LX)-EX\|_\infty \leq \frac{2\sup\limits_{t\in[a,b]}\omega_1\left(E\left(D_t^\alpha X\right),\frac{1}{(\alpha+1)}\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)}$$

$$\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}}. \qquad (46)$$

5 Application

Let $f \in C([0,1])$ and the Bernstein polynomials

$$B_N(f)(t) := \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \tag{47}$$

$\forall\ t \in [0,1]$, $\forall\ N \in \mathbb{N}$.

We have that $B_N 1 = 1$ and B_N is a positive linear operator.

We have that

$$B_N\left((\cdot - t)^2\right)(t) = \frac{t(1-t)}{N},\ \ \forall\ t \in [0,1], \tag{48}$$

and

$$\left\| B_N\left((\cdot - t)^2\right)(t)\right\|_{\infty}^{\frac{1}{2}} \leq \frac{1}{2\sqrt{N}},\ \ \forall\ N \in \mathbb{N}. \tag{49}$$

Define the corresponding stochastic application of B_N by

$$B_N(X)(t,\omega) := B_N(X(\cdot,\omega))(t) = \sum_{k=0}^{N} X\left(\frac{k}{N},\omega\right)\binom{N}{k} t^k (1-t)^{N-k}, \tag{50}$$

$\forall\ t \in [0,1]$, $\forall\ \omega \in \Omega$, $N \in \mathbb{N}$, where X is a stochastic process. Clearly $B_N(X)$ is a stochastic process and $B_N : C_\Omega([0,1]) \hookrightarrow C_\Omega([0,1])$. Notice that

$$(EB_N(X))(t) = \sum_{k=0}^{N} (EX)\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k} = (B_N(EX))(t), \tag{51}$$

$\forall\ t \in [0,1]$.

That is $EB_N = B_N E$, i.e. B_N is an E-commutative positive linear operator.

Proposition 11 *Let* $0 < \alpha < 1$ *and* $X \in C_{\Omega}^{\alpha,1}([0,1])$. *Then*

$$\|E(B_N X) - EX\|_{\infty} \leq \frac{2 \sup_{t\in[0,1]} \omega_1 \left(E\left(D_t^{\alpha} X\right), \frac{1}{(\alpha+1)} \left\| B_N\left(|s-t|^{\alpha+1}\right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)}{\Gamma(\alpha+1)}$$

$$\left\| B_N\left(|s-t|^{\alpha+1}\right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}}. \qquad (52)$$

Corollary 12 *Let* $X \in C_{\Omega}^{\frac{1}{2},1}([0,1])$. *Then*

$$\|E(B_N X) - EX\|_{\infty} \leq \frac{4}{\sqrt{\pi}} \sup_{t\in[0,1]} \omega_1 \left(E\left(D_t^{\frac{1}{2}} X\right), \frac{2}{3} \left\| B_N\left(|s-t|^{\frac{3}{2}}\right)(t) \right\|_{\infty}^{\frac{2}{3}} \right)$$

$$\left\| B_N\left(|s-t|^{\frac{3}{2}}\right)(t) \right\|_{\infty}^{\frac{1}{3}}, \quad \forall\ N \in \mathbb{N}. \qquad (53)$$

Remark 13 *We notice that*

$$B_N\left(|s-t|^{\frac{3}{2}}\right)(t)=\sum_{k=0}^{N}\left|t-\frac{k}{N}\right|^{\frac{3}{2}}\binom{N}{k}t^k(1-t)^{N-k}$$

(by discrete Hölder's inequality)

$$\leq\left(\sum_{k=0}^{N}\left|t-\frac{k}{N}\right|^{2}\binom{N}{k}t^k(1-t)^{N-k}\right)^{\frac{3}{4}} \tag{54}$$

$$\overset{(48)}{=}\left(\frac{1}{N}t(1-t)\right)^{\frac{3}{4}}\leq\frac{1}{(4N)^{\frac{3}{4}}},\quad\forall\ t\in[0,1].$$

That is

$$\left\|B_N\left(|s-t|^{\frac{3}{2}}\right)(t)\right\|_{\infty}\leq\frac{1}{(4N)^{\frac{3}{4}}}, \tag{55}$$

and

$$\left\|B_N\left(|s-t|^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{1}{3}}\leq\frac{1}{(4N)^{\frac{1}{4}}}, \tag{56}$$

and

$$\left\|B_N\left(|s-t|^{\frac{3}{2}}\right)(t)\right\|_{\infty}^{\frac{2}{3}}\leq\frac{1}{2\sqrt{N}}, \tag{57}$$

$\forall\ N\in\mathbb{N}.$

Proposition 14 *Let* $X \in C_{\Omega}^{\frac{1}{2},1}([0,1])$. *Then*

$$\left\| E\left(B_N X\right) - EX \right\|_{\infty} \leq \frac{\left(\sqrt{2}\right)^3}{\sqrt{\pi}\sqrt[4]{N}} \sup_{t\in[0,1]} \omega_1\left(E\left(D_t^{\frac{1}{2}} X\right), \frac{1}{3\sqrt{N}} \right), \qquad (58)$$

$\forall\ N \in \mathbb{N}$.

Hence $\lim_{N\to\infty} E\left(B_N X\right) = EX$, *uniformly.*

6 Caputo Fractional Stochastic Korovkin theory

Here L is meant as a sequence of positive E-commutative linear operators and all assumptions are as in Theorem 4.

Theorem 15 *We further assume that* $L(1)(t) \to 1$ *and* $L\left(|s-t|^{\alpha+1}\right)(t) \to 0$, *then* $(E(LX))(t) \to (EX)(t)$, *for any* $X \in C_{\Omega}^{\alpha,n}([a,b])$, $\forall\, t \in [a,b]$, *a pointwise convergence; where* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$.

Theorem 16 *We further assume that* $L(1)(t) \to 1$, *uniformly and* $\left\| L\left(|s-t|^{\alpha+1}\right)(t) \right\|_{\infty} \to 0$, *then* $E(LX) \to EX$, *uniformly over* $[a,b]$, *for any* $X \in C_{\Omega}^{\alpha,n}([a,b])$; *where* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$.

Remark 17 *The stochastic convergences of Theorems 15, 16 are derived by the convergences of the basic and simple real non-stochastic functions* $\left\{1, |s-t|^{\alpha+1}\right\}$, *an amazing fact!*

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Thank you!